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AN ITERATIVE METHOD FOR NONEXPANSIVE MAPPING IN BANACH SPACES

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Abstract. In this paper, we establish weak and strong convergence theorems of the three-step iterative sequences with errors for non-self nonexpansive mappings in uniformly convex Banach spaces. Our results extend and improve the recent ones announced by Naseer Shahzad and some others.

Key words : Nonexpansive non-self map; *Opial's* condition; Uniformly convex; Demiclosed
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1. Introduction and Preliminaries

Let E be a real Banach space, C a nonempty closed convex subset of E, and $T : C \to C$ a mapping. Recall that T is nonexpansive mapping if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by N the set of natural numbers and Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout this paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$.

Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors (see e.g., [3, 7, 9, 11, 12]). In [11], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space E. More precisely, They proved the following theorem.

Theorem TX (Tan and Xu [11,Theorem 1]). Let E be a a uniformly convex Banach space which satisfies Opial's condition or has a Frechet differentiable

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norm and C a nonempty closed convex bounded subset of E. Let $T : C \to C$ be a nonexpansive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in [0,1] such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n], \quad n \geq 1, \tag{1.1}$$

converges weakly to some fixed point of T.

In the above result, T remains self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space, if, however, the domain K of T is a proper subset of E(and this is the cases in several applications), and T maps K into E then iteration processes of Mann [5] and Ishikawa [2] studied by these authors may fail to be well defined.

Recently, Naseer Shahzad [10] studied the sequence $\{x_n\}$ defined by

$$x_1 \in K, \quad x_{n+1} = P((1-\alpha_n)x_n + \alpha_n TP[(1-\beta_n)x_n + \beta_n T x_n]), \tag{1.2}$$

where K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retraction. He proved weak and strong convergence theorems for non-self nonexpansive mappings in Banach spaces.

Motivated by the Nasser Shahzad [10], this paper study the iteration scheme as following

$$\begin{cases} z_n = P(\alpha''_n T_3 x_n + \beta''_n x_n + \gamma''_n w_n), \\ y_n = P(\alpha'_n T_2 z_n + \beta'_n x_n + \gamma'_n v_n), \\ x_{n+1} = P(\alpha_n T_1 y_n + \beta_n x_n + \gamma_n u_n), \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ are sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\epsilon \leq \alpha_n, \alpha'_n, \alpha''_n \leq 1-\epsilon$ for all $n \in N$ and some $\epsilon > 0$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are bounded sequence in K.

The purpose of this paper is to construct an iteration scheme with errors for approximating a common fixed point of nonexpansive non-self maps (when such a fixed point exists) and to prove some strong and weak convergence theorems for such maps. Our theorems improve and generalize some previous results.

A normed space E is said to satisfy $Opial's$ condition [6] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Let E be a real Banach space. A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to E$ such that $Px = x$ for all $x \in K$.

A map $P : E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

A mappingt T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

Recall that the mapping $T : K \to E$ with $F(T) \neq \emptyset$ where K is a subset of E, is said to satisfy condition A [10] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that for all $x \in K$

$$\|x - Tx\| \geq f(d(x, F(T))),$$

where $d(x, F(T) = inf\{\|x - p\| : p \in F(T)\}$.

Senter and Dotson [9] approximated fixed points of a nonexpansive mapping T by Mann iterates, Later on, Maiti and Ghosh [4] and Tan and Xu [11] studied the approximation of fixed points of a nonexpansive mapping T by Ishikawa iterates under the same condition (A) which is weaker than the requirement that T is demicompact. We modify this condition for three mappings T_1, T_2 and $T_3 : C \to C$ as follows:

Three mappings T_1, T_2 and $T_3 : C \to C$ where C a subset of E, are said to satisfy condition (A$'$) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $a\|x - T_1x\| + b\|x - T_2x\| + c\|x - T_3x\| \geq f(d(x, F(T)))$ for all $x \in C$ where $d(x, F(T)) = inf\{\|x - p\| : p \in F(T_1) \cap F(T_2) \cap F(T_3)\}$ and a, b and c are three nonnegative real numbers such that $a + b + c = 1$.

Note that condition (A$'$) reduces to condition (A) when $T_1 = T_2 = T_3$.

In order to prove our main results, we shall make use of the following Lemmas.

Lemma 1.1 (Schu [8]). Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in N$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences in E such that

$$\limsup_{n\to\infty} \|x_n\| \leq r, \quad \limsup_{n\to\infty} \|y_n\| \leq r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.2 (Browder [1]). Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E. Let T be nonexpansive mapping of K into E. Then $I - T$ is demiclosed with respect to zero.

Lemma 1.3 (Tan and Xu [11]). Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:

$$r_{n+1} \le r_n + t_n \qquad \text{for all } n \ge 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

2. **Convergence of The Iteration Scheme**

In this section, we shall prove the weak and strong convergence of the iterative scheme (1.3) to approximate a common fixed point of the nonexpansive mappings T_1, T_2 and T_3.

Lemma 2.1 Let E be a normed linear space and K a nonempty convex closed subset which is also a nonexpansive retract of E. Let T_1, T_2 and T_3 : $K \to E$ be nonexpansive mappings with $F(T) \neq \emptyset$, where $F(T)$ denotes the set of all common fixed points of T_1, T_2 and T_3. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in [0, 1] such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.3) with the restrictions $\sum_{n=1}^{\infty} \gamma''_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Let $p \in F(T)$. Since $\{w_n\}$, $\{v_n\}$ and $\{u_n\}$ are bounded sequences in C We set
$M_1 = \sup\{\|u_n - p\| : n \ge 1\}$, $\quad M_2 = \sup\{\|v_n - p\| : n \ge 1\}$,
$M_3 = \sup\{\|w_n - p\| : n \ge 1\}$, $\quad M = \max\{M_i : i = 1, 2, 3\}$.
It follows from (1.3) that

$$\begin{aligned}
\|z_n - p\| &= \|P(\alpha''_n T_3 x_n + \beta''_n x_n + \gamma''_n w_n) - p\| \\
&\le \|\alpha''_n T_3 x_n + \beta''_n x_n + \gamma''_n w_n - p\| \\
&\le \alpha''_n \|T_3 x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|w_n - p\| \\
&\le \alpha''_n \|x_n - p\| + \beta''_n \|x_n - p\| + \gamma''_n \|w_n - p\| \\
&\le \|x_n - p\| + \gamma''_n M.
\end{aligned}$$

That is,

$$\|z_n - p\| \le \|x_n - p\| + \gamma''_n M. \tag{2.1}$$

From (1.3) and (2.1) we get

$$
\begin{aligned}
\|y_n - p\| &= \|P(\alpha'_n T_2 z_n + \beta'_n x_n + \gamma'_n v_n) - p\| \\
&\le \|\alpha'_n T_2 z_n + \beta'_n x_n + \gamma'_n v_n - p\| \\
&\le \alpha'_n \|T_2 z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\le \alpha'_n \|z_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\le \alpha'_n \|z_n - p\| + (1 - \alpha'_n)\|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\le \alpha'_n (\|x_n - p\| + \gamma''_n M) + (1 - \alpha'_n)\|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\le \|x_n - p\| + \gamma''_n M + \gamma'_n M.
\end{aligned}
$$

That is,

$$
\|y_n - p\| \le \|x_n - p\| + \gamma''_n M + \gamma'_n M. \tag{2.2}
$$

Again, from (1.3) and (2.2) we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|P(\alpha_n T_1 y_n + \beta_n x_n + \gamma_n u_n) - p\| \\
&= \|\alpha_n T_1 y_n + \beta_n x_n + \gamma_n u_n - p\| \\
&\le \alpha_n \|T_1 y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\le \alpha_n \|y_n - p\| + \beta_n \|x_n - p\| + \gamma_n \|u_n - p\| \\
&\le \alpha_n \|y_n - p\| + (1 - \alpha_n)\|x_n - p\| + \gamma_n \|u_n - p\| \\
&\le \alpha_n (\|x_n - p\| + \gamma''_n M + \gamma'_n M) + (1 - \alpha'_n)\|x_n - p\| + \gamma_n M \\
&\le \|x_n - p\| + \gamma''_n M + \gamma'_n M + \gamma_n M \\
&\le \|x_n - p\| + \gamma''_n M + \gamma'_n M + \gamma_n M.
\end{aligned}
$$

That is,

$$
\|x_{n+1} - p\| \le \|x_n - p\| + (\gamma''_n + \gamma'_n + \gamma'_n)M. \tag{2.3}
$$

Therefore, by using Lemma 1.3, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. This completes the proof.

Notation. In the following, $\overline{\lim}$ stands for lim sup and $\underline{\lim}$ for lim inf.

Lemma 2.2 Let E be a uniformly convex Banach space and K a nonempty convex closed subset which is also a nonexpansive retract of E. Let T_1, T_2 and $T_3 : K \to E$ be nonexpansive mappings with $F(T) \ne \emptyset$, where $F(T)$ denotes the set of all common fixed points of T_1, T_2 and T_3. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in [0, 1] such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = \alpha''_n + \beta''_n + \gamma''_n = 1$ and $\epsilon \le \alpha_n, \alpha'_n, \alpha''_n \le 1 - \epsilon$ for all $n \in N$ and some $\epsilon > 0$, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.3).
Then $\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = \lim_{n\to\infty} \|x_n - T_3 x_n\| = 0$.

Proof. Take $p \in F(T)$, by Lemma 2.1 $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = c$. If $c = 0$, then by the continuity of T_1, T_2 and T_3 the conclusion follows. Now suppose $c > 0$. We claim $\lim_{n\to\infty} \|x_n - T_1x_n\| = \lim_{n\to\infty} \|x_n - T_2x_n\| = \lim_{n\to\infty} \|x_n - T_3x_n\| = 0$. Since $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded, it follows that $\{u_n - x_n\}$, $\{v_n - x_n\}$ and $\{w_n - x_n\}$ are all bounded. Now, we set
$r_1 = \sup\{\|u_n - x_n\| : n \geq 1\}$, $r_2 = \sup\{\|v_n - x_n\| : n \geq 1\}$,
$r_3 = \sup\{\|w_n - x_n\| : n \geq 1\}$, $r = \max\{r_i : 1 \leq i \leq 3\}$.
Taking limsup on both the sides in the inequality (2.1), we have

$$\overline{\lim}_{n\to\infty}\|z_n - p\| \leq c. \tag{2.4}$$

Similarly, taking limsup on both the sides in the inequality (2.2), we have

$$\overline{\lim}_{n\to\infty}\|y_n - p\| \leq c. \tag{2.5}$$

Next, we consider

$$\begin{aligned}\|T_1y_n - p + \gamma_n(u_n - x_n)\| &\leq \|T_1y_n - p\| + \gamma_n\|u_n - x_n\| \\ &\leq \|y_n - p\| + \gamma_n r.\end{aligned}$$

Taking limsup on both the sides in the above inequality and using (2.5), we get that

$$\overline{\lim}_{n\to\infty}\|Ty_n - p + \gamma_n(u_n - x_n)\| \leq c.$$

The inequalities

$$\begin{aligned}\|x_n - p + \gamma_n(u_n - x_n)\| &\leq \|x_n - p\| + \gamma_n\|u_n - x_n\| \\ &\leq \|x_n - p\| + \gamma_n r,\end{aligned}$$

yield

$$\overline{\lim}_{n\to\infty}\|x_n - p + \gamma_n(u_n - x_n)\| \leq c.$$

Again, $\lim\limits_{n\to\infty} \|x_{n+1} - p\| = c$ means that

$$\underline{\lim}_{n\to\infty}\|\alpha_n(T_1y_n - p + \gamma_n(u_n - x_n)) + (1-\alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \geq c. \tag{2.6}$$

On the other hand, we have

$$\begin{aligned}&\|\alpha_n(T_1y_n - p + \gamma_n(u_n - x_n)) + (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \\ \leq&\alpha_n\|T_1y_n - p\| + (1 - \alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ \leq&\alpha_n\|y_n - p\| + (1 - \alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ \leq&\alpha_n(\|x_n - p\| + \gamma''_n r + \gamma'_n r) + (1 - \alpha_n)\|x_n - p\| + \gamma_n\|u_n - x_n\| \\ \leq&\|x_n - p\| + \gamma''_n r + \gamma'_n r + \gamma_n r.\end{aligned}$$

Therefore, we obtain

$$\overline{\lim}_{n\to\infty}\|\alpha_n(Ty_n - p + \gamma_n(u_n - x_n)) + (1-\alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| \leq c. \tag{2.7}$$

Formulas (2.6) and (2.7) yield

$$\lim_{n\to\infty} \|\alpha_n(T_1y_n - p + \gamma_n(u_n - x_n)) + (1 - \alpha_n)(x_n - p + \gamma_n(u_n - x_n))\| = c.$$

Hence applying Lemma 1.1, we have

$$\lim_{n\to\infty}\|T_1y_n - x_n\| = 0. \tag{2.8}$$

Next, we observe that

$$\begin{aligned}\|x_n - p\| &\le \|T_1y_n - x_n\| + \|Ty_n - p\|\\ &\le \|T_1y_n - x_n\| + \|y_n - p\|.\end{aligned}$$

The latter yields

$$c \le \underline{\lim}_{n\to\infty}\|y_n - p\| \le \overline{\lim}_{n\to\infty}\|y_n - p\| \le c.$$

That is,

$$\lim_{n\to\infty}\|y_n - p\| = c.$$

Again, $\lim_{n\to\infty}\|y_n - p\| = c$ implies that

$$\underline{\lim}_{n\to\infty}\|\alpha_n'(T_2z_n - p + \gamma_n'(v_n - x_n)) + (1-\alpha_n')(x_n - p + \gamma_n'(v_n - x_n))\| \ge c. \tag{2.9}$$

Similarly, we obtain

$$\begin{aligned}&\|\alpha_n'(T_2z_n - p + \gamma_n'(v_n - x_n)) + (1-\alpha_n')(x_n - p + \gamma_n'(v_n - x_n))\|\\ \le&\alpha_n'\|T_2z_n - p\| + (1-\alpha_n')\|x_n - p\| + \gamma_n'\|v_n - x_n\|\\ \le&\alpha_n'\|z_n - p\| + (1-\alpha_n')\|x_n - p\| + \gamma_n'\|v_n - x_n\|\\ \le&\alpha_n'(\|x_n - p\| + \gamma_n''r) + (1-\alpha_n')\|x_n - p\| + \gamma_n'\|v_n - x_n\|\\ \le&\|x_n - p\| + \gamma_n''r + \gamma_n'r.\end{aligned}$$

Therefore, we have

$$\overline{\lim}_{n\to\infty}\|\alpha_n'(T_2z_n - p + \gamma_n'(v_n - x_n)) + (1-\alpha_n')(x_n - p + \gamma_n'(v_n - x_n))\| \le c. \tag{2.10}$$

Formulas (2.9) and (2.10) yield

$$\lim_{n\to\infty}\|\alpha_n'(T_2z_n - p + \gamma_n'(v_n - x_n)) + (1-\alpha_n')(x_n - p + \gamma_n'(v_n - x_n))\| = c. \tag{2.11}$$

Notice that

$$\begin{aligned}\|T_2z_n - p + \gamma_n'(v_n - x_n)\| &\le \|T_2z_n - p\| + \gamma_n'\|v_n - x_n\|\\ &\le \|z_n - p\| + \gamma_n'r.\end{aligned}$$

Taking limsup on both the sides in the above inequality and using (2.4), we have

$$\overline{\lim}_{n\to\infty}\|T_2z_n - p + \gamma_n'(v_n - x_n)\| \le c. \tag{2.12}$$

The inequalities

$$\begin{aligned}\|x_n - p + \gamma_n'(v_n - x_n)\| &\le \|x_n - p\| + \gamma_n'\|v_n - x_n\|\\ &\le \|x_n - p\| + \gamma_n'r,\end{aligned}$$

yield

$$\overline{\lim}_{n\to\infty}\|x_n - p + \gamma_n'(v_n - x_n)\| \le c. \tag{2.13}$$

Applying Lemma 1.1, it follows from (2.11), (2.12) and (2.13) that

$$\lim_{n\to\infty} \|T_2 z_n - x_n\| = 0. \tag{2.14}$$

The inequalities

$$\begin{aligned}\|x_n - p\| &\le \|T_2 z_n - x_n\| + \|T_2 z_n - p\| \\ &\le \|T_2 z_n - x_n\| + \|z_n - p\|,\end{aligned}$$

yield

$$c \le \underline{\lim}_{n\to\infty}\|z_n - p\| \le \overline{\lim}_{n\to\infty}\|z_n - p\| \le c.$$

That is,

$$\lim_{n\to\infty} \|z_n - p\| = c. \tag{2.15}$$

Using the same method, we have

(2.16)

$$\lim_{n\to\infty} \|\alpha''_n(T_3 x_n - p + \gamma''_n(w_n - x_n)) + (1-\alpha''_n)(x_n - p + \gamma''_n(w_n - x_n)) - p\| = c.$$

The inequalities

$$\begin{aligned}\|T_3 x_n - p + \gamma''_n(w_n - x_n)\| &\le \|T_3 x_n - p\| + \gamma''_n\|w_n - x_n\| \\ &\le \|x_n - p\| + \gamma''_n r,\end{aligned}$$

yield

$$\limsup_{n\to\infty} \|T_3 x_n - p + \gamma''_n(w_n - x_n)\| \le c. \tag{2.17}$$

Again,

$$\begin{aligned}\|x_n - p + \gamma''_n(w_n - x_n)\| &\le \|x_n - p\| + \gamma''_n\|w_n - x_n\| \\ &\le \|x_n - p\| + \gamma''_n r.\end{aligned}$$

Therefore, we obtain

$$\overline{\lim}_{n\to\infty}\|x_n - p + \gamma''_n(w_n - x_n)\| \le c. \tag{2.18}$$

Formulas (2.16), (17) and (2.18) yield

$$\lim_{n\to\infty} \|T_3 x_n - x_n\| = 0. \tag{2.19}$$

Next we prove

$$\lim_{n\to\infty} \|T_2 x_n - x_n\| = \lim_{n\to\infty} \|T_1 x_n - x_n\| = 0.$$

We note

$$\begin{aligned}\|T_1 x_n - x_n\| &\le \|x_n - T_1 y_n\| + \|T_1 y_n - T_1 x_n\| \\ &\le \|x_n - T_1 y_n\| + \|y_n - x_n\| \\ &= \|x_n - T_1 y_n\| + \|P(\alpha'_n T_2 z_n + \beta'_n x_n + \gamma'_n v_n) - P x_n\| \\ &\le \|x_n - T_1 y_n\| + \|\alpha'_n T_2 z_n + \beta'_n x_n + \gamma'_n v_n - x_n\| \\ &\le \|x_n - T_1 y_n\| + \|T_2 z_n - x_n\| + \gamma'_n r.\end{aligned}$$

It follows from (2.8), (2.14) and $\sum_{n=0}^{\infty}\gamma'_n<\infty$ that

$$\|T_1x_n-x_n\|\to 0,\qquad \text{as } n\to\infty. \tag{2.20}$$

Similarly, we have

$$\begin{aligned}\|T_2x_n-x_n\| &\le \|x_n-T_2z_n\|+\|T_2z_n-T_2x_n\|\\ &\le \|x_n-T_2z_n\|+\|z_n-x_n\|\\ &= \|x_n-T_2z_n\|+\|P(\alpha''_nT_3x_n+\beta''_nx_n+\gamma''_nr)-Px_n\|\\ &\le \|x_n-T_2z_n\|+\|\alpha''_nT_3x_n+\beta''_nx_n+\gamma''_nr-x_n\|\\ &\le \|x_n-T_2z_n\|+\|T_3x_n-x_n\|+\gamma''_nr.\end{aligned}$$

It follows from (2.14), (2.18) and $\sum_{n=0}^{\infty}\gamma''_n<\infty$ that

$$\|T_2x_n-x_n\|\to 0,\qquad \text{as } n\to\infty.$$

That is,

$$\lim_{n\to\infty}\|T_3x_n-x_n\|=\lim_{n\to\infty}\|T_2x_n-x_n\|=\lim_{n\to\infty}\|T_1x_n-x_n\|=0.$$

This completes the proof of the theorem.

Theorem 2.1 Let K be a nonempty closed convex subset of a uniformly convex Banach space E satisfying *Opial's* condition. Suppose T_1, T_2 and $T_3 : K\to E$ be nonexpansive mappings. Let $\{x_n\}$ be defined by (1.3), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ be real sequences in [0, 1] such that $\alpha_n+\beta_n+\gamma_n=\alpha'_n+\beta'_n+\gamma'_n=\alpha''_n+\beta''_n+\gamma''_n=1$ and $\epsilon\le\alpha_n,\alpha'_n,\alpha''_n\le 1-\epsilon$ for all $n\in N$ and some $\epsilon>0$. Then $\{x_n\}$ converges weakly to some common fixed point of T_1, T_2 and T_3.

Proof. For any $p\in F(T_1)\cap F(T_2)\cap F(T_3)$, it follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n-p\|$ exists. We now prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Firstly, let p_1 and p_2 be weak limits of subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemmas 1.2 and 2.2, we know that $p\in F(T)$. Secondly, assume $p_1\ne p_2$, then by Opial's condition, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|x_n-p_1\| &= \lim_{k\to\infty}\|x_{n_k}-p_1\|<\lim_{k\to\infty}\|x_{n_k}-p_2\|=\lim_{j\to\infty}\|x_{n_j}-p_2\|\\ &<\lim_{k\to\infty}\|x_{n_k}-p_1\|=\lim_{n\to\infty}\|x_n-p_1\|,\end{aligned}$$

which is a contradiction, hence $p_1=p_2$. Then $\{x_n\}$ converges weakly to some common fixed point of T_1, T_2 and T_3. The proof is completed.

Next, we shall prove a strong convergence theorem.

Theorem 2.2 Let E be a uniformly convex Banach space and K a nonempty convex closed subset which is also a nonexpansive retract of E. Let T_1, T_2 and $T_3 : K\to E$ be nonexpansive mappings with $F(T)\ne\emptyset$, where $F(T)$ denotes the set of all common fixed points of T_1, T_2 and T_3. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$, $\{\alpha''_n\}$, $\{\beta''_n\}$ and $\{\gamma''_n\}$ be as taken in Lemma 2.1, starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.3). Suppose T_1, T_2 and T_3 satisfies condition (A′). Then x_n converges strongly to some common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. By Lemma 2.2,

$$\lim_{n\to\infty} \|T_3 x_n - x_n\| = \lim_{n\to\infty} \|T_2 x_n - x_n\| = \lim_{n\to\infty} \|T_1 x_n - x_n\| = 0$$

and (2.3) gives that

$$\inf_{p\in F} \|x_{n+1} - p\| \leq \inf_{p\in F} \|x_n - p\| + (\gamma''_n + \gamma'_n + \gamma_n)M.$$

That is,

$$d(x_{n+1}, F) \leq d(x_n, F) + (\gamma''_n + \gamma'_n + \gamma_n)M$$

gives that $\lim_{n\to\infty} d(x_n, F)$ exists by virtue of Lemma 1.3. Now by condition (A′), $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now we can take a subsequence $\{x_{n_j}\}$ of x_n and sequence $y_j \subset F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. Then following the arguments from [11], we get that $\{y_j\}$ is a Cauchy sequence in $F(T)$ and so it converges. Let $y_j \to y$. Since $F(T)$ is closed, therefore $y \in F(T)$ and then $x_{n_j} \to y$. As $\lim_{n\to\infty} \|x_n - p\|$ exists, $x_n \to y \in F(T)$ thereby completing the proof.

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