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ON REALIZATIONS OF THE STEENROD ALGEBRAS

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ABSTRACT. The Steenrod algebra can not be realized as an enveloping of any Lie superalgebra. We list several problems that suggest a need to modify the definition of the enveloping algebra, for example, to get rid of certain strange deformations which we qualify as an artifact of the inadequate definition of the enveloping algebra in positive characteristic. P. Deligne appended our paper with his comments, hints and open problems.

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1. Introduction

Hereafter p is the characteristic of the ground field $\mathbb{K}$.

1.1. Motivations. In the mid-1970s, Bukhshtaber and Shokurov [BS] interpreted the Landweber-Novikov algebra as the universal enveloping algebra of the Lie algebra of the vector fields on the line with coordinate t, and with the coefficients of $\frac{d}{dt}$ vanishing at the origin together with their first derivative. Therefore, when (at about the same time) P. Deligne told one of us (DL) that Grothendieck told him what sounded (to DL) like a similar interpretation of the Steenrod algebra, it did not alert the listener, although one should be very careful when $p > 0$. From that time till recently DL remembered Deligne's information in the following form

"The Steenrod algebra $\mathfrak{A}(2)$ is isomorphic to the enveloping algebra $U(\mathfrak{g})$ of a subsuperalgebra $\mathfrak{g}$ of the Lie superalgebra of contact vector fields on the $1|1$-dimensional superline, whose generating functions vanish at the origin together with their first derivative" (1)

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but the precise statement was never published and with time DL forgot what at that time he thought he understood from Deligne what Grothendieck meant under $\mathfrak{g}$. (For a precise statement in Deligne's own words, quite distinct from (1), see §4.) Somewhat later, Bukhshtaber [Bu] published a paper whose title claims to interpret the Steenrod algebras $\mathfrak{A}(p)$ for $p > 2$ in terms similar to (1), namely as (isomorphic to)

"the enveloping algebra of the supergroup of p-adic diffeomorphisms of the line". (2)

The body of the paper [Bu] clarifies its cryptic title (it is deciphered as meant to be "the enveloping algebra of a subsuperalgebra of the Lie superalgebra of vector fields on the $1|1$-dimensional superline in characteristic $p > 2$"), but nowhere actually states that the Steenrod algebra $\mathfrak{A}(p)$ is identified with the $U(\mathfrak{g})$ for any $\mathfrak{g}$. Instead, $\mathfrak{A}(p)$ is realized by differential operators but no description of the totality of these operators in more "tangible" terms, e.g., like the graphic (but wrong, as we will see) descriptions (1), (2) is offered; this is an open (but perhaps unreasonable) problem, cf. [Wp] with §§4.

Our initial intention was to explicitly describe the subsuperalgebra $\mathfrak{g}$ of the Lie superalgebra of vector fields on the $1|1$-dimensional superline for which $\mathfrak{A}(p) \simeq U(\mathfrak{g})$ as we remembered (1); we also wanted to decipher (2). However, having started, we have realized that we do not understand even what $U(\mathfrak{g})$ is if $p > 0$. More precisely, it is well-known ([S]) that there are two versions of the enveloping algebras (a "usual" one and a restricted one), but it seems to us that there are many more versions. An **open problem** is to give the appropriate definition of $U(\mathfrak{g})$ (this is definitely possible, at least, for classical Lie superalgebras with Cartan matrix) and related notions, such as representations and (co)homology of $\mathfrak{g}$.

So we begin with a discussion of the notion of $U(\mathfrak{g})$, and next pass to realizations of the Steenrod algebras. We conclude that, under conventional definitions ([S]), there is no Lie superalgebra $\mathfrak{g}$ such that $\mathfrak{A}(p) \simeq U(\mathfrak{g})$.

This result is not appealing: we hoped to clarify known realizations, not make a negative statement that $\mathfrak{A}(p)$ is NOT something. There are, however, realizations of $\mathfrak{A}(p)$ by differential operators ([Bu, Wd]). These realizations, although accepted, still look somewhat mysterious to us. In his comments at the end of this note, Deligne suggests a positive characterization of $\mathfrak{A}(p)$.

1.2. Notations. Let $T(V)$ be the tensor algebra of the superspace V, let $S(V)$ and $\Lambda(V)$ be the symmetric and exterior algebra of the space V, respectively. For a set $x = (x_1, \dots, x_n)$ of indeterminates that span V, we write $T[x]$ or $S[x]$ or $\Lambda[x]$, respectively. Let $\mathbb{Z}_+$ be the set of nonnegative integers.

As an abstract algebra, the Steenrod algebra $\mathfrak{A}(p)$ is defined, for any prime p, as follows ($\deg\beta = 1$):

$$\mathfrak{A}(p) = \begin{cases} \left(T[P^i \mid \deg P^i = 2i(p-1) \ \text{ for } i \in \mathbb{Z}_+] \otimes \Lambda[\beta]\right)/I(p) & \text{for } p > 2 \\ \left(T[P^i \mid \deg P^i = i \ \text{ for } i \in \mathbb{Z}_+]\right)/I(2) & \text{for } p = 2, \end{cases} \tag{3}$$

where the ideal of relations $I(p)$ for $p > 2$ is generated by the *Adem relations*

$$\begin{aligned} P^aP^b &= \sum_{i=0}^{[a/p]} (-1)^{a+i}\tbinom{(p-1)(b-i)-1}{a-pi} P^{a+b-i}P^i \qquad \text{for } a < pb, \\ P^a\beta P^b &= \sum_{i=0}^{[a/p]} (-1)^{a+i}\tbinom{(p-1)(b-i)}{a-pi}\beta P^{a+b-i}P^i - \\ &\quad \sum_{i=0}^{[(a-1)/p]} (-1)^{a+i}\tbinom{(p-1)(b-i)-1}{a-pi-1} P^{a+b-i}\beta P^i \ \text{ for } a \leq pb, \end{aligned} \tag{4}$$

whereas $I(2)$ is generated by

$$P^aP^b = \sum_{i=0}^{[a/2]} (-1)^{a+i}\tbinom{b-i-1}{a-2i} P^{a+b-i}P^i \quad \text{for } a < 2b. \tag{5}$$

Remark. For $p = 2$, the P^i are usually denoted Sq^i.

1.3. Lie superalgebras for $p = 2$. Observe that, for $p \neq 2$, for any Lie superalgebra $\mathfrak{g}$ and any odd x, we have

$$[x, x] = 2x^2. \tag{6}$$

In other words, there is a *squaring* operation

$$x^2 = \frac{1}{2}[x, x] \tag{7}$$

and to define the bracket of odd elements is the same as to define the squaring, since

$$[x, y] = (x + y)^2 - x^2 - y^2 \text{ for any } x, y \in \mathfrak{g}_{\bar{1}}. \tag{8}$$

A *Lie superalgebra* for $p = 2$ is a superspace $\mathfrak{g}$ such that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, $\mathfrak{g}_{\bar{1}}$ is an $\mathfrak{g}_{\bar{0}}$-module (made into the two-sided one by symmetry) and on $\mathfrak{g}_{\bar{1}}$ a *squaring* (roughly speaking, the halved bracket) is defined

$$\begin{aligned} &x \mapsto x^2 \quad \text{such that } (ax)^2 = a^2x^2 \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } a \in \mathbb{K}, \\ &\text{and the map } (x, y) \mapsto (x + y)^2 - x^2 - y^2 \text{ is bilinear} \\ &\text{and } \mathfrak{g}_{\bar{0}}\text{-invariant, i.e., } [x, y^2] = (\mathrm{ad}_y)^2(x) \text{ for any } x \in \mathfrak{g}_{\bar{0}} \text{ and } y \in \mathfrak{g}_{\bar{1}}. \end{aligned} \tag{9}$$

Then the bracket (i.e., product in $\mathfrak{g}$) of odd elements is defined to be

$$[x, y] := (x + y)^2 - x^2 - y^2. \tag{10}$$

The Jacobi identity for three odd elements is replaced by the following relation:

$$[x, x^2] = 0 \qquad \text{for any } x \in \mathfrak{g}_{\bar{1}}. \tag{11}$$

This completes the definition unless the ground field is $\mathbb{Z}/2$ in which case we have to add the condition

$$[x, y^2] = (\mathrm{ad}_y)^2(x) \text{ for any } x \in \mathfrak{g} \text{ and } y \in \mathfrak{g}_{\bar{1}} \tag{12}$$

which makes (11) and the last line in (9) redundant and replaces them over any field. The *restricted* Lie superalgebras are naturally defined.

1.4. Divided powers. For $p > 0$, there are many analogs of the polynomial algebra. These analogs break into the two types: the infinite dimensional ones and finite dimensional ones. The *divided power algebra* in indeterminates $x_1, \dots, x_m$ is the algebra of polynomials in these indeterminates, so, as space, it is

$$\mathcal{O}(m) = \mathrm{Span}\{x_1^{(r_1)} \dots x_m^{(r_m)} \mid r_1, \dots, r_m \geq 0\}$$

with the following multiplication:

$$(x_1^{(r_1)} \dots x_m^{(r_m)}) \cdot (x_1^{(s_1)} \dots x_m^{(s_m)}) = \prod_{i=1}^{m} \binom{r_i + s_i}{r_i} x_i^{(r_i+s_i)}.$$

For a *shearing parameter* $\underline{N} = (N_1, \dots, N_m)$, set

$$\mathcal{O}(m, \underline{N}) \quad (\text{or } \mathbb{K}[u; \underline{N}]) := \mathrm{Span}\{x_1^{(r_1)} \dots x_m^{(r_m)} \mid 0 \leq r_i < p^{N_i}, i = 1, \dots, m\},$$

where $p^\infty = \infty$. If $N_i < \infty$ for all i, then $\dim \mathcal{O}(m, \underline{N}) < \infty$.

Observe that only the conventional polynomial algebra and the one with $\underline{N} = (1, \dots, 1)$ are generated by the indeterminates that enter its definition. For any other value of $\underline{N}$, we have to add $x_i^{(p^{k_i})}$ for every k_i such that $1 < k_i < N_i$.

If an indeterminate x is odd, then the corresponding shearing parameter is equal to 1 for $p = 2$; for $p > 2$, we postulate

$$x^2 = 0 \quad \text{and anticommutativity of odd elements.}$$

2. The enveloping algebras of Lie algebras for $p > 0$

It looks strange that the following problem was never discussed in the literature. For $p > 0$, it seems natural — in view of the Poincaré-Birkhof-Witt theorem — to have as many types of universal enveloping algebras, as there are analogs of symmetric algebras or algebras of divided powers.

Of the variety of such hypothetical definitions of enveloping algebras (the usual one and the ones labelled by various values of the shearing parameter $\underline{N}$), only two are being considered: the usual $U(\mathfrak{g})$ and the one corresponding to $\underline{N} = (1, \dots, 1)$.

We hope that there exist $U(\mathfrak{g}; \underline{N})$ — analogs of $U(\mathfrak{g})$, such that $\mathrm{gr} U(\mathfrak{g}; \underline{N}) \simeq \mathcal{O}(\dim \mathfrak{g}; \underline{N})$. Such $\underline{N}$-dependent definitions of $U(\mathfrak{g})$ do exist, at least for simple finite dimensional complex Lie algebras, see [St] (take the $\mathbb{Z}$-form of $U(\mathfrak{g})$ described in [St] and tensor by $\mathbb{K}$) and we hope that the **open problem** to generalize the definition to arbitrary algebras $\mathfrak{g}$ is not difficult to solve. We suspect that these $U(\mathfrak{g};\underline{N})$ were ignored because they are not generated by the initially declared indeterminates (or the space $\mathfrak{g}$ they span) and the necessity to add extra generators (depending on $\underline{N}$) was too unusual: to preserve a one-to-one-correspondence between representations of $U(\mathfrak{g})$ and of $\mathfrak{g}$, we have to amend the definition of the latter.

The notion of the universal enveloping algebra is motivated, first of all, by the representation theory. So let us give more reasons, other than the PBW theorem, to consider the non-conventional universal enveloping algebras corresponding to any value of the shearing parameter $\underline{N}$.

2.1. The induced and coinduced modules. Since any derivation of a given algebra is completely determined by its values on every generator of the algebra, the Lie algebra of all derivations of $\mathbb{K}[u;\underline{N}]$ is much larger than the Lie algebra of *special derivations* whose generators behave like partial derivatives:

$$\partial_i(u_j^{(k)}) = \delta_{ij}u_j^{(k-1)}. \tag{13}$$

In what follows, speaking about Lie algebras of vector fields (briefly: vectorial algebras) we only consider special derivations, e.g., in (14).

The simple vectorial Lie algebras for $p=0$ have only one parameter: the number of indeterminates. If $\mathrm{Char}\mathbb{K} = p > 0$, the vectorial Lie algebras acquire one more parameter: $\underline{N}$. For Lie superalgebras, $\underline{N}$ only concerns the even indeterminates. Let

$$\mathfrak{vect}(m;\underline{N}|n) \ \text{ a.k.a } W(m;\underline{N}|n) \ := \mathfrak{der}\mathbb{K}[u;\underline{N}] \tag{14}$$

be the general vectorial Lie algebra.

The induced and coinduced modules are natural classes of modules over Lie algebras. Over $\mathbb{C}$, the modules of (formal) tensor fields constitute a natural class of modules. In particular, a most natural — (x)-adic — filtration in the polynomial algebra $\mathbb{K}[x_1,\dots,x_m]$, induces a filtration (*Weisfeiler filtration*) in the Lie algebra $\overline{\mathfrak{vect}(m)} := \mathfrak{der}(\mathbb{K}[[x]])$. The associated grading is given by setting $\deg x_i = 1$ for all i. Let

$$\mathcal{L} = \mathcal{L}_{-1} \subset \mathcal{L}_0 \subset \mathcal{L}_1 \subset \dots$$

be the Weisfeiler filtration of $\mathfrak{L} := \mathfrak{vect}(m)$; let $L_i := \mathcal{L}_i/\mathcal{L}_{i+1}$.

Let V be a $\mathfrak{gl}(m)$-module, considered as a $\mathcal{L}_0$-module such that $\mathcal{L}_iV = 0$ for $i>0$. We define the induced and coinduced modules over $\mathcal{L}$ as

$$\mathrm{Ind}_{\mathcal{L}_0}^{\mathcal{L}}(V) = U(\mathcal{L}) \otimes_{U(\mathcal{L}_0)} V, \qquad \mathrm{Coind}\mathcal{L}_{\mathcal{L}_0}(V) = \mathrm{Hom}_{U(\mathcal{L}_0)}(U(\mathcal{L}), V). \tag{15}$$

In particular, the spaces of tensor fields of type V (with fiber V) are coinduced $\mathcal{L}$-modules.

The spaces $\mathcal{O}(m;\underline{N})$ and $\mathfrak{vect}(m;\underline{N})$ **are** $\mathfrak{vect}(m;\underline{M})$-modules (well-defined if $M_i \leq N_i$ for every i), but if we want to consider them as coinduced modules, we need the unconventional universal enveloping algebras, namely for the commutative Lie (super)algebra $\mathcal{L}_{-1}/\mathcal{L}_0 \simeq \mathrm{Span}(\partial_1, \dots, \partial_m)$, we define

$$U(\mathcal{L}_{-1}/\mathcal{L}_0;\underline{N}) := \mathcal{O}(m;\underline{N}). \tag{16}$$

The computation of deformations of $\mathfrak{vect}(m;\underline{N})$ is currently performed either by painstaking calculations ([DK, Dz]) or with the help of computer but both with the same — conventional — definition of $U(\mathfrak{g})$. Grozman [GL] used **SuperLie** package to verify the rigidity of $\mathfrak{vect}(m;\underline{N})$ for small m and $\underline{N} = (1, \dots, 1)$ for $p = 3$, whereas, for $\underline{N} \neq (1, \dots, 1)$, Dzhumadildaev and Kostrikin [DK] found lots and lots of infinitesimal deformations (2-cocycles) all of which are mysterious.

On the other hand, recall that, for $p = 0$ and any $\mathfrak{h}$-module M, we have ([Fu])

$$H^q(\mathfrak{g}; \mathrm{Coind}_{\mathfrak{h}}^{\mathfrak{g}}(M)) \simeq H^q(\mathfrak{h}; M); \qquad H_q(\mathfrak{g}; \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(M)) \simeq H_q(\mathfrak{h}; M). \tag{17}$$

Now, observe that, over vectorial Lie superalgebras $\mathfrak{g}$, the modules of tensor fields are precisely the coinduced ones: $T(M) := \mathrm{Coind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}}(M)$, where $\mathfrak{g}_{\geq 0} = \mathop{\oplus}\limits_{i \geq 0} \mathfrak{g}_i$ and M is any $\mathfrak{g}_{\geq 0}$-module such that $\mathfrak{g}_{>0}M = 0$ for $\mathfrak{g}_{>0} = \mathop{\oplus}\limits_{i>0} \mathfrak{g}_i$, i.e., M is, actually, a $\mathfrak{g}_0$-module. In particular, let $\mathrm{id}_{\mathfrak{gl}(m)}$ be the identity $\mathfrak{gl}(m)$-module. Then

$$W(m;\underline{N}) \simeq \mathcal{O}(m;\underline{N}) \otimes \mathrm{id}_{\mathfrak{gl}(m)}$$

which is a coinduced module if we define $U(\mathcal{L}_{-1}/\mathcal{L}_0;\underline{N})$ my means of (16).

So, for a conjectural cohomology theory "H"$^{\bullet}_{\underline{N}}$, the following rigidity theorem would be an corollary of the general theorem (17) and the mysterious infinitesimal deformations found in [DK, Dz] should be considered as "artifacts" (except, perhaps, certain values of p and m for which "H"$^2(\mathfrak{gl}(m); \mathrm{id}_{\mathfrak{gl}(m)}) \neq 0$) because "$H$"$^i_{\underline{N}}(\mathfrak{gl}(m); \mathrm{id}_{\mathfrak{gl}(m)}) = H^i(\mathfrak{gl}(m); \mathrm{id}_{\mathfrak{gl}(m)})$ for i small, and

2.2. Corollary (Conjecture)**.** *We have*

$$\text{“}H\text{”}^2_{\underline{N}}(W(m;\underline{N}); W(m;\underline{N})) \simeq \text{“}H\text{”}^2_{\underline{N}}(\mathfrak{gl}(m); \mathrm{id}_{\mathfrak{gl}(m)}) = 0.$$

2.3. How to quantize? The Poisson Lie (super)algebra $\mathfrak{po}(2n|m)$ realized on polynomials admits only one deformation as a Lie (super)algebra, cf. [LS]. After Dirac, physicists interpret this deformation as quantization. Quantization deforms $\mathfrak{po}(0|2m)$ into $\mathfrak{gl}(\Lambda(m))$. What is the analog of this statement for $p > 0$ and $\mathfrak{po}(2n;\underline{N}|2m)$? The answer depends on how we understand $U(\mathfrak{g})$.

3. Main result

3.1. Theorem. *For any $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus \mathfrak{g}_k$, we consider on $U(\mathfrak{g})$ the induced $\mathbb{Z}$-grading $U(\mathfrak{g}) = \oplus U(\mathfrak{g})_k$. For any $p > 0$, if P^0 (or Sq^0) is a scalar, there is no grading preserving isomorphism $f : \mathfrak{A}(p) \longrightarrow U(\mathfrak{g})$ between the Steenrod algebra $\mathfrak{A}(p)$ and the (common or restricted) universal enveloping algebra of any $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}$ with the parity of elements of $\mathfrak{g}_k$ being the same as that of k.*

Proof. Suppose that such an isomorphism exists. First, let us show that $\dim \mathfrak{g}_k$ is uniquely determined by the information on $\dim \mathfrak{A}(p)_i$. Let

$$\mathfrak{G}_k = \bigoplus_{i \leq k} \mathfrak{g}_k$$

as a ($\mathbb{Z}$-graded) linear superspace. Clearly, $\dim \mathfrak{G}_0 = 0$ (since $\dim \mathfrak{A}(p)_i = 0$ for $i < 0$). So, according to PBW theorem, $\dim U(\mathfrak{g})_k = \dim \mathfrak{g}_k + d_k$, where d_k is equal to the dimension of the space of (super)symmetric polynomials on $\mathfrak{G}_{k-1}$ of weight k, if we consider a non-restricted (common) universal enveloping algebra, or to the dimension of the space of (super)symmetric polynomials on $\mathfrak{G}_{k-1}$ of weight k and degree $< p$ w.r.t. any even basic element, if we consider a restricted universal enveloping algebra.

Since $\dim U(\mathfrak{g})_k = \dim \mathfrak{A}(p)_k$, and d_k is determined by dimensions and parities of $\mathfrak{g}_i$ for $i < k$, one can find $\dim \mathfrak{g}_k$ for any k by induction. The following table illustrates this for $p = 2$, a non-restricted algebra and small values of k. In the table, the first row contains k; the second row contains bases of $\mathfrak{A}(2)_k$; the third row contains bases of the spaces of (super)symmetric polynomials on $\mathfrak{G}_{k-1}$ of degree k, where L_i denotes a non-zero element of $\mathfrak{g}_k$; the fourth row contains $\dim \mathfrak{g}_k$.

1	2	3	4	5	6	7	8
Sq^1	Sq^2	Sq^3 Sq^2Sq^1	Sq^4 Sq^3Sq^1	Sq^5 Sq^4Sq^1	Sq^6 Sq^5Sq^1 Sq^4Sq^2	Sq^7 Sq^6Sq^1 Sq^5Sq^2 $Sq^4Sq^2Sq^1$	Sq^8 Sq^7Sq^1 Sq^6Sq^2 $Sq^5Sq^2Sq^1$
$-$	$-$	L_2L_1	L_2^2 L_3L_1	$L_2^2L_1$ L_2L_3	L_2^3 $L_2L_3L_1$	L_6L_1 $L_2^2L_3$ $L_2^3L_1$	L_6L_2 L_2^4 $L_2^2L_3L_1$ L_7L_1
1	1	1	0	0	1	1	0

Now let us first consider the case $p = 2$. From the table and similar computations for a hypothetical restricted algebra $\mathfrak{g}$, we see that $L_1 = Sq^1, L_2 = Sq^2$ are elements of $\mathfrak{g}$. If $Sq^0 = 0$, then $L_1L_2 = 0$, which can not be true. If Sq^0 is a non-zero scalar, then, up to a non-zero scalar factor,

$$(L_2)^2 = Sq^3Sq^1 \neq 0; \qquad (L_2)^2L_1 = 0.$$

This can hold only if we consider a restricted algebra, and Sq^3Sq^1 is an element of $\mathfrak{g}$, proportional to Sq^1 — which can not be true, since these two non-zero elements have different weights.

Now we consider the case $p > 2$. The computations of dimensions similar to the above ones show that the minimal weights in which $\mathfrak{g}$ has non-zero elements are:

$1,\ 2(p-1),\ 2p-1,\ 2p^2-2,\ 2p^2-1,\ 2p^3-2$ for $\mathfrak{g}$ non-restricted
$1,\ 2(p-1),\ 2p-1,\ 2(p-1)p,\ 2p^2-2,\ 2p^2-1,\ 2(p-1)p^2$ for $\mathfrak{g}$ restricted.

Since $\dim \mathfrak{A}(p)_{2(p-1)} = 1$, we see that $L_{2(p-1)} = P^1$ is an element of $\mathfrak{g}$. If $P^0 = 0$, then, $(L_{2(p-1)})^2 = (P^1)^2 = 0$, which is false. It follows from the Adem relations that

$$(L_{2(p-1)})^p = (P^1)^p = 0,$$

which can hold only in a restricted algebra. Then, since $\dim \mathfrak{A}(p)_{2(p-1)p} = 1$, we see that $L_{2(p-1)p} = P^p$ is an element of $\mathfrak{g}$. If $P^0 = \lambda \neq 0$, it follows from the Adem relations that

$$\begin{aligned}
&[L_{2(p-1)}, L_{2(p-1)p}] = [P^1, P^p] = P^1P^p - P^pP^1 = \lambda P^{p+1} - P^pP^1;\\
&[L_{2(p-1)}, [L_{2(p-1)}, L_{2(p-1)p}]] = [P^1, [P^1, P^p]] =\\
&P^1(\lambda P^{p+1} - P^pP^1) - (\lambda P^{p+1} - P^pP^1)P^1 =\\
&2(\lambda^2 P^{p+2} - \lambda P^{p+1}P^1 + \lambda P^pP^2) = 2\lambda P^{p+1}P^1 \neq 0.
\end{aligned}$$

The last expression must be an element of $\mathfrak{g}$, but $\mathfrak{g}$ does not have non-zero elements of weight $2(p-1)(p+2)$, so we get a contradiction. □

4. Pierre Deligne's comments in a letter to DL, May 23, 2006

About Steenrod. What Grothendieck saw is the following (for p odd).

a) Quillen [Q1]: for complex cobordism, one has

$$\Omega U(B\mathbb{C}^*) = \Omega U(Pt)[\eta], \text{ where } \deg(\eta) = 2$$

(with $B\mathbb{C}^* = \mathbb{P}^\infty(\mathbb{C})$), and the group law of $B\mathbb{C}^*$, deduced from that of $\mathbb{C}^*$, induces on $\operatorname{Spec}\Omega U(B\mathbb{C}^*)$ a structure of formal group over $\operatorname{Spec}\Omega U(Pt)$. This turns $\operatorname{Spec}\Omega U(Pt)$ into the scheme of formal group laws on the pointed formal disc $\operatorname{Specf}(\mathbb{Z}[\![t]\!])$:

$$\Omega U(Pt) = \mathbb{Z}[a_{i,j} \mid i, j \geq 0,\ i + j > 0]/\text{identities},$$

the identities expressing that $F(t,u) = \sum a_{ij} t^i u^j$ is a formal group law. The group scheme of automorphisms of the pointed formal disc hence acts by transport of structures on $\Omega U(Pt)$. It is the group of

$$t \longmapsto \sum a_i t^i \qquad (i \geq 1,\ a_1 \text{ invertible}). \tag{18}$$

The action of the subgroup $\mathbb{G}_m \colon t \mapsto at$ gives the half degree. If we consider the subgroup with $a_1 = 1$, this action extends to a functorial action on $\Omega U(X)$, compatible with products (Landweber operations, see [BS]). The group scheme of transformations (18) has a double covering, with coordinates $\sqrt{a_1}$ and the a_i ($i \geq 2$). This double covering is again a group scheme, and it contains the $\mathbb{G}_m$-subgroup "$a_i = 0$ for $i \geq 2$" (coordinate $\sqrt{a_1}$). The action of this $\mathbb{G}_m$ gives the degree.

b) This suggests that for any commutative ring R, and any 1-dimensional formal group G over R, possibly given with a trivialization of its Lie algebra: $\mathrm{Lie}(G) \xrightarrow{\sim} R$, there could be a corresponding cohomology theory, functorial in G. If t is a parameter for G (compatible with the trivialization of the Lie algebra), G is given by

$$\Omega U(Pt) \to R$$

and the theory would be obtained from complex cobordism by some "derived extension of scalars", while the Landweber operations would ensure that the result is independent of the choice of t, up to unique isomorphisms.

I am rather naive here; we are playing with (ringed) spectrum, not with rings and their derived categories. I don't know what has been done, but results are known: As I remember being told, the case where $\mathrm{Spec}(R)$ is a complete intersection in $\mathrm{Spec}(\Omega U(P^t))$ is OK. This allows for the construction of Morava's K-theories using this philosophy.

c1) $G = \mathbb{G}_m^\wedge/\mathbb{Z}$: in each characteristic, we are in the open orbit of the action on $\mathrm{Spec}\,\Omega U(Pt)$, so that the extension of scalars to $\mathbb{Z}$ is an exact functor, and one gets K-theory (Conner and Floyd [CF]).

c2) For a formal group over a field of char p, the (geometric) invariant is the height, and one gets the Morava K-theories ([DMea]).

c3) For $\mathbb{G}_a^\wedge/\mathbb{F}_p$, one gets the ordinary mod p cohomology. The group scheme of automorphisms of $\mathbb{G}_a^\wedge$ (which are 1 on the Lie algebra) should hence act. It is the group scheme

$$A = \left\{ t \longmapsto \sum b_i t^{p^i},\ b_0 = 1 \right\},$$

whose affine algebra is $\mathbb{Z}[b_1, b_2, \dots]$ (denoted $\mathcal{O}(A)$) the coproduct (giving the group law) being defined by

$$b_0 := 1,$$
$$\Delta b_k = \sum_{\ell+m=k} b_\ell \otimes b_{k-\ell}^{p^\ell}.$$

As shown by Milnor, this group indeed acts functorially on $H^*(\cdot, \mathbb{F}_p)$, see [Mi] Th. 3, page 162. "Action" means "comodule structure $H^* \to H^* \otimes \mathcal{O}(A)$".

This does not capture the odd part of the story, for which I lack understanding. What Milnor says is that (for p odd)

$$\mathrm{Spec}(H^*(B\,\mathbb{Z}/p, \mathbb{Z}/p)),$$

with the group law coming from that of $B\mathbb{Z}/p$, is[1)] $\mathbb{G}_a^+ \times \mathbb{G}_a^-$, i.e., $H^\bullet = \mathbb{Z}/p\,[t, \tau]$ with t even and τ odd, and the group law

$$(t', \tau') + (t'', \tau'') = (t' + t'', \tau' + \tau'').$$

If B is the super group scheme (for the definition, see [SuSy]) of automorphisms of $G = \mathbb{G}_a^+ \times \mathbb{G}_a^-$, respecting the filtration $\mathrm{Lie}\,\mathbb{G}_a^+ \subset \mathrm{Lie}\,G$ and acting trivially on the successive quotients, the group B acts functorially on $H^*(X, \mathbb{Z}/p)$, respecting the cup-product [and one could add to it a $\mathbb{G}_m$ giving the degree]. The action on $H^*(B\mathbb{Z}/p, \mathbb{Z}/p)$ is the one defining B, and the affine algebra $\mathcal{O}(B)$ is the dual of the Steenrod algebra [Milnor loc. cit. Th. 2, page 159].

I would hope that the odd part of the story is analogous to the following fact: if k is a quotient of a ring R, then $\mathrm{Ext}^i_R(k, k)$ acts on $H^*(M \overset{\mathbb{L}}{\otimes}_R k)$ for any M in $D^-(R)$.

Other comments on the text with Lebedev.

Other convenient definitions of the space of quadratic forms on a projective module M:

- $\mathrm{Sym}^2 M^\vee$, where Sym^2 = covariants of S_2 acting on 2-tensors.

equivalently: the space of quadratic form is the cokernel of $C \mapsto C(X, Y) - C(Y, X)$ on the space of bilinear maps,

- the dual of $\Gamma^2(M)$ (divided power, = symmetric 2-tensors)

— If G is a smooth algebraic group on $\mathrm{Spec}(R)$, a reasonable analog of what $U(\mathfrak{g})$ is in characteristic 0 is the algebra of left-invariant differential operators on G. As a coalgebra, it is the dual of the completion of G at the unit element. It is, I think, what Dieudonné calls the *hyperalgebra of the group.* It cannot be constructed from the Lie algebra. For instance, $\mathrm{Lie}\,\mathbb{G}_a = \mathrm{Lie}\,\mathbb{G}_m = R$, but

1) Obviously one factor is an "even" group, the other one is an "odd" one representing, respectively, the functors $C \longmapsto C_{\bar 0}$ and $C \longmapsto C_{\bar 1}$ for any supercommutative C.

for G_a one gets the $\frac{\partial_x^i}{i!}$, and for $\mathbb{G}_m$ the binomial $\binom{x\partial_x}{i}$, where the choice of generator is crucial.

4.1. Pierre Deligne's comments in a letter to DL, September 1, 2006. As I am a geometer, groups are more congenial to me than Lie algebras, and it does not bother me that in characteristic p the Lie algebra of a group does a poor job of controlling it. If I want to have all relevant "divided powers" for a given group, I just take as starting point the bialgebra of left invariant differential operators. This "is the same" as giving the formal group ($\mathcal{O}(G^\wedge) =$ dual) and is, if I remember right, what Dieudonné calls a hyper (Lie?) algebra.

Lie algebras with a pth power operation (= restricted), on the other hand, are exactly the same things as algebraic groups equal to the Kernel of Frobenius.

So, I am more happy with Steenrod "being" a (super) group scheme than it being some kind of enveloping algebra.

Even in the even case, characteristic 2 and 3 are tricky, and I am not sure one definition is suitable for all applications.

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Theorem. *The Landweber-Novikov algebra S is isomorphic with the algebra of left invariant differential operators in the group $\mathit{Diff}_1(Z)$.*

Corollary. *The tensor product $S \otimes R$ is isomorphic with the enveloping algebra of the Lie algebra of formal vector fields on the line, which vanish at the origin with the first derivative. In addition, the ring A^U of all stable cohomology operations of complex cobordism theory is identified with a certain ring of differential operators on $\mathit{Diff}_1(R)$.*

The arguments rely on techniques developed previously by the first author [Mat. Sb. (N.S.) 83(125) (1970), 575–595; Two-valued formal groups. Algebraic theory and applications to cobordism. I. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 5, 1044–1064; English translation: Math. USSR-Izv. 9 (1975), no. 5, 987–1006 (1976).].

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