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ON SOME SUBFIELDS OF $K((X))$

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ABSTRACT. Let K be a commutative field and let $K((X))$ be the field of Laurent series in one variable X, consider with its natural X-adic topology. In this paper we prove that any closed subfield $K \subset L \subset K((X))$ is of the form $L = K((f))$ and $K((X))$ is algebraic over L of degree $ord_X(f)$. Some other properties of L are studied.

Key words : Field of Laurent series, Krull valuation, algebraic extension, completeness
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1. Introduction

Let K be a commutative field and let $K((X))$ be the field of Laurent series in one variable X with the coefficients in K. If $g \in K((X))$, then $g = X^m u$ where $m \in Z$ and u is a unit in $K[[X]]$, the ring of formal power series. We call m the order of g and denote it by $ord_X(g)$. The mapping $g \mapsto ord_X(g) \in Z \cup \{\infty\}$, $ord_X(0) = \infty$ is a Krull valuation on $K((X))$ with the valuation ring $K[[X]]$.

Lemma 1. *Let f be a non constant series of $K((X))$ with $ord_X(f) \geq 1$. Then the restriction of the X-adic topology to $K((f))$ is the same as its natural f-adic topology.*

Proof. It is sufficient to see that $ord_X(g) = ord_X(f) ord_f(g)$ for any $g \in K((f))$. So $ord_x(g) \to \infty$ if and only if $ord_f(g) \to \infty$ □

Lemma 2. *Let $f \in K((X))$ such that $ord_X(f) \geq 1$. Then any $g \in K[[X]]$ can be written in a unique way as: $g = r_0 + r_1 f + ... + r_n f^n + ...$, where $r_i = r_i(X)$ are polynomials in X with $degr_i(X) < ord_X(f)$, for any $i = 0, 1, 2....$*

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Proof. If the order of g is greater or equal to the order n of f, then $g = X^m v$, with v a unit in $K[[X]]$. But $f = X^n u$, where u is a unit in $K[[X]]$. So, $g = X^{m-n} f u^{-1} v = fh$, where $h = X^{m-n} u^{-1} v$, has the order less than $ord_X(g)$. If $ord_X(h) \geq ord_X(f)$ we take h instead of g and repeat the above reasoning up to a point when $g = f^h h_1$ and $ord_X(h_1) < ord_X(f)$. Hence we may assume: $ord_X(g) < ord_X(f)$. Write $g = a_0 + a_1 X + ... + a_{n-1} X^{n-1} + X^n g_1$ and denote $a_0 + a_1 X + ... + X_{n-1} X^{n-1}$ by $r_0(X)$. Since $X^n = f u^{-1}$, we obtain that $g = r_0(X) + f u^{-1} g_1$. Take now $u^{-1} g_1$ and repeat the above reasoning. We find $g = r_0(X) + r_1(X) f + f^2 g_2$, where $g_2 \in K[[X]]$ and $degr_1(X) < ord_X(f)$. If we continue in this way we get $g = r_0(X) + r_1(X) f + r_2(X) f^2 + ... + r_t(X) f^t + f^{t+1} g_{t+1}$, for any $t = 0, 1, 2, ...$ Since $ord_X(f) \geq 1$, the series $r_0(X) + r_1(X) f + r_2(X) f^2 + ... + r_t(X) f^t + ...$ is convergent in $K[[X]]$ to g. □

Theorem 3. *Let $f \in K((X))$ such that $ord_X(f) \geq 1$. Then $K((X))$ is an algebraic extension of $K((f))$ and $[K((X)) : K((f))] = n$, where $n = ord_X(f)$.*

Proof. Let $V = K((f)) + K((f))X + ... + K((X))X^{n-1}$ be the $K((f))$-vector subspace of $K((X))$ generated by $\{1, X, ..., X^{n-1}\}$. Let us consider $g \in K[[X]]$. From Lemma 2 we write $g = (a_0 + a_1 X + ... + a_{n-1} X^{n-1}) + (b_0 + b_1 X + ... + b_{n-1} X^{n-1}) f + ... + (w_0 + w_1 X + ... + w_{n-1} X^{n-1}) f^t + ...$ After rearrangement of the terms, using the convergence in $K((X))$, we obtain $g = (a_0 + b_0 f + ... + w_0 f^t + ...) + (a_1 + b_1 f + ... + w_1 f^t + ...) X + ... + (a_{n-1} + b_{n-1} f + ... + w_{n-1} f^t + ...) X^{n-1}$, i.e. $g \in V$. In particular, $X^n \in V$, i.e. $X^n - s_{n-1}(f) X^{n-1} - s_{n-2}(f) X^{n-2} - ... - s_0(f) = 0$, where $s_j(f) \in K((f))$ for any $j = 0, 1, ..., n-1$. So that $X^{-1} \in V$. Now, any $h \in K((X))$ can be written as $h = \frac{g}{X^t}$, with $g \in V$. Since V is also a subfield (being a finite extension of a field) we get that $h \in V$, $K((X)) = V$. It results from here that $[K((X)) : K((f))] \leq n$. Since $K \subset K((f)) \subset K((X))$ and since the residue fields of $K((f))$ and $K((X))$ coincide with K, then the degree $[K((X)) : K((f))]$ is the ramification index of $K((f)) \subset K((X))$ which is exactly n. Hence $[K((X)) : K((f))] = n$ □

Corollary 4. *For any $f \in K((X))$ with $ord_X(f) = n \geq 1$, one has $K((f)) = K((X^n))$ if and only if $X^n \in K((f))$*

Theorem 5. *Let $L \supset K$ be a closed subfield of $K((X)$ (relative to the X-adic topology). Then $K((X))$ is a finite algebraic extension of L and $L = K((f))$, for a $f \in K((X))$.*

Proof. Let $f \in L$ with $ord_X(f) \geq 1$. Since $K \subset K((f)) \subset L \subset K((X))$ (here we use the fact that L is closed in $K((X))$), from Theorem 3 one has that $K((X))/L$ is algebraic and $[K((X)) : L] \leq n$, where $n = ord_X(f)$. Now, since L is a closed subfield of $K((X))$, it is complete. Then, by 4. $L = K((f))$, where $f \in L$ such that $ord_X(f) = min\{ord_X(g) > 0 : g \in L\}$.

□

Remark 1. *If L is not closed, it is possible that L cannot be generated by one element. Take for instance $K = Q$, the rational number field, and take*

$L = Q(X, e^X) \subset Q((X))$, where $e^X = \sum_{x=0}^{\infty} \frac{1}{n!}X^n$. Since $Q(X) \subset L \subset Q((X))$, we have $\tilde{L} = Q((X))$. If L is closed then L must be equal to $Q((X))$, which is impossible (as $e^{X^2} \notin L$). Since in L we have rational functions in X and in e^X, let us assume that $L = Q(f)$, $f \in Q((X))$. Then $X = A(f)/B(f)$, where $A(X), B(X) \in Q[X]$. This means that f is algebraic over $Q(X)$. Since $e^X = U(f)/V(f)$, where $U(X), V(X) \in Q[X]$ and since f is algebraic over $Q(X)$ we get that e^X is algebraic over $Q(X)$. Hence e is algebraic over Q, which is a contradiction (see 2., for instance), e being a transcendental number.

Remark 2. *The mapping $X \to f$, where $ord_X(f) \geq 1$ gives a field K-endomorphism of $K((X))$. Since $K((f)) \subset K((X))$ is an algebraic extension of degree $n = ord_X(f)$, this last K-endomorphism is K-automorphism if and only if $ord_X(f) = 1$, i.e. $f = a_1X + a_2X^2 + ...,\ a_1 \neq 0$. This last result was directly obtained by Shaheen Nazir in 5.*

Let $E \supset K$ be a subfield of $K(X)$. From Lüroth theorem (see 3.), we know that $E = K(g)$ for a rational function $g(X)$ of $K(X)$. If $ord_X(g) < 0$ we shall change g with $1/g$. If $ord_X(g) = 0$ we write $g(X) = \frac{a_0+a_1X+...+a_kX^k}{b_0+b_1X+...+b_lX^l}$ and change g with $g - \frac{a_0}{b_0}$, which has the order ≥ 1. Hence we can always consider g in $E = K(g)$ with $ord_X(g) \geq 1$.
Since the X-adic topology on E is the same with g-adic topology of it, the completion of E in $K((X))$ is exactly $K((g))$. Thus we have the following proposition:

Proposition 6. *Let $E \supset K$ be a subfield of $K(X)$. Let $\omega(E) = ord_X(g) \geq 1$ where g is any generator of E in $K(X)$ i.e. $E = K(g)$, then the X-adic completion of E in $K((X))$ is exactly $\tilde{E} = K((g))$ and $K((X)) = \widetilde{K(X)}$ is algebraic over $\tilde{E}$, $[K((X)) : \tilde{E}] = \omega(E)$.*

Definition 1. *Let $K \subset E \subset K(X)$. An element $g \in K(X)$ s.t. $E = K(g)$ is said to be a Lüroth generator of E. If L is a closed subfield of $K((X))$, an element $q \in K(X)$ s.t. $L = K((q))$ is said to be Lüroth generator of L.*

If L is a closed subfield of $K((X))$, it may not have a Lüroth generator. For example $L = Q((f))$, $f = e^X - 1 - X$. We shall prove that $L \cap Q(x) = Q$. If not, take $q(X) = \frac{P(X)}{R(X)} \in L \cap Q(X)$, $q(X) \notin Q$. We can assume that $ord_X(q(X)) \geq 1$ (see the above reasoning). Suppose $q(X) = \frac{A_0+A_1f+A_2f^2+...}{f^k}$. Since $ord_X(f) = 2$ and $ord_X(q) \geq 1$, we must have $k = 0$ and $A_0 = 0$. Take a natural number ≥ 2 such that $R(m) \neq 0$ then $R(m) \in Q$, but $A_1f(m) + A_2f^2(m) + ...$ is not a convergent series except $A_{n_0+1} = A_{n_0+2} = ... = 0$ for a natural number n_0. So that $q(X) = A_1f(X) + A_2f^2(X) + ... + A_{n_0}f^{n_0}(X)$ with $A_1, A_2, ..., A_{n_0} \in Q$. From here one can see that $e^X - 1 - X$ is algebraic over $Q(q(X))$ i.e. e^X is algebraic over $Q(q(X))$, since X is algebraic over $Q(q(X))$(see Lüroth theorem). Hence does exist a relation of the following type: $A_0(X) + A_1(X)e^X + ... + A_h(X)e^{hx} = 0$ for any X, where $A_0(X), ..., A_h(X) \in Q[X]$. Take them to be coprime and then take $X = 1$, we obtain a nontrivial relation of the form: $b_0 + b_1 e + b_2 e^2 + ... + b_k e^k = 0$ where $b_0, b_1, ..., b_h \in Q$. This means that e is algebraic over Q which is a contradiction. Hence $L \cap Q(X) = Q$. Therefore L cannot have a Lüroth generator. The following result will clarify the general situation.

Theorem 7. *A closed subfield $L \supset K$ of $K((X))$, has a Lüroth generator if and only if $\widetilde{L \cap K(X)} = L$*

Proof. Let $L = K((g))$, where $ord_X(g) \geq 1$ and $g \in K(X)$. Since $g \in L \cap K(X) \Rightarrow K(g) \subset L \cap K(X)$, taking the completion, we have $L = K((g)) \subseteq \widetilde{L \cap K(X)}$. Moreover, $\widetilde{L \cap K(X)} \subseteq L$. Thus $\widetilde{L \cap K(X)} = L$. Conversely, since $L \neq K$, $L \cap K(X) \neq K$ so there exists $q(X) \in K(X) \backslash K$ such that $ord_X(q) \geq 1$ and $K(q) = L \cap K(X)$. Then $L = \widetilde{L \cap K(X)} = \widetilde{K(q)} = K((q))$, because the X-adic topology on $L((q))$ is the same with the q-adic topology of it. $\square$

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