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PROBABILITY OF THE MODERATE DEVIATIONS FOR THE SUM-FUNCTIONS OF SPACINGS

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Abstract. Let $0 = U_{0,n} \leq U_{1,n} \leq, ... \leq U_{n-1,n} \leq U_{n,n} = 1$ be an ordered sample from uniform [0,1] distribution,$D_{in} = U_{i,n} - U_{i-1,n},\ i = 1, 2, ...n;\ n = 1, 2, ...,$ be their spacings,and let $f_{1n}, ..., f_{nn}$be a set of measurable functions. In this paper theorems on the probabilities of deviations in the moderate zones for $R_n(D) = f_{1n}(nD_{1n}, +... + f_{nn}(nD_{nn}$ are presented. Application of these results to study an intermediate efficiencies of the tests based on statistic $R_n(D)$ are also considered.

Key words : Spacings, uniform distribution, large deviations,goodness-of-fit,asymptotic efficiencies
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1. Introduction

Let $U_1, U_2, ...$ be a sequence of independent uniform (0,1) random variables (r.v.), $0 = U_{0n} \leq U_{1n} \leq ... \leq U_{n-1,n} \leq U_{nn} = 1$ the ordered of $U_1, U_2, ..., U_{n-1}$; $D_{in} = U_{in} - U_{i-1,n}, i = 1, 2, ..., n; n = 1, 2, ...$ their spacings and let $D = (D_{1n}, ..., D_{nn})$. Let $f_m(y) = f_{mn}(y), m = 1, 2, ..., n$ be measurable functions of nonnegative argument y. We consider the statistics of the type

$$R_n(D) = \sum_{m=1}^{n} f_m(nD_{mn}), n = 1, 2, ... \tag{1.1}$$

Statistics of this form are used for several tasks, e.g. for goodness of fit tests, testing the dispersive ordering, for estimation of unknown parameters, in the problems of random coverage of the circle. An extensive survey on the distribution theory of these statistics and their applications are given in Pyke (1965,1972) and Deheuvels (1985). We also refer to Beirlant, Janssen and Veraverbeke (1991), Does, R.J.M.M., klaassen, C.A.J. (1984), Does, Helmers, Klaassen (1987). In

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these papers the asymptotic normality, estimation and Edgeworth's type asymptotical expansion of the remainder term in the central limit theorem (C L T) for symmetric statistic (1.1) (i.e. when $f_m(u) = f(u)$ does not depend on m) are proved. The Lindeberg's type condition for CLT and lower estimation of the remainder term in CLT for $R_n(D)$ has been obtained by Mirakhmedov (2005). Many other authors have studied the Pitman asymptotic efficiency of test based on statistic of type (1.1), e.g. Del Pino (1979), Holst and Rao (1981), Jammalamadaka, Zhou and Tiwari(1989). For special types of statistic (1.1) Bahadur's exact asymptotic efficiency has been studied by Xian Zhou and Jammalamadaka (1989) and Bahadur's approximate efficiency have been studied by Rao (1972), Bartoszewicz (1995). Although a large number of authors have studied statistic of type (1.1) but still there is a lot of interest in it, see, for example Dehevels and Derzka (2003), Jammalamadaka and Goria (2004). The literature about the topic of our interest in this work is not readily available. Here we will prove the probability of deviation theorems on the moderate zones. We also consider the application of these theorems to the study of asymptotic intermediate efficiencies (Kallenberg (1983)) of the tests based on statistics of type (1.1). These results complements the results of the authors mentioned above. In what follows C, C_k are positive constants, ϵ is arbitrary positive small constants, $\sum_m$ is summation over m from 1 to n. All asymptotic relations are considered as $n \to \infty$.

2. **Results**

Let $Y_2, Y_2, ...$ be a sequence of independent $\exp\{1\}$ r.v's and $Y = (Y_1, ..., Y_n)$, $S_n = Y_1 + ... + Y_n$, $R_n(Y) = \sum_m f_m(Y_m)$, ρ=corr$(R_n(Y), S_n)$, $g_m(u) = f_m(u) - Ef_m(Y_m) - (u-1)\rho\sqrt{DR_n(Y)/n}$, $T_n(D) = \sum_m g_m(nD_m)$, $T_n(Y) = \sum_m g_m(Y_m)$. Note that $\sum_m Eg_m(Y_m) = 0$ and $\sigma_n^2 \equiv DT_n(Y) = (1-\rho^2)DR_{n(Y)}$. Also, obviously, that $T_n(D) = R_n(D) - ER_n(Y)$. We note that $f_m(y)$ can random functions. In such cases we suppose that $f_1(y_1), ..., f_n(y_n)$ is sequence of independent random variables not depending on D and Y. Put $P_n(x) = P(T_n(D) < x\sigma_n)$ and $\Phi(x)$ stand for standard normal distribution function.

Theorem 2.1 Let

$$\underline{lim}\frac{1}{n}\sigma_n^2 > 0 \tag{2.1}$$

$$\overline{lim}\frac{1}{n}\sum_m E|g_m(Y_m)|^{2+\delta} < \infty \tag{2.2}$$

for some $\delta > 0$. Then for all x such that $0 \le x \le \sqrt{\delta \ln n}$ we have

$$1 - P_n(x) = (1 - \Phi(x))(1 + o(1)), P_n(-x)(1 + o(1)) \tag{2.3}$$

Theorem 2.2. Put $\chi_m = Eexp(H|g_m(Y_m)|)$. Let condition (2.1) is fulfilled and for some $H > 0$

$$\overline{lim}\frac{1}{n}\sum_{m}\chi_m < \infty, \chi = o(n^{2/3})m = 1, 2, ..., n \tag{2.4}$$

Then for $x \geq 0$ and $x = o(n^{1/6})$ relations (2.3) hold true.

Remarks

1. Theorem 2.1 and 2.2 complete analog of the well known results on the probability of the moderate deviations of sum of independent variables.
2. From Beirlant, Janssen and Veravarbake (1991), Does and Klaassen (1984), and Mirakhmedov (2005) it follows that the normal approximation of the $P_n(x)$ under conditions (2.1) and (2.2) is valid for fixed x. Theorem 2.1 and 2.2 shows that such approximation is still true for $x = O(\sqrt{\ln n})$ under (2.1) and (2.2) and for $x = o(n^{1/6})$ under (2.1) and (2.4).

3. Proof of Theorems

The method which we shall use here is based on the Cramer's transformation (see below) and on the following well known property of the vector spacings $D : \Im(nD) = \Im(Y/S_n = n)$, where $\Im(X)$ denote distribution of a random vector X. Therefore, for arbitrary measurable function $L(x_1, ..., x_n)$ of nonnegative arguments such that $\int_{-\infty}^{\infty} |EL(Y_1, ..., Y_n)\exp(i\tau S_n)|d\tau < \infty$, we have

$$EL(nD_{1n}, ..., nD_{nn}) = \frac{1}{2\pi P_n(n)}\int_{-\infty}^{\infty} E(L(Y_1, ..., Y_n))\exp\{i\tau(S_n - n)\}d\tau \tag{3.1}$$

Where $P_n(z)$ isis the density function of r.v. S_n

The Cramer,s transform: Given r.v. Y such that $\mathrm{E}e^{H|Y|} < \infty$,for some $H > 0$ r.v. X with distribution function $P\{X < 0\} = E(\exp\{hY\}1\{Y < u\})/E\exp\{hY\}$ is called Cramer's transform with parameter h of r.v. Y, where $|h| < H$. We have

$$P\{Y > u\} = Ee^{hY}E(\exp\{-hX\}1\{X > u\}) \tag{3.2}$$

Proof of Theorem 2.1.We will prove first relation from (2.3).Second relation can be obtained from first one by substitution $-g_m(u)$ instead of $g_m(u), m = 1, 2, ...$ From corollary 3 of Mirakhmedov (2005) it follows that Theorem 2.1 holds true for $0 \leq x \leq \sqrt{0.5\delta' \ln n}$,where $\delta' = min(1, \delta)$. Therefore, now on we suppose that $\sqrt{0.5\delta' \ln n} \leq x \leq \sqrt{\delta \ln n}$.

Let $\tilde{g_m} = g_m(u)1\{|g_m(u)| \leq \epsilon x\sigma_n\}$ be truncated function, where $1\{A\}$ is the indicator of event A,and $\epsilon > o$ will be chosen sufficiently small. Putting $\tilde{T_m}(D) = \sum_m \tilde{g_m}(nD)$, $\tilde{P_n}(x) = P(\tilde{T_n}(D) < x\sigma_n)$ we have

$$|1 - P_n(x) - (1 - \Phi(x))| \leq |(1 - \tilde{P_n}(x)) - (1 - \Phi(x))| + |P_n(x) - \tilde{P_n}(x)| \equiv \nabla_1 + \nabla_2 \tag{3.3}$$

(i) Estimation of ∇_1. For a complex variable z we denote $\tilde{\varphi_n}(z) = E\exp\{z\tilde{T_n}(D)\}$.

Let $\tilde{G_n}$ be the Cramer's transform with parameter $h = x/\sigma_n$ of the r.v. $\tilde{T}_n(D)$. Then from (3.2) we have

$$\begin{aligned} P(\tilde{T}_n(D) > x\sigma_n) = & \quad \tilde{\varphi}_n(h)E[\exp\{-h\tilde{G}_n\}1\{\tilde{G}_n > x\sigma_n\}] \\ = & \quad \exp\{\ln\tilde{\varphi}_n(h) - x^2\}E[\exp\{-x\tilde{G}_n^*\}1\{\tilde{G}_n^* > 0\}] \\ = & \quad \exp\{\ln\tilde{\varphi}_n(h) - x^2\}[\textstyle\int_0^\infty \exp\{-xu\}d\Phi(u) \\ & + \textstyle\int_0^\infty \exp\{-xu\}d(P\{\tilde{G}_n^* < u\} - \Phi(u))] \end{aligned} \quad (3.4)$$

Where $\tilde{G}_n^* = (\tilde{G}_n - x\sigma_n)/\sigma_n$,since $\tilde{\varphi}_n(h) \geq 1/2$ for sufficiently large n. We denote first and second summand inside the square bracket by A_1 and A_2 respectively. It can be readily shown that $A_1 = \Phi(-x)\exp\{x^2/2\}$. The estimation of A_2 rest on the following.

Lemma 1. Under conditions Theorem 2.1 we have
1. $\sup|P\{\tilde{G}_n^* < u\} - \Phi(u)| = O(n^{-\delta'/3(2+\delta')})$
2.$\tilde{\varphi}_n(h) = \exp\{x^2/2\}(1 + O(n^{-\delta'/3(2+\delta')}))$ $\delta' = min(1, \delta)$.

Proof (see Appendix)

Integration part and using Lemma 1 (1) we get $A_2 = O(n^{-\delta'/3(2+\delta')})$. By Lemma 1(2) we have $\exp\{\ln\tilde{\varphi}_n(h) - x^2\} = \exp\{-x^2/2\}(1 + O(n^{-\delta'/3(2+\delta')}))$. These jointly with (3.4) and

$$1 - \Phi(x) = \left(\sqrt{2\pi}x\exp\{x^2/2\}\right)^{-1}(1 + O(x^{-1})) \quad (3.5)$$

yields
$P(\tilde{T}_n(D) > x\sigma_n) = \exp\{x^2/2\}(1+O(n^{-\delta'/3(2+\delta')}))[\Phi(-x)\exp\{x^2/2\}+O(n^{-\delta'/3(2+\delta')})] = (1 - \Phi(x))O(xn^{\delta'/3(2+\delta')})$ i.e. $\nabla_1 = (1 - \Phi(x))O(n^{\delta'/3(2+\delta')})$.

ii) *Estimation of* ∇_2. Note that $p_n(n) = n^n(n!)^{-1}e^n$, hence using Stirling 's formula we obtain

$$\sqrt{2\pi n}p_n(n) = 1 + O(n^{-1}) \quad (3.6)$$

We have
$\{\tilde{T}_n(D) > u\} = \{T_n(D) > u, \bigcap_{m=1}^n\{|g_m(nD_{mn})| \leq \varepsilon c\sigma_n\}\} \subseteq \{T_n(D) > u\}$.
$\{T_n(D) > u\} = \{T_n(D) > u, \bigcap_{m=1}^n\{|g_m(nD_{mn})| \leq \varepsilon c\sigma_n\}\bigcup\{T_n(D) > u,$
$\bigcup_{m=1}^n\{|g_m(nD_{mn})| > \varepsilon x\sigma_n\}\} \subseteq \{\tilde{T}_n(D) > u\}\bigcup_{m=1}^n\{|g_m(nD_{mn})| > \varepsilon x\sigma_n\}$.
Hence putting $X_m(u) = 1\{|g_m(u) > \varepsilon x\sigma_n|\}$ and using formula (3.1) we get

$$\begin{aligned} \nabla_2 \leq & \quad \textstyle\sum_m P\{|g_m(nD_{mn})| > \varepsilon x\sigma_n\} \\ = & \quad \textstyle\sum_m EX_m(nD_{mn}) \\ \leq & \quad \textstyle\sum_m \frac{1}{2\pi p_n(n)}\int_{-\infty}^\infty |E(X_m(Y_m)\exp\{i\tau(S_n - n)\})|d\tau \\ \leq & \quad \textstyle\sum_{m=1}^n \frac{EX_m(Y_m)}{2\pi p_n(n)}\int_{-\infty}^\infty |E\exp\{i\tau(S_n - Y_m)\}|d\tau \end{aligned}$$

(3.7)

Note that

$$|E\exp\{i\tau Y_k\}| = (1+\tau^2)^{-1/2}. \tag{3.8}$$

Therefore quite clear calculations show that the integral on the right hand side of (3.7) does not exceed π. Thus taking into account (3.6) we get
$\nabla_2 \leq C\sum_m EX_m(Y_m)$.
Using Chebishev's inequality, relation (3.5) and condition (2.2) we obtain

$$\begin{aligned}\nabla_2 \leq & \frac{C}{(\varepsilon x\sigma_n)^{2+\delta}}\sum_m |g_m(Y_m)|^{2+\delta} \\ = & O(n^{-\delta/2}x^{-(2+\delta)}) \\ = & (1-\Phi(x))O(n^{-\delta/2}x^{-(1+\delta)}\exp\{x^2/2\}) \\ = & (1-\Phi(x))O(x^{-(1+\delta)})O((\ln n)^{-(1+\delta)/2})\end{aligned}$$

The estimates of ∇_1 and ∇_2 jointly with (3.3) complete the proof of the Theorem 2.1.
Proof of the Theorem 2.2. Due to condition (2.4) in this case we need not to truncate the functions $g_m(u)$. Put $\varphi_n(z) = E\exp\{zT_n(D)\}$. Let G_n be the Cramer's transformation of the r.v. $T_n(D)$ with parameter $h = x/\sigma_n$ and $G_n^* = (G_n - x\sigma_n)/\sigma_n$. We have from (3.2)

$$\begin{aligned}P\{T_n(D) > x\sigma_n\} = & \varphi_n(h)E[\exp\{-hG_n\}1\{G_n > x\sigma_n\}] \\ = & \exp\{\ln\varphi_n(h) - x^2\}E[\exp\{-xG_n^*\}1\{G^{*_n} > 0\}]\end{aligned}$$

Since $\varphi_n(h) > 1/2$ for sufficiently large n. Therefore Theorem 2.2 can be proved by same way as estimation of ∇_1 applying following Lemma 2 instead of Lemma 1.
Lemma 2. If $x \geq 0$ and $x = O(n^{1/6})$, then under conditions of the Theorem 2.2 we have
1. $\sup|P\{G_n^* < u\} - \Phi(u)| = O(\frac{x^2}{\sqrt{n}})$,
2. $\varphi_n(h) = \exp\{\frac{x^2}{2} + \frac{x^3}{6\sigma_n^3}\sum_m Eg_m^3\}\{1 + O(\frac{x^2}{\sqrt{n}})\} = \exp\{\frac{x^2}{2}\}\{1 + O(\frac{x^3}{\sqrt{n}})\}$.
Proof (see Appendix).

4. Application

Let $X_{1n}, X_{2n}, ..., X_{(n-1)n}$ be the ordered statistics of sample from population having distribution function F, and $T_{kn} = X_{kn} - X_{(k-1)n}k = 1, 2, ..., n$ be their spacings, with notation $X_{0n} = 0$ and $X_{nn} = 1$. If sample is from uniform distribution on [0,1] (i.e. under H_o), then $T_{nk} = D_{nk}$ (see above). We wish to test null hypothesis $H_o : F(x) = x, 0 \leq x \leq 1$, versus the sequence of alternatives H_{1n}.

$$F_n(x) = x + L(x)\delta(n), 0 \leq x \leq 1, \tag{4.1}$$

Where $\delta(n) \to 0$ as $n \to \infty$ and L(x) satisfy some smoothness conditions.
There are several approaches to the definition of asymptotically properties of statistical tests, which differ by the conditions imposed on the asymptotic behavior of the size w_n, powerβ_n and sequence of alternatives H_{1n}. The most common is the Pitman's approach where the alternatives converge to H_0 at a rate necessary to keep $\omega_n \to \omega$, $\beta_n \to \beta$, $0 < \omega < \beta 1$. This is one of extreme alternatives, because more fast rate of closeness of the hypothesis H_{1n} and H_0 indistinguishable, since $\beta_n \to \omega$. Another extreme case is Bahadur's approach under which the asymptotically power $\beta < 1$ and the alternatives $H_{1n} = H_0$ are fixed (more precisely, H_1 does not approach H_0), and test is characterized by the rate of decrease of the size ω_n. Similarly one can fix ω and H_{1n} and measure the performance of the test by the rate of convergence β_n to 1. This is Hodges-Lehman approach. Finally, one can consider two intermediate settings: first is $\beta_n \to \beta, 0 < \beta < 1$ while $\omega_n \to 0$ and $H_1 \to H_0$ "not too fast", and second $\omega_n \to \omega, 0 < \omega < 1$, while $\beta_n \to 1$ and $H_1 \to H_0$ "not too fast". These situation give rise to the concept of intermediate asymptotic efficiency (IAE) due to Kallenberg (1983) (see also Ivchenco and Mirakhmedov (1995)) viz. ω -IAE in the first case (intermediate between Pitman's and Bahadur's settings) and β-IAE in the second case (intermediate between Pitman's and Hodges-Lehman settings).
We shall study intermediate efficiency of the tests statistics of type $R_n(T)$, where function $f_{kn}(u)$ is of form f(u,r_{kn}), where r_{kn}=(k-0.5)/n, function f(u,y) is defined on [0,∞] × [0,1]. Statistics $R_n(T)$ is called symmetric if function $f(y, u) = f(y)$ not depends on u, in other cases statistics $R_n(T)$ is non-symmetric. A test based on statistics $R_n(T)$ with kernel function f(u,y) is called f-test. It was shown by Holst and Rao (1981) that non-symmetric f-test may discriminate alternatives (4.1) with $\delta(n) = n^{-1/2}$, and linear test i.e. test based on $L_n = \sum_{k=1}^{n} l(r_{kn})T_{kn}$,where $L'(x) = l(x)$, is most efficient in Pitman sense. Application of the Theorem 2.1 and 2.2 shows that linear test is still most efficient in the intermediate sense due to Kallenberg (1983),i.e. for alternatives (4.1) with $\delta(n) = o(n^{-1/6}), \delta(n)\sqrt{n} \to \infty$. Symmetric f-test can distinguish the alternatives that far from H_0 on distance $\delta(n) = n^{-1/4}$ only, and asymptotical efficiency in Pitman sense characterized through $c(f) = corr(g(Y), Y^2)$,where $g(Y) = f(Y) - (Y - 1)cov(f(Y), Y)$. Hence asymptotical most effective tests in this case are well known Greenwood test based on $R_n(T)$ with kernel function $f(u) = u^2$. From Theorem2.1 and 2.2 it follows that the same conclusion is still true for middle intermediate alternatives (hence for weak intermediate also) and (due to Kallenberg (1983) and Ivchenko and Mirakhmedove (1995)),i.e. for alternatives (4.1) with $\delta(n) = o(n^{-1/12}), \delta(n)\sqrt[4]{n} \to \infty$.

5. Appendix

In this section we give the schematic proofs of Lemma 1 and 2. We will use the notations from section 3. Thereto we put $u = \sigma_n^{-1} it + h$, where $h = x/\sigma_n$, and for notational convenience we will use g_m and $\tilde{g_m}$ instead of $g_m(Y_m)$ and $\tilde{g_m}(Y_m)$ correspondingly.
Let F and V be the distribution functions and f and v their characteristic functions correspondingly. We have
$|f(t) - v(t)| \leq |\int_0^t D_s(f(s) - v(s))ds| = \leq |t| \sup_{|s| \leq |t|} |D_s(f(s) - v(s))|$.
Apply this in the well known Essseen's inequality to get

$$|F(x) - V(x)| \leq \frac{1}{\pi}[\int_{1 \leq |t| \leq T} |f(t) - v(t)|dt + \sup_{|t| \leq 1} |D_s(f(t) - v(t))| + \frac{24}{T} max_x |D_x V(x)|]. \tag{A.1}$$

Proof of Lemma 1. We denote
$\tilde{\psi_{mn}}(u, \tau) = E \exp\{u\tilde{g_m} + i\tau(Y_m - 1)n^{-1/2}\}$, $\tilde{\Psi_n}(u, \tau) = \prod_m \tilde{\psi_{mn}}(u, \tau)$,
$\tilde{Q}(t, \tau, x) = itx + \frac{x^2}{2} - \frac{t^2}{2} - \frac{\tau^2}{2}$ we have
$e^{i\tau} \tilde{\psi_{mn}}(u, \tau\sqrt{n}) = E \exp\{u\tilde{g_m} + i\tau Y_m\} = \int_0^\infty e^{i\tau y} e^{-y} E \exp\{u\tilde{g_m}(y)\}dy$
$= \int_{-\infty}^\infty e^{i\tau y}[e^{-y} E \exp\{u\tilde{g_m}(y)\} 1\{y \geq 0\}]$
That is $e^{i\tau} \tilde{\psi_{mn}}(u, \tau\sqrt{n})$ is the Fourier transformation of the function $k(y) = [e^{-y} E \exp\{U\tilde{g_m}(y)\} 1\{y \geq 0\}]$ and k(y) and $|k(y)|^2$ are integrable functions. Therefore by Plancheral's identity the function $|e^{i\tau} \tilde{\psi_{mn}}(u, \tau\sqrt{n})|$ is integrable and moreover

$$\int_{-\infty}^\infty |e^{i\tau} \tilde{\psi_{mn}}(u, \tau\sqrt{n})|^2 d\tau = 2\pi \int_{-\infty}^\infty |k(y)|^2 dy \leq 2\pi n^{2\varepsilon\delta} \tag{A.2}$$

recalling definition of $\tilde{g_m}$ and that $0 \leq x\sqrt{\delta \ln n}$ we have
$|E \exp\{u\tilde{g_m}\}| \leq E \exp\{h\tilde{g_m}\} \leq n^{\varepsilon\delta}$.From this and Holder's inequality we see
$\int_{-\infty}^\infty |\tilde{\Psi_n}(t, \tau)|d\tau \leq 2\pi n^{2\varepsilon\delta + 0.5}$
Therefore from (3.1) we have

$$\tilde{\varphi_n} = \frac{1}{2\pi\sqrt{n} p_n(n)} \int_{-\infty}^\infty \tilde{\Psi_n}(u, \tau)d\tau \tag{A.3}$$

Denoting $\varphi_n(\tilde{t}, h) \equiv E \exp\{it\tilde{G_n^*}\}$ we obtain

$$\tilde{\varphi_n} = \frac{\exp\{-itx\}\tilde{\varphi_n}(u)}{\tilde{\varphi_n}(h)} \tag{A.4}$$

Putting $a_n = C_1 n^{\delta'/3(2+\delta')}$, $b_n = C_2 n^{\frac{1}{2}-\frac{\varepsilon\delta}{\delta'}}$ from (A.3) we get

$$\begin{aligned}
\exp\{-itx\}\tilde{\varphi_n}(u) = & \frac{1}{2\pi\sqrt{n}p_n(n)}[\int_{-\infty}^{\infty} \exp\{-itx + \tilde{Q}(t,\tau,x)\}d\tau \\
+ & \int_{|\tau|\le a_n} e^{-itx}\{\tilde{\Psi_n}(u,\tau) - e^{\tilde{Q}(t,\tau,x)}\}d\tau \\
+ & \int_{a_n\le|\tau|\le b_n} e^{-itx}\tilde{\Psi_n}(u,\tau)d\tau \\
+ & \int_{b_n\le|\tau|} e^{-itx}\tilde{\Psi_n}(u,\tau)d\tau \\
- & \int_{a_n\le|\tau|} e^{-itx+\tilde{Q}(t,\tau,x)}d\tau] \\
\equiv & \frac{1}{2\pi\sqrt{n}p_n(n)}[\sum_{j=1}^{5} B_j]
\end{aligned} \quad (A.5)$$

Where B_j denote j-th integral inside the square brackets. It is obvious that

$$B_1 = \sqrt{2\pi}\exp\{\frac{x^2}{2} - \frac{t^2}{2}\} \quad (A.6)$$

For estimates of B_2 and B_3 we will use

Lemma A.1. Under conditions of the Lemma 1 there constants C_1, C_2 such $C_1 < C_2$ and for $j = 0, 1$ the following assertions are true :

1) if $|t| \le a_n, |\tau| \le a'_n$ then

$D_t^j\tilde{\Psi_n}(t,\tau) = (-t+ix)^j \exp\{\tilde{Q}(t,\tau,x)\}(1 + O((x+|t|+|\tau|)^{2+\delta'} n^{-\frac{\delta'}{2}+\varepsilon\delta}))$

where $a'_n = C_2 a_n/C_1$

2) if $|t| \le a_n, a'_n \le |\tau| \le b_n$ then

$|D_t^j\tilde{\Psi_n}(u,\tau)| \le C\exp\{\frac{x^2}{2} - \frac{t^2}{2} - \frac{\tau^2}{4}\}(|t|+|\tau|+hn^{\varepsilon\delta})^j$.

The proof of the Lemma A.1 is like the proof of Lemma 2 from Mirakhmedov(1992), and we omit it. Using Lemma A.1 and taking $\varepsilon < \delta'/4\delta$ after simple and quite clear calculations we obtain

$$B_2 + B_3 = \exp\{\frac{x^2-t^2}{2}\}\left(1 + O((x+t^{2+\delta'})n^{-\frac{\delta'}{4}})\right) \quad (A.7)$$

Let $|t| \le a_n, |\tau| > b_n$. Using inequalities $x < e^{x-1}$ and $x - 1 \le \frac{1}{2}(x^2-1)$ we have

$$\begin{aligned}
|\tilde{\psi_{mn}}(u,\tau)| \le & |E\exp\{i\tau\overline{Y_m}\}(\exp\{iu\tilde{g_n}\}-1) + E\exp\{i\tau\overline{Y_m}\}| \\
\le & |\tilde{\psi_m}(0,\tau)| + |u|E|\tilde{g_m}| \\
\le & \exp\{-(1-|\tilde{\psi_m}(0,\tau)|)\} \\
\le & \exp\{-\tfrac{1}{2}(1-|\tilde{\psi_m}(0,\tau)|^2) + (|t|\sigma_n^{-1}+h)E|\tilde{g_m}|\}
\end{aligned} \quad (A.8)$$

since $u = t\sigma_n^{-1} + h$. By (3.8) for $|\tau| > b_n$ and $n \ge C_2^{\delta'/\varepsilon\delta}$ we get

$$1 - |\tilde{\psi_m}(0,\tau)|^2 \ge 0.5C_2 n^{-\frac{2\varepsilon\delta}{\delta'}} \quad (A.9)$$

Using Holder's inequality we see that $\sum_m E|\tilde{g_m}| \leq \sigma_n \sqrt{n}$. Therefore for $|t| \leq a_n$

$$\frac{|t|}{\sigma_n} \sum_m E|\tilde{g_m}| \leq C_1 n^{\frac{1}{2}+\frac{\delta'}{3(2+\delta')}}, h\sum_m E|\tilde{g_m}| \leq \sqrt{\delta n \ln n} \qquad (A.10)$$

From (A.8),(A.9),(A.10) it follows that for any integer k and s such that $1 \leq k,s \leq n$ and $n \geq C_2^{\frac{\delta'}{\varepsilon\delta}}$ we have $\prod_m^{(k,s)} |\tilde{\psi_{mn}}(u,\tau)| \leq \exp\{-\frac{C_2}{2}(n-2)n^{-2\varepsilon\delta/\delta'} + C_1 n^{\frac{1}{2}+\frac{\delta'}{3(2+\delta')}} + \sqrt{\delta n \ln n}\}$ here $\prod_m^{(k,s,\ldots)}$ is the product over $m = 1, ..., n$ such that $m \neq k, s, ...$ we choose $\varepsilon < \delta'/6\delta$, and $C_2 > 2C_1$ then we get

$$\prod_m^{(k,s)} |\tilde{\psi_{mn}}(u,\tau)| \leq C_3 \exp\{-C_4 n^{2/3}\} \qquad (A.11)$$

Using Holder,s inequality and (A.2) we have

$$\int_{-\infty}^{\infty} |\tilde{\psi_{sn}}(u,\tau)\tilde{\psi_{kn}}(u,\tau)| d\tau \leq C n^{\varepsilon\delta+0.5}. \qquad (A.12)$$

From this and (A.11) for B_4 we obtain

$$|B_4| \leq C \exp\{-cn^{2/3}\} \qquad (A.13)$$

It is easy looking that

$$|B_5| \leq C \exp\{\frac{x^2}{2} - \frac{t^2}{2} - \frac{a_n^2}{2}\} \qquad (A.14)$$

Substituting (A.6),(A.7),(A.13),(A.14) into (A.5) we get : if $|t| \leq C_1 n^{\delta'/3(2+\delta')}$, then

$$\exp\{-itx\}\tilde{\varphi_n}(u) = \frac{\exp\{(x^2-t^2)/2\}(1+O(n^{-\delta'/4}))}{\sqrt{2\pi n} p_n(n)} \qquad (A.15)$$

Particularly, at $t = 0$ from (A.15) and (3.8) we have

$$\tilde{\varphi_n}(h) = \frac{\exp\{(x^2/2)\}(1+O(n^{-\delta'/4}))}{\sqrt{2\pi n} p_n(n)} = \exp\{x^2/2\}(1+O(n^{-\delta'/4})). \qquad (A.16)$$

This proved the second assertion of Lemma 1.
Since (A.4) from (A.15) and (A.16) we obtain

$$\tilde{\varphi_n}(t,h) = \exp\{-t^2/2\}(1+O(n^{-\delta'/4})) \qquad (A.17)$$

From (A.3),(A.15),(A.16) it follows that

$$\begin{aligned} D_t\tilde{\varphi}_n(u,h) = & \exp\{-itx\}[D_t\tilde{\varphi_n}(u) - it\tilde{\varphi_n}(u)]/\tilde{\varphi_n}(h) \\ = & \frac{1}{\sqrt{2\pi}} \exp\{-itx - \frac{x^2}{2}\}(1+O(n^{-\delta'/4})) \int_{-\infty}^{\infty} [D_t\tilde{\Psi}_n(u,\tau) - ix\tilde{\Psi}_n(u,\tau)] d\tau \\ = & \frac{1}{\sqrt{2\pi}} \exp\{-itx - \frac{x^2}{2}\}(1+O(n^{-\delta'/4}))[J_1 + J_2] \end{aligned} \qquad (A.18)$$

where $J_1 = \int_{|\tau|\le a_n}[D_t\tilde{\Psi}_n(u,\tau) - ix\tilde{\Psi}_n(u,\tau)]d\tau$,

$J_2 = \int_{|\tau|>a_n}[D_t\tilde{\Psi}_n(u,\tau) - ix\tilde{\Psi}_n(u,\tau)]d\tau$

We have

$J_1 = \int_{|\tau|\le a_n} D_t(\tilde{\Psi}_n(u,\tau) - \exp\{\tilde{Q}(t,\tau,x)\}d\tau) - ix\int_{|\tau|\le a_n}(\tilde{\Psi}_n(u,\tau) - \exp\{\tilde{Q}(t,\tau,x)\}d\tau) +$
$\int_{|\tau|\le a_n}(D_t\exp\{\tilde{Q}(t,\tau,x)\} - ix\exp\{\tilde{Q}(t,\tau,x)\})d\tau.$

Hence using first assertion of Lemma A.1 we obtain

$$\frac{1}{\sqrt{2\pi}}\exp\{-itx - \frac{x^2}{2}\}J_1 = -t\exp\{-\frac{t^2}{2}\} + \exp\{-\frac{t^2}{2}\}(1 + O((x+|t|)^{2+\delta'}n^{-\frac{\delta'}{2}+\varepsilon\delta})) \qquad (A.19)$$

since

$\frac{1}{\sqrt{2\pi}}\exp\{-itx - \frac{x^2}{2}\}\int_{|\tau|\le a_n}(D_t\exp\{\tilde{Q}(t,\tau,x)\} - ix\exp\{\tilde{Q}(t,\tau,x)\})d\tau$
$= -t\exp\{-\frac{t^2}{2}\}(1 + \frac{1}{\sqrt{2\pi}}\int_{|\tau|>a_n} e^{-\frac{\tau^2}{2}}d\tau)$

since $n > 2$ there are integer k,s such that $1 \le k,\ s \le n,\ k \ne j,\ s \ne j$

$$\begin{aligned}\int_{|\tau|>a_n}|D_t\tilde{\Psi}_n(u,t)|d\tau \le \quad & \sum_{j=1}^n \int_{|\tau|>a_n}|D_t\tilde{\psi}_j(u,\tau)|\prod_m^{(j)}|\tilde{\psi}_m(u,\tau)|d\tau \\ \le \quad & \frac{1}{\sigma_n}\sum_{j=1}^n E|\tilde{g}_j|[\int_{a_n<|\tau|<b_n}\prod_m^{(j)}|\tilde{\psi}_m(u,\tau)|d\tau \\ + \quad & \int_{|\tau|>b_n}|\tilde{\psi}_k(u,\tau)\tilde{\psi}_s(u,\tau)|\prod_m^{(j,k,s)}|\tilde{\psi}_m(u,\tau)|d\tau]\end{aligned}$$

Therefore,using second assertion of Lemma A.1 for first integral inside of bracket, and (A.11) and (A.12) for second integral, and that
$\sum_m E|\tilde{g_m}| \le \sigma_n\sqrt{n}$ we get
$\int_{|\tau|>a_n}|D_t\tilde{\Psi}_n(u,t)|d\tau \le Cn^{1+\varepsilon\delta}\exp\{-cn^{2/3}\}.$
Analogous using second assertion of Lemma A.1, and (A.11) and (A.12) we get
$\int_{|\tau|>a_n}|\tilde{\Psi}_n(u,t)|d\tau \le Cn^{0.5+\varepsilon\delta}.$
Thus

$$|J_2| \le Ce^{-cn^{\frac{1}{3}}}. \qquad (A.20)$$

Now, from (A.18), (A.19) and (A.20) it follows that

$$D_t\tilde{\varphi}_n(u,h) = -t\exp\{-\frac{t^2}{2}\} + \exp\{-\frac{t^2}{2}\}(1 + O((x+|t|)^{2+\delta'}n^{-\frac{\delta'}{2}+\varepsilon\delta})) + C\theta\exp\{-cn^{\frac{1}{3}}\}. \quad (A.21)$$

Putting in (A.1) $F(x) = P(\tilde{G}_n^* < x)$, $V(x) = \Phi(x)$, $f(t) = \tilde{\varphi}_n(t,h)$, $v(t) = \exp\{-t^2/2\}$, and $T = n^{\delta'/3(2+\delta')}$, and using (A.17) and (A.21) we complete the proof of Lemma 1.

Proof of Lemma 2. Let $\psi_{mn}(u,\tau) = E\exp\{ug_m + i\tau(Y_m - 1)n^{-1/2}\}$, $\Psi_n(u,\tau) = \prod_m \psi_{mn}(u,\tau)$, $Q(t,\tau,x) = \frac{x^2}{2} + \frac{x^3}{6\sigma_n^3}\sum_m Eg_m^3 - \frac{t^2}{2} - \frac{\tau^2}{2}$.

Lemma A.2 Under conditions of the Theorem 2.2 there are C_1 and C_2 such that

i) if $|t| \le C_1 n^{1/6}$, $|\tau| \le C_2 n^{1/6}$ then
$|e^{itx}\Psi_n(u,\tau) - e^{Q(t,\tau,x)}| \le C\frac{x^2+|t|^3+|\tau|^3}{\sqrt{n}} \exp\left(Q(t,\tau,x) + \frac{t^2+\tau^2}{4}\right)$
ii) if $|t| \le C_1\sqrt{n}$, $|\tau| \le C_2\sqrt{n}$ then
$|\Psi_n(u,\tau)| \le C\exp\left(\frac{x^2}{2} - \frac{t^2}{2} - \frac{\tau^2}{2}\right)$,
iii) if $|t| \le C_1\sqrt{n}$ and $|\tau| \ge C_2\sqrt{n}$, then for any integer k,s: $1 \le k,\ s \le n$,
$\prod_m^{(k,s)} |\psi_{mn}(t,\tau)| \le C_{11}\exp\{-C_{12}n\}$.
Proof. From condition (2.4) for $k \ge 0$ and $|h| \le H/2$ we have

$$E|g_m|^k e^h g_m \le 2^{k+1}k!H^{-k}\chi_m. \qquad (A.22)$$

From this and (2.1) we get $\sigma_n^{-1}\sum_m E|g_m|^3 \le C_1(H)n^{-1/2}$ and

$$|\psi_{mn}(u,\tau)-1| \le 0.5|u|^2 Eg_m^2 e^{hg_m} + 0.5n^{-1}\tau^2 E(Y_m-1)^2 e^{hg_m} \le 16n^{-1}\chi_m\left(H^{-2}(x^2+t^2)+\tau^2\right). \quad (A.23)$$

Therefore under conditions of Lemma 2 for sufficiently large n we have $|\psi_{mn}(u,\tau)-1| \le 0.5$, i.e. $|\psi_{mn}(u,\tau)| \ge 0.5$, since $\chi_m = o(n^{2/3})$. Using Taylor's expansion idea it is easy to see that

$$\psi_{mn}(u,\tau)-1 = 0.5u^2Eg_m^2 + in^{-1/2}u\tau Eg_m(Y_m-1) - 0.5\tau^2 n^{-1} + 6^{-1}h^3Eg_m^3 + r_m(x,t,\tau). \qquad (A.24)$$

where $|r_m(x,t,\tau)| \le C_2(H)n^{-3/2}((x+|t|+|\tau|)^3 - x^3)\chi_m =^{def} \alpha_m(x,t,\tau,n)$ and Holder's inequality and (A.22) are used. Apply Holder's inequality to get $\left(E(Y_m-1)^2e^{hg_m}\right)^2 \le 4!\chi_m$ and $\left(Eg_m^2e^{hg_m}\right)^2 \le 3^4 4!\chi_m$, for $|h| \le H/3$. Using these inequalities and first inequality in (A.23) we get
$|\psi_{mn}(t,\tau)-1|^2 \le |u|^4(Eg_m^2e^{hg_m})^2 + n^{-2}\tau^4(E(Y_m-1)^2e^{hg_m})^2$
$\le 3^4 4! H^{-4}\chi_m n^{-2}(x^4+t^4+\tau^4) = O(\alpha_m(x,t,\tau,n))$.
Therefore taking into account (A.24) we have
$\ln\psi_{mn}(u,\tau) = \psi_{mn}(u,\tau)-1+O(|\psi_{mn}(u,\tau)-1|^2) = 0.5x^2\sigma_n^{-2}Eg_m^2 + itx\sigma_n^{-2}Eg_m^2 - 0.5t^2\sigma_n^{-2}Eg_m^2 + i\tau n^{-1/2}uEg_m(Y_m-1) - 0.5\tau^2n^{-1} + 6^{-1}h^3Eg_m^3 + O(\alpha_m(x,t,\tau,n))$
Thus taking into account that $\sum_m E(Y_m-1)g_m = 0$ under conditions of Lemma A.2 we have

$$\begin{aligned}|e^{-itx}\Psi_n(u,\tau) - \exp\{Q(x,t,\tau)\}| \le & \quad C_3\sum_m |\alpha_m(x,t,\tau,n)\exp[Q(x,t,\tau)\\ + & \quad O(\textstyle\sum_m \alpha_m(x,t,\tau,n))]|\\ \le & \quad C_4 n^{-1/2}((x+|t|+|\tau|)^3 - x^3)\exp\{Q(x,t,\tau)\}\end{aligned}$$

since $\sum_m \alpha_m(x,t,\tau,n) = O(1)$.
Part (i) of Lemma A.2 follows.
ii) Let $|t| \le C_1\sqrt{n}$, $|\tau| \le C_2\sqrt{n}$. Given r.v. X we let X and X' be mutually independent r.v. having common distribution. We have

$$\begin{aligned}|\psi_m(u,\tau)|^2 = &\quad E\exp\{h(g_m+g'_m)+it\sigma_n^{-2}(g_m-g'_m)+i\tau(Y_m-Y'_m)\}\\ \le &\quad 1+(x^2-t^2)\sigma_n^{-2}Eg_m^2-\tau^2n^{-1}6(\sigma_n^{-3}(x^3\\ + &\quad |t|^3))E|g_m|^3\exp\{2h|g_m|\}+n^{-3/2}|\tau|^3E|Y_m|^3\exp\{2h|g_m|\}\\ \le &\quad \exp\{(x^2-t^2)\sigma_n^{-2}Eg_m^2-\tau^2n^{-1}\\ + &\quad 6.3^3.3!H^{-3}(\sigma_n^{-3}(x^3+|t|^3)+n^{-3/2}|\tau^3|)\chi_m\}\end{aligned}$$

since $Eg_m=0$ and (A.22). Hence for sufficiently large n there exist C_5 such that for $|t|\le C_5\sqrt{n}$ and $|\tau|\le C_5\sqrt{n}$ we have

$$\prod_m^{(k,j)}|\psi_m(u,\tau)|\le C_6\exp\{0.5(x^2-t^2-\tau^2)+C_7n^{-1/2}(x^3+|t|^3+|\tau|^3)\}\le C_8\exp\{0.5x^2-0.25(t^2+\tau^2)\}. \quad (A.25)$$

Part (ii) follows.
Let $|t|\le C_5\sqrt{n}$ and $|\tau|>C_5\sqrt{n}$. From (3.8) we have $1-|\psi_m(0,\tau)^2|\ge C_9$. Also $|t|\sigma_n^{-1}\sum E|g_m|\le C_5\sigma_n\sqrt{n}$ and $h\sum_m E|g_m|\le n^{2/3}$. Like (A.8) we have $|\psi_m(u,\tau)|\le\exp\{-\frac{1}{2}(1-\|\psi_m(0,\tau)|^2)+(|t|\sigma^{-1}+h)E|g_m|\}$. Use these relations choosing C_5 sufficiently small to get

$$\prod_m^{(k,s)}|\psi_{mn}(u,\tau)|\le C_{10}\exp\{-0.5C_9n+C_5\sigma_n\sqrt{n}+n^{2/3}\}C_{11}\exp\{-C_{12}n\}. \quad (A.26)$$

Part (iii) follows this complete the proof of Lemma A.2.
Note that like (A.2) it is easy to find that

$$\int_{-\infty}^{\infty}|\psi_m(u,\tau)|^2d\tau\le 2\pi\chi_m\sqrt{n}\le cn^{7/6},\int|\psi_j(u,\tau)\psi_k(u,\tau)|d\tau\le cn^{7/6}. \quad (A.27)$$

From this using Holder's inequality we see that
$\int_{-\infty}^{\infty}|\Psi_n(t,\tau)|d\tau\le Cn^{7/6}$.
Therefore for $\varphi_n=E\exp\{zT_n(D)\}$ from formula (3.1) we have

$$\varphi_n(u)=\frac{1}{2\pi\sqrt{n}p_n(n)}\int_{-\infty}^{\infty}\Psi_n(u,\tau)d\tau. \quad (A.28)$$

Denoting $\varphi_n\equiv E\exp\{itG_n^*\}$ we obtain

$$\varphi_n(t,h)=\frac{\exp\{-itx\}\varphi_n(u)}{\varphi_n(h)}. \quad (A.29)$$

Lemma A.3. Under conditions of Theorem 2 we have
i) There is a constant C_1 such that for $|t|\le C_1\sqrt{n}$
$|\varphi_n(t,h)-\exp\{-t^2/2\}|\le C\frac{x^2+|t|^3}{\sqrt{n}}\exp\{-t^2/2\}$.
ii) for $|t|\le 1$
$|D_t(\varphi_n(t,h)-\exp\{-t^2/2\})|\le C\frac{x^2}{\sqrt{n}}$.

Proof. By (A.28) we have

$$
\begin{aligned}
2\pi\sqrt{n}p_n(n)[\exp(-itx)\varphi_n(u)] = & \int_{-\infty}^{+\infty}\exp\{Q(t,\tau,x)\}d\tau \\
+ & \int_{|\tau|\le C_2 n^{1/6}}\left(e^{-itx}\Psi_n(u,\tau)-e^{Q(t,\tau,x)}\right)d\tau \\
+ & \int_{C_2 n^{1/6}\le|\tau|\le C_2\sqrt{n}} e^{-itx}Psi_n(u,\tau)d\tau \\
+ & \int_{C_2\sqrt{n}\le|\tau|} e^{-itx}Psi_n(u,\tau)d\tau - \int_{C_2 n^{1/6}\le|\tau|} e^{\tilde{Q}(t,\tau,x)}d\tau \\
\equiv & \left[\sum_{j=1}^{5} B_j\right] \qquad (A.30)
\end{aligned}
$$

where B_j denote the j-th integral inside the square brackets. Obviously
$B_1=\exp\left[\frac{x^2}{2}+\frac{x^3}{6\sigma_n^3}\sum_m Eg_m^3-\frac{t^2}{2}\right]$ and $|B_5|\le C\exp\{-Cn^{1/3}-0.5t^2+0.5x^2\}$,
since (A.22) and (2.1) .
Apply part i) and part ii) of Lemma A.3 for estimation of B_2 and B_3 respectively also part iii) of Lemma A.3 and relation (A.27) for estimation of B_4 to get
$|B_2|+|B_3|+|B_4|\le C\frac{|t|^3+x^2}{\sqrt{n}}\exp\left[\frac{x^2}{2}+\frac{x^3}{6\sigma_n^3}\sum_m Eg_m^3-\frac{t^2}{2}\right]$.
Thus

$$\exp\{-itx\}\phi_n(\sqrt{2\pi}p_n(n))^{-1}\exp\left[\frac{x^2}{2}+\frac{x^3}{6\sigma_n^3}\sum_m Eg_m^3-\frac{t^2}{2}\right]\left[1+\frac{|t|^3+x^2}{\sqrt{n}}\exp\{0.25t^2\}\right]. \quad (A.31)$$

Putting $t=0$ we get

$$\phi_n(h)=(\sqrt{2\pi n}p_n(n))^{-1}\exp\left[\sqrt{x^2}2+\sqrt{x^3}6\sigma_n^3\sum_m Eg_m^3\right]\left[1+\frac{x^2}{\sqrt{n}}\right]. \quad (A.32)$$

and taking into account (3.6) and (A.22) we get the second part Lemma 2. Apply (A.31) and second part.
Thus the proof of Lemma 2 can be completed using Lemma A.2 and reasons like those used for the proof of Lemma 1.

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