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UPPER BOUNDS FOR THE SIZE RAMSEY NUMBERS FOR P_3 VERSUS C_3^t OR P_n

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Abstract. In this paper, we derive an upper bound for the size Ramsey number for a path P_3 versus a friendship graph C_3^t. Furthermore, some minimal Ramsey graph for a combination (P_3, C_3^t) is presented. We also give an upper bound of the size Ramsey number for P_3 versus P_n.

Key words : Size Ramsey number, path, friendship graph.
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1. Introduction

All graphs considered here are finite, simple, and undirected. The graph G has a vertex set $V(G)$ and edge set $E(G)$. For a general reference for graph theoretic notions, see [8]. A 2-coloring of a graph F always means a 2-coloring on the edges of F with red and blue. A (G, H)-coloring of a graph F is a 2-coloring of F such that F contains neither a red copy of G nor a blue copy of H.

For any pair of graphs G and H, notation $F \rightarrow (G, H)$ means that in any 2-coloring of F there exists a monochromatic G or H in F. Let $\mathcal{R}(G, H)$ be the set of all graphs F satisfying $F \rightarrow (G, H)$ and $F \setminus \{e\} \not\rightarrow (G, H)$ for any edge $e \in E(F)$. Each graph $F \in \mathcal{R}(G, H)$ is called a *minimal Ramsey graph* for a combination of G and H. The *size Ramsey number* $\hat{r}(G, H)$ is defined as $\min\{E(F) \mid F \rightarrow (G, H)\}$. The size Ramsey number was first introduced and studied by P. Erdös, R. J. Faudree, C. C. Rousseau in 1978 [5]. They proved that the size Ramsey number $\hat{r}(K_n, K_m) = \binom{r(K_n, K_m)}{2}$ and

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determined the size Ramsey numbers for stars and for the products of two graphs. The results on size Ramsey number for paths, trees and cycles can be found in [1, 2, 9, 10]. Additionally, the size Ramsey number for graph pairs with one is either a matching or a star, respectively can be found in [3, 4, 6, 11].

Let P_n denote a path of n vertices and C_3^t denote a friendship graph, i.e., a graph obtained by connecting a vertex c (called a *hub*) to all vertices of tK_2. In this paper, we derive an upper bound for the size Ramsey number for a path P_3 on three vertices versus a friendship graph C_3^t. Then, we show that W_{3t+1} is in $\mathcal{R}(P_3, C_3^t)$. We also give an upper bound of the size Ramsey number for P_3 versus P_n. Formally, we will prove the following theorems.

Theorem 1. *For all* $t \geq 1$, $\hat{r}(P_3, C_3^t) \leq 6t + 2$.

Theorem 2. *For all* $t \geq 1$, W_{3t+1} *is in* $\mathcal{R}(P_3, C_3^t)$.

Theorem 3. *For all* $n \geq 3$, $\hat{r}(P_3, P_n) \leq 2n - 1$.

2. The Proof of Theorem 1

In this section, we show that the size Ramsey number $\hat{r}(P_3, C_3^t) \leq 6t + 2$. To do so, consider any 2-labeling on the edges of a wheel W_{3t+1} of $6t + 2$ edges as follows.

Let W_{3t+1} be a wheel of $3t + 1$ spokes, with

$$V(W_{3t+1}) = \{c\} \cup \{v_i | 1 \leq i \leq 3t + 1\}$$

and

$$E(W_{3t+1}) = E_1 \cup E_2$$

where $E_1 = \{cv_i | 1 \leq i \leq 3t + 1\}$ and $E_2 = \{v_i v_{i+1} | 1 \leq i \leq 3t\} \cup \{v_{3t+1} v_1\}$.

Let χ be any 2-coloring of W_{3t+1} such that there is no a red P_3. We shall show that χ induces a blue C_3^t. To do so, let us consider the following two cases.

Case 1. There exists one red edge in E_1.
Without loss of generality, let cv_1 be a red edge in E_1. Consider a subgraph $W_{3t+1} - v_1 \cong \{c\} + P_{3t}$. Since under bi-coloring χ, W_{3t+1} contains no red P_3, the red edges in E_2 (if any) are independent and there are at most $\lfloor \frac{3t}{2} \rfloor$ of this color. Therefore, we have at least t blue edges in E_2. These edges together with c forms a blue C_3^t in W_{3t+1}.

Case 2. No red edges in E_1.
By similar argument in Case 1, we have at most $\lfloor \frac{3t+1}{2} \rfloor$ red edges and other

edges must be blue. It is easy to verify that these blue edges together with c forms a blue C_3^t in W_{3t+1}. As a consequence of these two cases, we have $\hat{r}(P_3, C_3^t) \leq 6t + 2,$ for all $t \geq 1$. □

3. The Proof of Theorem 2

Now, to prove Theorem 2 we have to show that for any fixed edge e there exists a 2-labeling on $F \cong W_{3t+1} - e$ such that F contains neither a red P_3 nor a blue C_3^t.

Let us consider the graph $F \cong W_{3t+1} - e$ for any fixed edge $e \in E(W_{3t+1})$. Before the removal the edge e, label the vertices and edges of W_{3t+1} as in the proof of Theorem 1. Consider the following two cases.

Case 1. $e \in E_1$.
Without loss of generality, let $e = cv_1$. If $t = 1$, then color cv_3 by red and other edges by blue. Therefore, $W_4 \setminus cv_1$ has no red P_3 neither blue C_3^1. If $t \geq 2$, let

$$E_3 = \{v_{3t+1}v_1, v_1v_2, v_3v_4, v_4v_5, v_{3t-1}v_{3t}\} \cup (E_1 \setminus cv_4),$$
$$E_4 = \{cv_4, v_{3t}v_{3t+1}, v_2v_3\},$$

and

$$E_5 = E(F) \setminus (E_3 \cup E_4).$$

Color the edges of E_3 by blue and E_4 by red. Color the edges of E_5 alternatingly two blue and one red (in some order). Under this 2-coloring F contains neither a red P_3 nor a blue C_3^t.

Case 2. $e \in E_2$.
Again, without loss of generality, let $e = v_1v_2$. In this case, F is a fan $\{c\} + P_{3t+1}$. Let

$$E_6 = \{v_{3t+1}v_1, cv_3\},$$
$$E_7 = \{v_2v_3, v_3v_4\} \cup \{E_1 \setminus cv_3\},$$

and

$$E_8 = E(F) \setminus (E_6 \cup E_7).$$

Color the edges of E_6 by red and E_7 by blue. Color the edges of E_8 alternatingly two blue and one red starting from the edge $v_{3t+1}v_{3t}$. Consequently, F contains neither a red P_3 nor a blue C_3^t. □

The next result presents an upper bound of the size Ramsey number for a combination of P_3 and P_n, where $n \geq 3$.

4. **The Proof of Theorem 3**

To prove Theorem 3, consider the two graphs drawn in Figure 1: (a) for the case of even n and (b) for the case of odd n.

Let $k = \lfloor \frac{n}{2} \rfloor$. Denote the graph by G, where $|V(G)| = n + 1$, and $|E(G)| = 2n - 1$, with

$V(G) = \{u_i |\ i = 0, \ldots, k\} \cup \{v_i |\ i = 0, \ldots, s\}$, $s = k - 1$ for even n or k for odd n.

$E(G) = E_1 \cup E_2 \cup E_3 \cup E_4$, where

$E_1 = \{m_i = v_i u_{i+1} |\ i = 0, \ldots, k - 1\}$,

$E_2 = \{d_i = u_i v_i |\ i = 0, \ldots, s\}$, $s = k - 1$ for even n or k for odd n.

$E_3 = \{l_i = u_i u_{i+1} |\ i = 0, \ldots, k - 1\}$,

$E_4 = \{r_i = v_i v_{i+1} |\ i = 0, \ldots, k - 2\}$.

Define $L_i = \{u_i, v_i\}$, $i = 0, \ldots, k - 1$, and $L_k = \{u_k\}$.

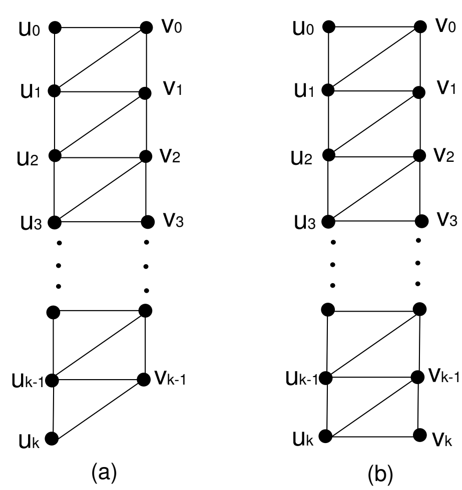

FIGURE 1. The graph G: (a) case for even n, and (b) case for odd n

Let χ be *any* 2-coloring of G. Suppose G contains no red P_3. We will show that the coloring χ creates a blue P_n in G. To do so, we use the following algorithm.

Algoritma 1. *Constructing a blue path P_n in G*

(1) Input n

(2) $P := (u_0)$
(3) $i := 0, x := 0$
(4) **while** $i \leq k-1$ **do**
  if edge d_i is blue
  then extend the path P by attaching $d_i = \{u_i v_i\}$, namely
    $P := P \cup \{u_i v_i\}$
  else if $i = k-1$ and n is even
    then $u_{k-1}u_k$ and $u_k v_{k-1}$ must be blue. Then, extend path P by involving these two edges. The resulting path is $P := P \cup \{u_{k-1}u_k\} \cup \{u_k v_{k-1}\}$, $i := i+1$, return to step 3.
    else $u_i u_{i+1}$, $u_{i+1}v_i$, and $v_i v_{i+1}$ must be blue. Then, extend path P by involving these edges. The resulting path is $P := P \cup \{u_i u_{i+1}\} \cup \{u_{i+1} v_i\} \cup \{v_i v_{i+1}\}$, and $i := i+1$.

  if m_i is blue
  then extend path P by involving edge m_i. The resulting path is
    $P := P \cup \{v_i u_{i+1}\}$, and $i := i+1$.
  else if $i = k-1$
    then if n is even
      then $i := i+1$, return to step 3.
      else $v_{k-1}v_k$ and $v_k u_k$ must be blue. Then, extend path P by adding the two edges. The resulting path is $P := P \cup \{v_{k-1}v_k\} \cup \{v_k u_k\}$, $i := i+1$, and $x := 1$.
    else $v_i v_{i+1}$, $v_{i+1}u_{i+1}$, and $u_{i+1}u_{i+2}$ are blue. Then, extend path P by involving these edges. The resulting path is $P := P \cup \{v_i v_{i+1}\} \cup \{v_{i+1} u_{i+1}\} \cup \{u_{i+1} u_{i+2}\}$, and $i := i+2$.
(5) **if** d_k is blue and $x := 0$
  then extend path P by involving d_k. The resulting path is
    $P := P \cup \{u_k v_k\}$.
(6) **End.**

The resulting path P of the above algorithm will contain exactly two vertices in each $L_i = \{u_i, v_i\}$, $i = 0, \ldots, k-1$ and additionally for odd n it has at least one vertex in L_k. So, the path P has length of at least n. Hence,

$\hat{r}(P_3, P_n) \leq 2n - 1$. □

5. Open problems and conjectures

To conclude this paper, let us state the following conjecture and give some open problems to work on.

Conjecture 1. *For all* $t \geq 1$, $\hat{r}(P_3, C_3^t) = 6t + 2$.

Open problem 1. *For* $n \geq 3$, *find a better upper bound for the size Ramsey number* $\hat{r}(P_3, P_n)$. *Find the size Ramsey number* $\hat{r}(P_m, P_n)$, *in general.*

For small cases, we are able to show that $\hat{r}(P_3, P_k) = 3, 5, 6, 8, 10, 12, 13, 16, 16, 19$ for $k = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ respectively. The proofs refer to [12].

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