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UNSTEADY MHD OSCILLATING FLOW WITH GENERAL FREE STREAM VELOCITY

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Abstract. Unsteady magnetohydrodynamics viscous problem is solved by Laplace transform technique when fluid at lower plate is oscillating in time and fluid at infinity is assumed as general free stream velocity. Some special cases with their physical significance are also discussed and are compared with already known results.

Key words : Unsteady flow, MHD, Laplace transform, suction/blowing, velocity amplitude, oscillating flow.
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1. **Introduction**

Fluid Mechanics has fascinated many generations of scientists and engineers. Although many years of research have been devoted to the study of fluids of low molecular weight, which are well described by the Navier-Stokes equations [1, 2], many challenging problems in both theory and applications remain. The motion of any fluid is described by the equations of conservation of mass, momentum, and energy. Physically, the equation of continuity states that within a small fixed volume there can be no net rate of addition of mass. The equation of momentum describes that the mass-time-acceleration of a fluid element equals the sum of the pressure, viscous, and gravitational forces acting on the element, and the energy equation interprets that the temperature of a fluid element changes as it moves along with the fluid because of heat conduction and heat production by viscous heating [3, 4]. In general, these three governing equations are used to describe the flow behaviour but

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there are few situations under which the energy equation is not important. For example, in sufficiently slow flows, viscous heating is not essential.

Also in our analysis, we use the equations for incompressible Newtonian fluid under isothermal conditions. For such flow, the continuity and the momentum equations are enough to describe the flow behaviour. These are four partial differential equations for the four unknowns; pressure and three components of velocity. Extensive experimental testing has shown that these equations describe the incompressible flow of Newtonian fluids exactly. Analytical solutions are, not always easy to obtain but there are numerous flow situations for which the equations are simplified by making assumptions on the velocity field. In fact the equations are among the most challenging and extensively studied equations of mathematical physics. As a consequence we have available numerous treatises giving analytical solutions and solution procedures for Newtonian fluid mechanics [5, 6].

In most of the problems, the body forces, in the Navier-Stokes equations are neglected for simplicity and convenience. It is observed theoretically [7 − 9] and experimentally [10] that when magnetohydrodynamics (MHD) forces acts as the body forces in the flow field phenomena, it controls the boundary layers. Also MHD is the theory of the macroscopic interaction of electrically conducting fluids with a magnetic field and it acts perpendicular to the velocity field. It is significant applications in many engineering problems, geophysics and astronomy.

In the present work, we discuss the viscous or Newtonian problem of an unsteady MHD flow past an infinite oscillating porous plate at $y = 0$ with the velocity $U_0 e^{(\beta - i\omega)t}$ and at $y = \infty$ with the general free stream velocity $f_1(t)$. The Laplace transform technique has been used to find the exact solution of the problem. Graphs are also sketched for the special cases.

2. Formulation of the equations

According to Stokes the constitutive equation for the Newtonian fluid is:

$$\Upsilon_{ij} = -p\delta_{ij} + \Lambda D_{kk}\delta_{ij} + 2\mu D_{ij} \tag{1}$$

Where Λ is the bulk viscosity, μ is the dynamic viscosity, δ_{ij} is the Kroneker delta and

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{2}$$

is the strain tensor. Stoke assumed that $\Lambda = -\frac{2}{3}\mu$, so that pure volumetric change does not affect the stress ($tr\Upsilon_{ij}$ is independent of trD_{ij}). Furthermore, the terms $\Lambda D_{kk}\delta_{ij}$ can be absorbed in the pressure term. This leads to the familiar constitutive equation for Newtonian fluid:

$$\Upsilon_{ij} = -p\delta_{ij} + 2\mu D_{ij}. \tag{3}$$

Using the constitutive equation in the balance of linear momentum and considering only the fact that the fluid can undergo only isochoric motion, we obtain

$$\frac{\partial v_i}{\partial x_i} = 0, \tag{4}$$

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = \rho g_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j}, \tag{5}$$

where v_i, p, and ρ are velocity vector, pressure, and density respectively.

3. **Solution of the problem**

Let us assume the $x-$axis parallel to the oscillating porous plate and the $y-$axis perpendicular to it and also that the fluid initially is at rest. The velocity field thus becomes

$$v_i = \left[u\left(y,t\right), V_0, 0\right] \tag{6}$$

where $u(y,t)$ is the velocity of the fluid in the x-direction, $V_0 < 0$ is the suction velocity, and $V_0 > 0$ is the blowing velocity. For $t > 0$ the flat plate is moved periodically with the following velocity

$$u(0,t) = U(t) = U_0\ e^{(\beta - i\omega)t}, (\ \omega > 0,\ t > 0,\ \beta = \text{const.} \neq 0), \tag{7}$$

where ω is the oscillating frequency of the plate at $y = 0$. We also consider that the fluid is electrically conducting and there is transversal magnetic field in the flow region. It is also assumed that the magnetic field is perpendicular to the velocity field and that there is no imposed external electric field such that the magnetic Reynolds number is small [7, 8]. Therefore, the body forces are replaced by MHD forces and we obtain from Maxwell's equations [7, 8]

$$\mathbf{J} \times \mathbf{B} = -\sigma B_0^2 \mathbf{V}, \tag{8}$$

where σ is the electrical conductivity, $\mathbf{J}$ is the electric current density and $\mathbf{B}$ is the total magnetic field in which induced magnetic field is neglected compared to the applied external magnetic field [7, 8]. Inserting equations (6) and (8) into (5) we obtain the following partial differential equation

$$\frac{\partial^2 u}{\partial y \partial t} + V_0 \frac{\partial^2 u}{\partial y^2} - \nu \frac{\partial^3 u}{\partial y^3} + n \frac{\partial u}{\partial y} = 0. \tag{9}$$

Integrating with respect to y

$$\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial y} - \nu \frac{\partial^2 u}{\partial y^2} + nu = c\left(t\right). \tag{10}$$

where $c\left(t\right)$ is integration constant and

$$n = \frac{\sigma B_0^2}{\rho},\ \nu = \frac{\mu}{\rho} \tag{11}$$

are MHD parameter and kinematic viscosity respectively.

Suppose at $y \to \infty$, $u(y,t) \to f_1(t)$. This implies from (10) that

$$c\left(t\right) = \frac{\partial f_1}{\partial t} + nf_1\left(t\right). \tag{12}$$

Using the value of $c\left(t\right)$ from (12) in (10) we obtain

$$\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial y} = \frac{\partial f_1}{\partial t} - n\left(u - f_1\right) + \nu \frac{\partial^2 u}{\partial y^2}, \tag{13}$$

where $f_1(t)$ is the free stream velocity.

The boundary and initial conditions for the problem are given by

$$\begin{aligned} u\left(0,t\right) &= U_0 e^{(\beta - i\omega)t}, \\ u\left(\infty,t\right) &= f_1\left(t\right), \\ u\left(y,0\right) &= 0, \end{aligned} \tag{14}$$

where U_0 is the reference velocity.

Applying the Laplace transform to (13) and (14) and using initial condition, we obtain

$$\frac{d^2\overline{u}}{dy^2} - \frac{V_0}{\nu}\frac{d\overline{u}}{dy} - \frac{n+s}{\nu}\overline{u} = -\frac{n+s}{\nu}F_1\left(s\right), \tag{15}$$

$$\begin{aligned} \overline{u}\left(0,s\right) &= \frac{U_0}{s - (\beta - i\omega)}, \\ \overline{u}\left(\infty,s\right) &= F_1\left(s\right). \end{aligned} \tag{16}$$

The solution of (15), after applying the boundary conditions (16), is given by

$$\begin{aligned} \overline{u}\left(y,s\right) &= \frac{U_0}{s - (\beta - i\omega)} e^{\frac{1}{2\nu}\left[V_0 - \sqrt{V_0^2 + 4\nu(n+s)}\right]y} \\ &\quad + F_1\left(s\right)\left[1 - e^{\frac{1}{2\nu}\left[V_0 - \sqrt{V_0^2 + 4\nu(n+s)}\right]y}\right]. \end{aligned} \tag{17}$$

The Laplace inversion of (17) is obtained by applying convolution theorem and is given as

$$\begin{aligned} u(y,t) &= \frac{U_0}{2} e^{\frac{y}{2\nu}V_0+(\beta-i\omega)t} \qquad (18)\\ &\times \left[\begin{array}{c} e^{\frac{y}{2\nu}\sqrt{V_0^2+4\nu[n+(\beta-i\omega)]}} \\ \times erfc\left\{ \frac{y}{2\sqrt{\nu t}} + \sqrt{V_0^2+4\nu\left[n+(\beta-i\omega)\right]}\sqrt{\frac{t}{4\nu}} \right\} \\ +e^{-\frac{y}{2\nu}\sqrt{V_0^2+4\nu[n+(\beta-i\omega)]}} \\ \times erfc\left\{ \frac{y}{2\sqrt{\nu t}} - \sqrt{V_0^2+4\nu\left[n+(\beta-i\omega)\right]}\sqrt{\frac{t}{4\nu}} \right\} \end{array} \right] \\ &+f_1(t) - \frac{y}{4\sqrt{\pi}} e^{\frac{V_0}{2\nu}y} \int_0^{\frac{t}{a}} f_1(t-a\tau)\,\tau^{\frac{-3}{2}} e^{-b\tau-\frac{y^2}{16\tau}}\,d\tau, \end{aligned}$$

where

$$a=\frac{4}{\nu}, \quad b=\frac{V_0^2}{\nu^2}+\frac{4n}{\nu}.$$

We now discuss some specific cases in order to understand some physical aspects of the solution (18).

3.1. $f_1(t)=0$**.** which means that the general free stream velocity $f_1(t)$ vanishes far from the plate (i.e. $y\rightarrow\infty$). The solution in this case is given as

$$\begin{aligned} u(y,t) &= \frac{U_0}{2} e^{\frac{y}{2\nu}V_0+(\beta-i\omega)t} \qquad (19)\\ &\times \left[\begin{array}{c} e^{\frac{y}{2\nu}\sqrt{V_0^2+4\nu[n+(\beta-i\omega)]}} \\ \times erfc\left\{ \frac{y}{2\sqrt{\nu t}} + \sqrt{V_0^2+4\nu\left[n+(\beta-i\omega)\right]}\sqrt{\frac{t}{4\nu}} \right\} \\ +e^{-\frac{y}{2\nu}\sqrt{V_0^2+4\nu[n+(\beta-i\omega)]}} \\ \times erfc\left\{ \frac{y}{2\sqrt{\nu t}} - \sqrt{V_0^2+4\nu\left[n+(\beta-i\omega)\right]}\sqrt{\frac{t}{4\nu}} \right\} \end{array} \right]. \end{aligned}$$

In order to see the effects of suction/blowing, amplitude, and the MHD, we first non-dimensionalize (19) and then make some graphs so that we can have some physical phenomena. Introducing

$$\eta=\sqrt{\frac{\omega}{2v}}y,\ c=\frac{\beta}{\omega},\ d=\frac{V_0}{2\sqrt{v\omega}},\ \tau=\omega t,\ f=\frac{u}{U_0},\ \alpha=\frac{n}{\omega} \qquad (20)$$

into (19) we obtain the non-dimensional form as follows

$$f(\eta,\tau) = \frac{1}{2} e^{\sqrt{2} d\eta + (c-i)\tau} \times \left[\begin{array}{c} e^{\sqrt{2}\eta\sqrt{d^2+\alpha+c-i}} erfc \left\{ \frac{\eta}{\sqrt{2\tau}} + \sqrt{d^2+\alpha+c-i}\sqrt{\tau} \right\} \\ + e^{-\sqrt{2}\eta\sqrt{d^2+\alpha+c-i}} erfc \left\{ \frac{\eta}{\sqrt{2\tau}} - \sqrt{d^2+\alpha+c-i}\sqrt{\tau} \right\} \end{array} \right]. \tag{21}$$

In order to see the physical behaviour of equation (21) we have made graphs in Figs. 1-4.

Discussion on graphs Figs. $1-4$ are plotted for non-dimentional velocity f against non-dimentional variable η. In Figs. $1-4$ we have shown the effects of suction/blowing, acceleration/deceleration and MHD. In Fig. 1 the suction/blowing parameter $d = \frac{V_0}{2\sqrt{v\omega}}$ is given values $-2,-1,0,0.5,1$ whereas acceleration/deceleration parameter $c = \frac{\beta}{\omega}$ is fixed to zero and MHD parameter α is fixed at 0.1. It is observed from Fig. 1 that in the caseof suction $(d = -2,-1)$ the boundary layer thickness decreases and hence the velocity increases whereas in the case of blowing $(d = 0.5, 1)$ the boundary layer thickness becomeslarge as it is expected physically. The case $d = \alpha = 0$ refers to the classical viscous case. In Fig. 2 similar observations are obtained as in Fig.1 except with the difference that MHD parameter α is increased to 0.5 and it is noted that with this increase the boundary layer thickness is controlled i.e. it decreases with increase in α. In Fig. 3 the velocity amplitude is discussed for $(c = -0.6, -0.5, -0.4, 0, 0.5,\ d = 0,\ \alpha = 0.1)$. It is clear from Fig. 3 that the amplitude increases in the case of deceleration $(c = -0.6, -0.5, -0.4)$ and decreases for acceleration $(c = 0.5)$. Again the case $c = \alpha = 0$ gives the classical viscous case. Fig. 4 also shows the similar results as Fig. 3 only with the difference that the MHD parameter is increased to 0.2 and with this difference the boundarylayer thickness is decreased, and which is in agreement to the experimental results (see for instance, the refs. [7 − 10]).

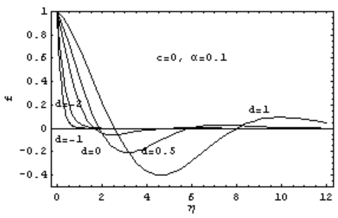

Fig 1.

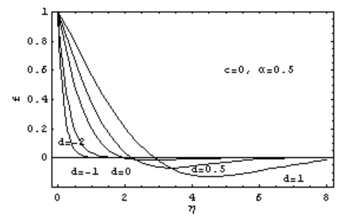

Fig 2

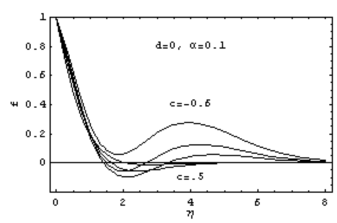

Fig 3.

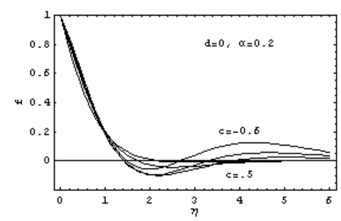

Fig 4.

3.2. $f_1(t) = 0$, $n = V_0 = \beta = 0$**.** Under these assumptions the solution reduces to the Stokes second problem. Because far from the plate when $t \to \infty$, we have

$$erfc\left\{\frac{y}{2\sqrt{\nu t}} + \sqrt{V_0^2 + 4\nu\left[n + (\beta - i\omega)\right]}\sqrt{\frac{t}{4\nu}}\right\} \to 0 \tag{22}$$

$$erfc\left\{\frac{y}{2\sqrt{\nu t}} - \sqrt{V_0^2 + 4\nu\left[n + (\beta - i\omega)\right]}\sqrt{\frac{t}{4\nu}}\right\} \to 2 \tag{23}$$

and we get the following solution [2]

$$u(y,t) = U_0 e^{-i\omega t} e^{-\frac{y}{2\nu}\sqrt{-i\omega}}. \tag{24}$$

The dimensionless form of (24) is given as

$$f_s(\eta, \tau) = e^{-i\tau} e^{\eta(1-i)}. \tag{25}$$

3.3. **Constant free stream velocity** $f_1(t) = U_0$**.** The solution when the free stream velocity is of magnitude U_0, is given by

$$\begin{aligned} u(y,t) &= \frac{U_0}{2} e^{\frac{y}{2\nu}V_0 + (\beta - i\omega)t} && (26) \\ &\times \left[\begin{array}{c} e^{\frac{y}{2\nu}\sqrt{V_0^2+4\nu[n+(\beta-i\omega)]}} \\ \times erfc\left\{ \frac{y}{2\sqrt{\nu t}} + \sqrt{V_0^2+4\nu\left[n+(\beta-i\omega)\right]}\sqrt{\frac{t}{4\nu}} \right\} \\ +e^{\frac{-y}{2\nu}\sqrt{V_0^2+4\nu[n+(\beta-i\omega)]}} \\ \times erfc\left\{ \frac{y}{2\sqrt{\nu t}} - \sqrt{V_0^2+4\nu\left[n+(\beta-i\omega)\right]}\sqrt{\frac{t}{4\nu}} \right\} \end{array} \right] \\ &+U_0\left[1 - \frac{y}{2\sqrt{\nu\pi t^3}} e^{\frac{y}{2\nu}V_0 - \left(\frac{V_0^2}{4\nu}+n\right)t - \frac{y^2}{4\nu t}}\right] \end{aligned}$$

The non-dimensional form of (23) is

$$\begin{aligned} f(\eta,\tau) &= \frac{1}{2} e^{\sqrt{2}d\eta + (c-i)\tau} && (27) \\ &\times \left[\begin{array}{c} e^{\sqrt{2}\eta\sqrt{d^2+\alpha+c-i}} erfc\left\{ \frac{\eta}{\sqrt{2\tau}} + \sqrt{d^2+\alpha+c-i}\sqrt{\tau} \right\} \\ +e^{-\sqrt{2}\eta\sqrt{d^2+\alpha+c-i}} erfc\left\{ \frac{\eta}{\sqrt{2\tau}} - \sqrt{d^2+\alpha+c-i}\sqrt{\tau} \right\} \end{array} \right] \\ &+1 - \frac{\eta}{\sqrt{2\pi\tau^3}} e^{\sqrt{2}d\eta - \left(d^2+\alpha\right)\tau - \eta^2/2\tau}. \end{aligned}$$

In order to see the physical behaviour of equation (27) we have plotted graphs in Figs. $5-8$ for non-dimentional velocity f against non-dimentional variable η.

The effects of suction/blowing, acceleration/deceleration and MHD are the same as in section 3.1, except that the range of the velocity is increased from 1 to 2 and that the velocity is strictly positive.

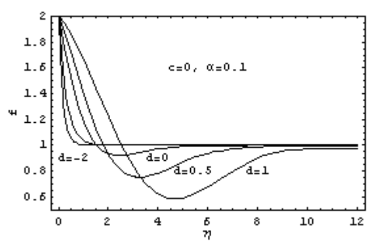

Fig 5

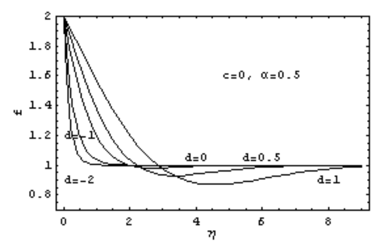

Fig 6

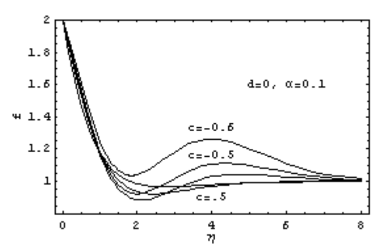

Fig 7

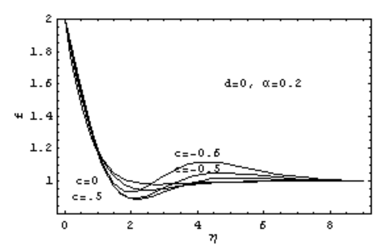

Fig 8

3.4. $f_1(t) = 0,\ \alpha = \frac{n}{\omega} = 0$**.** The non-dimensional solution in this case is given by

$$
\begin{aligned}
f(\eta,\tau) &= \frac{1}{2} e^{\sqrt{2} d\eta + (c-i)\tau} \\
&\quad \times \left[\begin{array}{c} e^{\sqrt{2}\eta\sqrt{d^2+c-i}} erfc \left\{ \frac{\eta}{\sqrt{2\tau}} + \sqrt{d^2+c-i}\sqrt{\tau} \right\} \\ + e^{-\sqrt{2}\eta\sqrt{d^2+c-i}} erfc \left\{ \frac{\eta}{\sqrt{2\tau}} - \sqrt{d^2+c-i}\sqrt{\tau} \right\} \end{array} \right].
\end{aligned} \tag{28}
$$

The solution (25) is in agreement to the solution given in [11].

4. **Conclusion**

We have solved unsteady viscous problem when plate at $y = 0$ is oscillating in time with amplitude β and plate at $y = \infty$ is general free stream velocity $f_1(t)$. A uniform magnetic field is applied perpendicular to the velocity field. The general free stream velocity is discussed in different situations and their results are explained with physical implications.

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