Journal of Prime Research in Mathematics Vol. **2***(2006), 203-207*

ON SOME NUMERICAL INVARIANTS ASSOCIATED TO A COMPACT SETS (IN METRIC SPACES)

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ABSTRACT. In this note we associate to a compact subset of a metric space a subset of natural numbers. We give some interesting properties of this last subset. We also introduce a configuration matrix for a given compact set and propose a conjecture related to this.

Key words : Compact set, covering function, uniform covering, configuration matrix, conjecture.
AMS SUBJECT: 54D99.

1. Introduction

Let C be a compact subset of a metric space (X, d). Let $D = \{D_i\}_{i\epsilon I}$ be any covering of C consisting of open subsets. Since C is a compact subset , we can extract a finite covering $D' = \{D_1, ..., D_k\}$ from D. We can assume that this last covering is a reduced one i.e. if we exclude anyone of $D_j; j = 1, ..., k$, then the remaining will not be a covering of C.

For a covering $D = \{D_i\}_{i\epsilon I}$ of C, we denote by $n_C(D)$ the least cardinal number of all finite coverings of C extracted from D. For instance, if $C = \{a, b\}, a \neq b$ where $a, b \in X$, then $n_C(D)$ is equal to 1 or 2. Denote:

$$A(C) = \{n_C(D) : D \text{ is a reduced covering of } C\}$$

Then $A(C) \subset \mathbb{N}$. If C is not the empty set, then $\inf A(C) = 1$, and $\sup A(C) \leq \infty$.

Let us call $C \mapsto A(C)$, the covering function of (X, d) and the number $\sup A(C)$, the norm of C. We shortly denote it by $\parallel C \parallel$ and if $C = \phi$ we put $\parallel C \parallel = 0$.

Let D' and D'' be two reduced coverings of C. We say that D' *is richer than* D'' and write $D' \succeq D''$ if for all $D_1 \in D'$, there exists $D_2 \in D''$ such that $D_1 \subset D_2$.

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2. The norm and the covering function

Proposition 1

Let D_1 and D_2 be two reduced coverings of a compact subset C of a metric space (X, d), with $D_1 \succeq D_2$. Then $n_C(D_1) \geq n_C(D_2)$.

Proof

Let $D_1 = \{U_1, ..., U_m\}$ and $D_2 = \{V_1, ..., V_n\}$ be the two reduced coverings of C. To the contrary, suppose $m = n_C(D_1) < n_C(D_2) = n$. Since $D_1 \succeq D_2$, so for any $U_i \in D_1$ there exists $V_j \in D_2$ such that $U_i \subset V_j$ also by our supposition $n_C(D_1) < n_C(D_2)$, so there exists $V_k \in D_2$ such that for any $U_l \in D_1, U_l \not\subset V_k$. Now

$$C \subset \bigcup_{i=1}^{m} U_i \subset \bigcup_{j=1,\, j \neq k}^{n} V_j$$

This implies that D_2 is not reduced, a contradiction.

Theorem 1

Let C be a compact subset of (X, d) and D be a reduced covering of C. Then there exists a uniform reduced covering D^* of C (i.e. a reduced covering which consists of open balls of the same radii) such that $D^* \succeq D$ Moreover $\| C \| = \sup n_C(D^*)$ where D^* runs over the set of all uniform reduced coverings of C .

Proof

Let $D = \{D_1, ..., D_n\}$ be a finite reduced covering of C. Since each D_i is open, for any $x \in D_i$, there exists an open ball $B(x, r_{i,x})$ such that $B(x, r_{i,x}) \subset D_i$. Since C is a compact set we can assume that x runs on a finite subset M_i of D_i. Let $r = \min\{r_{i,x} : x \in M_i, i = 1, ..., n\}$. Clearly $\{B(x, r) : x \in M_i; i = 1, ..., n\}$ is a uniform covering of C. Since C is compact, this covering has a finite (reduced) subcovering D^* consisting of open balls of same radii. Clearly $D^* \succeq D$ and $n_C(D^*) \geq n_C(D)$. We also have:

$$\begin{aligned} \|C\| &= \sup\{n_C(D) : D \text{ is a reduced covering of } C\} \\ &= \sup\{n_C(D) : D \text{ is a uniform covering of } C\} \end{aligned}$$

This is because uniform coverings D^* are also included in the class of arbitrary reduced coverings.

Theorem 2

Let C, C_1 and C_2 be compact subsets of (X, d). Then the following assertions hold:

(i) $\|C\| = 0 \Leftrightarrow C = \phi$

(ii) $\|C_1 \cup C_2\| \leq \|C_1\| + \|C_2\|$

(iii) If $C_1 \cap C_2 = \phi$ then $\|C_1 \cup C_2\| = \|C_1\| + \|C_2\|$

Proof

(i) It is obvious.

(ii) Any reduced open covering of $C_1 \cup C_2$ is also a covering of C_1 as well as of C_2 , therefore

$$n_{C_1 \cup C_2}(D) \leq n_{C_1}(D) + n_{C_2}(D) \leq \|C_1\| + \|C_2\|$$

Taking the supremum over all the coverings D of $C_1 \cup C_2$, we have:

$$\|C_1 \cup C_2\| \leq \|C_1\| + \|C_2\|$$

(iii) If $C_1 \cap C_2 = \phi$, we can consider in the definition of $\|C_1\|$ and of$\|C_2\|$ only reduced coverings of the following type:

Let $D^{(1)} = \{D_1^{(1)}, D_2^{(1)}, ..., D_m^{(1)}\}$ and $D^{(2)} = \{D_1^{(2)}, D_2^{(2)}, ..., D_n^{(2)}\}$ be reduced coverings of C_1 and C_2 respectively such that

$$D_i^{(1)} \cap D_j^{(2)} = \phi \text{ for all } i = 1, ...m \text{ and } j = 1, ...m$$

Now it is clear that $D^{(1)} \cup D^{(2)}$ is a reduced covering of $C_1 \cup C_2$, and also

$$n_C(D^{(1)} \cup D^{(2)}) = n_C(D^{(1)}) + n_C(D^{(2)})$$

By the theorem 2 , there exist uniform reduced coverings $D^{*(1)}$ and $D^{*(2)}$ of C_1 and C_2 , richer than $D^{(1)}$ and $D^{(2)}$ respectively, satisfying:

$$n_C(D^{*(1)} \cup D^{*(2)}) = n_C(D^{*(1)}) + n_C(D^{*(2)})$$

By taking supremum over all uniform reduced coverings of the type $D^{*(1)}$ and $D^{*(2)}$, we have:

$$\|C_1 \cup C_2\| = \|C_1\| + \|C_2\|$$

Theorem 3

For any compact subset C of (X, d), the following statements hold:

(i) $\|C\| < \infty \Leftrightarrow C$ is a finite subset of X.

(ii) $\|C\| = \infty \Leftrightarrow A(C) = \mathbb{N}$.

Proof

Let $\|C\| = n < \infty$, then we shall show that C contains not more than n points. Let C contains more than n points. Then there is an open set in the reduced covering of C containing at least two points and we can separate them by two open balls of suitably smaller radii. In this way, we get a reduced covering having more than n open sets contrary to the assumption that $\|C\| = n$. The converse is straightforward.

Let $A(C) = \mathbb{N}$, then $\|C\| = \sup A(C) = \sup \mathbb{N} = \infty$.

Let $\|C\| = \infty$, then C must have an infinite number of points. Take an $m \in \mathbb{N}$ and consider a reduced covering $D = \{D_1, ...D_k\}$ with $k > m$. We can take a reduced covering $D^* = \{D_1, ..., D_{m-1}, D'\}$ where $D' = D_m \cup D_{m+1} \cup ... \cup D_k$, with m open subsets. Hence $n_C(D^*) = m$ and the theorem is proved.

3. Configuration Matrices

Let C be a compact subset of (X, d), so there exists an open ball $B(X, \varepsilon) \supset C$, where $x \epsilon X$ and $\varepsilon > 0$. Take $n_1 = 1$.

Let

$$\varepsilon_{n_1} = \inf\{\varepsilon \in \mathbb{R}_+ : C \subset B(x, \varepsilon), x \in X\}$$

Then $C \not\subset B(x, \varepsilon_{n_1})$ for all $x \in X$. We introduce more balls to cover C. Let n_2 be the least number of balls required to cover C with the same radii, less than or equal to ε_{n_1} .
Let

$$\varepsilon_{n_2} = \inf\{\varepsilon \in \mathbb{R}_+ : C \subset \bigcup_{i=1}^{n_2} B(x_i, \varepsilon), x_i \in X\}$$

Then

$$C \not\subset \bigcup_{i=1}^{n_2} B(x_i, \varepsilon_{n_2}),\ \ \forall x_i \in X$$

Continuing in this way, we get a decreasing (in some cases finite) sequence of real numbers:

$$\varepsilon_{n_1} > \varepsilon_{n_2} > \varepsilon_{n_3} > ...$$

and the corresponding monotonically increasing sequence of natural numbers:

$$1 = n_1 \leq n_2 \leq n_3 \leq ...$$

We write

$$\begin{pmatrix} \varepsilon_{n_1} & \varepsilon_{n_2} & \varepsilon_{n_3} & . & . & . \\ n_1 & n_2 & n_3 & . & . & . \end{pmatrix}$$

and call it the configuration matrix associated with the compact set C (see also [3] for the particular case of an ultrametric space).

Conjecture
Given any configuration matrix:

$$\begin{pmatrix} \varepsilon_{n_1} & \varepsilon_{n_2} & \varepsilon_{n_3} & . & . & . \\ n_1 & n_2 & n_3 & . & . & . \end{pmatrix}$$

there exists a compact set C in some metric space having this configuration matrix.

Example
Let $C = [a, b] \subset \mathbb{R}$ where $l = b - a > 0$, Then the configuration matrix associated to C is

$$\begin{pmatrix} \frac{l}{2} & \frac{l}{4} & \frac{l}{6} & . & . & . & \frac{l}{2n} & . & . & . \\ 1 & 2 & 3 & . & . & . & n & . & . & . \end{pmatrix}$$

Remark
In the case of an ultrametric space all the balls in a reduced covering (of a compact set) are pairwise disjoint so the number of balls as well as the ball themselves are uniquely specified.

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