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VERTEX-MAGIC TOTAL LABELINGS OF DISCONNECTED GRAPHS

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Abstract. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$ and let $e = |E(G)|$ and $v = |V(G)|$. A one-to-one map λ from $V \cup E$ onto the integers $\{1, 2, ..., v+e\}$ is called *vertex magic total labeling* if there is a constant k so that for every vertex x,

$$\lambda(x) \ + \ \sum \lambda(xy) \ = \ k$$

where the sum is over all vertices y adjacent to x. Let us call the sum of labels at vertex x the *weight* $w_\lambda(x)$ of the vertex under labeling λ; we require $w_\lambda(x) = k$ for all x. The constant k is called the *magic constant* for λ.

In this paper, we present the vertex magic total labelings of disconnected graph, in particular, two copies of isomorphic generalized Petersen graphs $2P(n, m)$, disjoint union of two non-isomorphic suns $S_m \cup S_n$ and t copies of isomorphic suns tS_n.

Key words : Vertex magic total labeling, disconnected graph, generalized Petersen graph, sun.
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1. Introduction

In this paper all graphs are finite, simple and undirected. The graph G has vertex set $V = V(G)$ and edge set $E = E(G)$ and we let $e = |E(G)|$ and $v = |V(G)|$. A general reference for graph theoretic notions is [10].

MacDougall *et al.* [6] introduced the notion of a *vertex-magic total labeling*. This is an assignment of the integers from 1 to $v+e$ to the vertices and edges of G so that at each vertex the vertex label and the labels on the edges incident at that vertex add to a fixed constant. More formally, a one-to-one map λ

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from $V \cup E$ onto the integers $\{1, 2, ..., v+e\}$ is a *vertex-magic total labeling* if there is a constant k so that for every vertex x,

$$\lambda(x) \ + \ \sum \lambda(xy) \ = \ k \tag{1}$$

where the sum is over all vertices y adjacent to x. Let us call the sum of labels at vertex x the *weight* of the vertex; we require $w(x) = k$ for all x. The constant k is called the *magic constant* for λ.

If a regular graph G possesses a vertex-magic total labeling, we can create a new labeling λ' from λ by setting

$$\lambda'(x) \ = \ v + e + 1 - \lambda(x)$$

for every vertex x, and

$$\lambda'(xy) \ = \ v + e + 1 - \lambda(xy)$$

for every edge xy. Clearly, λ' is also a one-to-one map from the set $V \cup E$ to $\{1, 2, ..., v+e\}$ and we will call λ' the *dual* of λ. If r is the degree of each vertex of G, then

$$k' \ = \ (r+1)(v+e+1) - k \tag{2}$$

is the new magic constant.

Since the introduction of this notion, there have been several results on vertex magic total labeling of particular classes of graphs. For example, MacDougall *et al.* [6] proved that cycle C_n for $n \geq 3$, path P_n for $n \geq 2$, complete graph K_n for odd n, complete bipartite graph $K_{n,n}$ for $n > 1$, have vertex magic total labelings. Bača, Miller and Slamin [1] proved that for $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, every generalized Petersen graph $P(n, m)$ has a vertex-magic total labeling with the magic constant $k = 9n + 2$, $k = 10n + 2$, and $k = 11n + 2$. The complete survey of the known results on vertex magic total labeling of graphs can be found in [4].

Most of the known results are concerning on vertex magic total labeling of connected graphs. For the case of disconnected graph, Wallis [8] proved the following theorem.

Theorem 1. *Suppose G is regular graph of degree r which has a vertex magic total labeling.*

(i) *If r is even, then tG is vertex magic whenever t is an odd positive integer.*
(ii) *If r is odd, then tG is vertex magic for every positive integer t.* □

The result above concerns on vertex-magic total labeling of disconnected graph whose components are regular and isomorphic graphs. For the case of disconnected graph whose components are not regular graphs, Gray *et al.* [5] proved that t copies of stars on 3 vertices $tK_{1,2}$ has a vertex magic total labeling. In this paper, we present the vertex magic total labelings of two copies of isomorphic generalized Petersen graphs $2P(n, m)$, disjoint union of two non-isomorphic suns $S_m \cup S_n$ as well as t copies of isomorphic suns tS_n.

The *generalized Petersen graph* $P(n, m)$, $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, consists of an outer n-cycle $u_0u_1...u_{n-1}$, a set of n spokes u_iv_i, $0 \leq i \leq n-1$, and n inner edges v_iv_{i+m} with indices taken modulo n. The standard Petersen graph is the instance $P(5, 2)$. Generalized Petersen graphs were first defined by Watkins [9]. The t copies of generalized Petersen graphs, denoted by $tP(n, m)$, has vertex set $V(tP(n, m)) = \{u_i^j \mid 0 \leq i \leq n-1,\ 1 \leq j \leq t\} \cup \{v_i^j \mid 0 \leq i \leq n-1,\ 1 \leq j \leq t\}$ and edge set $E(tP(n, m)) = \{u_i^j v_i^j \mid 0 \leq i \leq n-1,\ 1 \leq j \leq t\} \cup \{v_i^j v_{i+m}^j \mid 0 \leq i \leq n-1,\ 1 \leq j \leq t\}$.

A sun S_n is a cycle C_n with an edge terminating in a vertex of degree 1 attached to each vertex. The sun S_n consists of vertex set $V(S_n) = \{v_i \mid 1 \leq i \leq n\} \cup \{a_i \mid 1 \leq i \leq n\}$ and edge set $E(S_n) = \{v_iv_{i+1} \mid 1 \leq i \leq n\} \cup \{v_ia_i \mid 1 \leq i \leq n\}$. We note that if $i = n$ then $i + 1 = 1$. The t copies of suns, denoted by tS_n, has vertex set $V(tS_n) = \{v_i^j \mid 1 \leq i \leq n,\ 1 \leq j \leq t\} \cup \{a_i^j \mid 1 \leq i \leq n,\ 1 \leq j \leq t\}$ and edge set $E(tS_n) = \{v_i^j v_{i+1}^j \mid 1 \leq i \leq n,\ 1 \leq j \leq t\} \cup \{v_i^j a_i^j \mid 1 \leq i \leq n,\ 1 \leq j \leq t\}$. Thus tS_n has $2nt$ vertices and $2nt$ edges.

2. Main Results

We start this section with a construction of vertex magic total labeling of 2 copies of generalized Petersen graphs as a consequence of the Theorem 1 as stated in the previous section.

Corollary 1. *For $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, the 2 copies generalized Petersen graphs $2P(n, m)$ has a vertex-magic total labeling with the magic constant $k = 19n + 2$.*

Proof. Label the first graph of $2P(n, m)$ according to the labeling of $P(n, m)$ given by Baca, Miller and Slamin [1]. Continue to label the second graph of $2P(n, m)$ by adding each label with $5n$ (the labels are $\{5n+1, 5n+2, ..., 10n\}$). Interchange the correspondence edge labels of the outer cycle as well as the inner cycle in the first graph with the ones in the second graph of $2P(n, m)$.

This way gives the labeling of $2P(n,m)$ as described in the following formula.

$$\begin{aligned}
\lambda(u_i^j) &= (4n+1)\alpha(i,0)+(5n+1-i)\alpha(1,i)+(j-1)5n \\
\lambda(v_i^j) &= (2n+m-i)\alpha(i,m-1)+(3n+m-i)\alpha(m,i) \\
&\quad +(j-1)5n \\
\lambda(u_i^j u_{i+1}^j) &= 1+i+(2-j)5n \\
\lambda(u_i^j v_i^j) &= 4n-i+(j-1)5n \\
\lambda(v_i^j v_{i+m}^j) &= n+1+i+(2-j)5n
\end{aligned}$$

for $i \in Z_n$ and $j = 1, 2$, where

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{if } x > y. \end{cases}$$

It is simple to verify that the labeling λ is a bijection from the set $V(2P(n,m)) \cup E(2P(n,m))$ onto the set $\{1, 2, ..., 10n\}$.

Let us denote the weights (under labeling λ) of vertices u_i^j of $2P(n,m)$ by

$$w_\lambda(u_i^j) = \lambda(u_i^j) + \lambda(u_i^j u_{i+1}^j) + \lambda(u_i^j v_i^j) + \lambda(u_{i-1}^j u_i^j)$$

and the weights of vertices v_i^j by

$$w_\lambda(v_i^j) = \lambda(v_i^j) + \lambda(v_i^j v_{i+m}^j) + \lambda(u_i^j v_i^j) + \lambda(v_{n+i-m}^j v_i^j)$$

for $j = 1, 2$ and $i \in Z_n$, where all indices are taken modulo n.

It is clearly true that $w_\lambda(u_i^j) = 19n + 2$ and $w_\lambda(v_i^j) = 19n + 2$ for all $j = 1, 2$ and $i \in Z_n$. Consequently, λ is the vertex-magic total labeling of $2P(n,m)$ with the magic constant $k = 19n + 2$. □

Figure 1 shows an example of vertex-magic total labeling of 2 copies of generalized Petersen graphs $2P(6,2)$ with the magic constant $k = 116$.

By duality, we have the following corollary.

Corollary 2. *For $n \geq 3$, $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$, the 2 copies of generalized Petersen graphs $2P(n,m)$ has a vertex-magic total labeling with the magic constant $k = 21n + 2$.* □

The results above concern on vertex-magic total labeling of disconnected graph whose components are regular and isomorphic graphs. The following theorem presents vertex-magic total labeling of disconnected graph whose components are non-regular and non-isomorphic graphs.

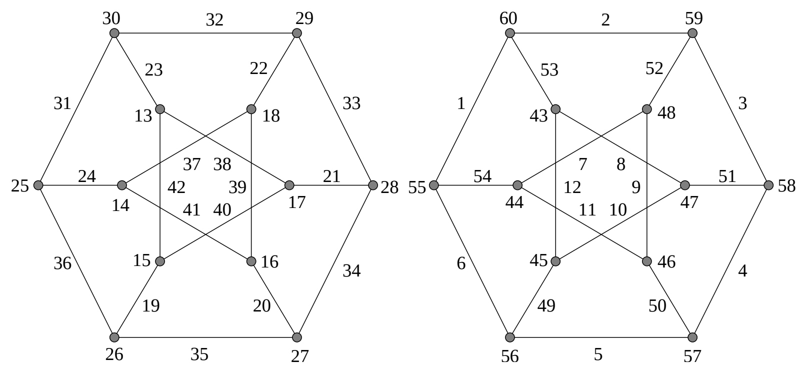

FIGURE 1. Vertex-magic total labeling of $2P(6,2)$ with $k = 116$

Theorem 2. *For $m \geq 3$ and $n \geq 3$, the disjoint union of two suns $S_m \cup S_n$ has a vertex-magic total labeling with the magic constant $k = 6(m+n)+1$.*

Proof. For any positive integer $m, n \geq 3$, label the vertices and edges of the first sun S_m in the following way:

$$\begin{aligned}
\lambda(v_i^1) &= 2i && \text{for } i = 1, 2, ..., m \\
\lambda(v_i^1 v_{i+1}^1) &= \begin{cases} 2(m+n-i)+1 & \text{for } i = 1, 2, ..., m-1 \\ 2n+1 & \text{for } i = m \end{cases} \\
\lambda(a_i^1) &= \begin{cases} 2(m+2n+1) & \text{for } i = 1 \\ 4(m+n+1)-2i & \text{for } i = 2, 3, ..., m \end{cases} \\
\lambda(v_i^1 a_i^1) &= \begin{cases} 2(2m+n)-1 & \text{for } i = 1 \\ 2(m+n+i)-3 & \text{for } i = 2, 3, ..., m \end{cases}
\end{aligned}$$

Then label the vertices and edges of the second sun S_n for $n \geq 3$ in the following way:

$$\begin{aligned}
\lambda(v_j^2) &= 2(m+j) && \text{for } j = 1, 2, ..., n \\
\lambda(v_j^2 v_{j+1}^2) &= \begin{cases} 2(n-j)+1 & \text{for } j = 1, 2, ..., n-1 \\ 1 & \text{for } j = n \end{cases} \\
\lambda(a_j^2) &= \begin{cases} 2(m+n+1) & \text{for } j = 1 \\ 2(m+2n-j+2) & \text{for } j = 2, 3, ..., n \end{cases} \\
\lambda(v_j^2 a_j^2) &= \begin{cases} 4(m+n)-1 & \text{for } j = 1 \\ 2(2m+n+j)-3 & \text{for } j = 2, 3, ..., n \end{cases}
\end{aligned}$$

It is easy to verify that the labeling λ is a bijection from the set $V(S_m \cup S_n) \cup E(S_m \cup S_n)$ onto the set $\{1, 2, ..., 4(m+n)\}$.

Let us denote the weights (under labeling λ) of vertices v_i^1 of $S_m \cup S_n$ by

$$w_\lambda(v_i^1) = \lambda(v_i^1) + \lambda(v_i^1 v_{i+1}^1) + \lambda(v_i^1 a_i^1) + \lambda(v_{i-1}^1 v_i^1)$$

the weights of vertices a_i^1 by

$$w_\lambda(a_i^1) = \lambda(a_i^1) + \lambda(v_i^1 a_i^1)$$

the weights of vertices v_j^2 by

$$w_\lambda(v_j^2) = \lambda(v_j^2) + \lambda(v_j^1 v_{j+1}^2) + \lambda(v_j^2 a_j^2) + \lambda(v_{j-1}^2 v_j^2)$$

and the weights of vertices a_j^2 by

$$w_\lambda(a_j^2) = \lambda(a_j^2) + \lambda(v_j^2 a_j^2)$$

It is clearly true that $w_\lambda(v_i^1) = 6(m+n)+1$, $w_\lambda(a_i^1) = 6(m+n)+1$, $w_\lambda(v_j^2) = 6(m+n)+1$ and $w_\lambda(a_j^2) = 6(m+n)+1$, for all $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. Thus λ is the vertex-magic total labeling of $S_m \cup S_n$ for any positive integer $m, n \geq 3$ with the magic constant $k = 6(m+n) + 1$. □

Figure 2 shows an example of vertex-magic total labeling of $S_4 \cup S_6$ with the magic constant $k = 61$.

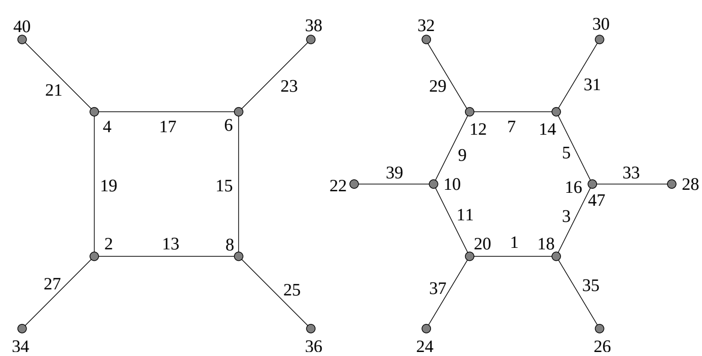

Figure 2. Vertex-magic total labeling of $S_4 \cup S_6$ with $k = 61$

We note that if $m = n$, then S_m is isomorphic to S_n. In this case Theorem 2 shows the vertex magic total labeling of 2 copies of isomorphic suns. General result on the vertex magic total labeling of t copies of isomorphic suns, for any integer $t \geq 1$, is given below.

Theorem 3. *For $n \geq 3$ and $t \geq 1$, the t copies of sun tS_n has a vertex-magic total labeling with the magic constant $k = 6nt + 1$.*

Proof. For all $j = 1, 2, ..., t$, label vertices and edges of tS_n in the following way:

$$\begin{aligned}
\lambda(v_i^j) &= 2n(j-1) + 2i && \text{for } i = 1, 2, ..., n \\
\lambda(v_i^j v_{i+1}^j) &= \begin{cases} 2nt - 2n(j-1) - 2i + 1 & \text{for } i = 1, 2, ..., n-1 \\ 2nt - 2nj + 1 & \text{for } i = n \end{cases} \\
\lambda(a_i^j) &= \begin{cases} 4nt - 2n(j-1) - 2(n-1) & \text{for } i = 1 \\ 4nt - 2n(j-1) - 2(i-2) & \text{for } i = 2, 3, ..., n \end{cases} \\
\lambda(v_i^j a_i^j) &= \begin{cases} 2nt + 2nj - 1 & \text{for } i = 1 \\ 2nt + 2n(j-1) + 2i - 3 & \text{for } i = 2, 3, ..., n \end{cases}
\end{aligned}$$

It is easy to verify that the labeling λ is a bijection from the set $V(tS_n) \cup E(tS_n)$ onto the set $\{1, 2, ..., 4nt\}$.

Let us denote the weights (under labeling λ) of vertices v_i^j of tS_n by

$$w_\lambda(v_i^j) = \lambda(v_i^j) + \lambda(v_i^j v_{i+1}^j) + \lambda(v_i^j a_i^j) + \lambda(v_{i-1}^j v_i^j)$$

and the weights of vertices a_i^j by

$$w_\lambda(a_i^j) = \lambda(a_i^j) + \lambda(v_i^j a_i^j)$$

Then, for all $j = 1, 2, ..., t$, the weights of vertices v_i^j can be determined as follows:

- for $i = 1$

$$\begin{aligned}
w_\lambda(v_1^j) &= (2n(j-1) + 2) + (2nt - 2n(j-1) - 2 + 1) + \\
&\quad (2nt + 2nj - 1) + (2nt - 2nj + 1) \\
&= (2nj - 2n + 2) + (2nt - 2nj + 2n - 1) + \\
&\quad (2nt + 2nj - 1) + (2nt - 2nj + 1) \\
&= 6nt + 1.
\end{aligned}$$

- for $i = 2, 3, ..., n-1$

$$\begin{aligned}
w_\lambda(v_i^j) &= (2n(j-1) + 2i) + (2nt - 2n(j-1) - 2i + 1) + \\
&\quad (2nt + 2n(j-1) + 2i - 3) + (2nt - 2n(j-1) - 2(i-1) + 1) \\
&= (2nj - 2n + 2i) + (2nt - 2nj + 2n - 2i + 1) + \\
&\quad (2nt + 2nj - 2n + 2i - 3) + (2nt - 2nj + 2n - 2i + 2 + 1) \\
&= 6nt + 1.
\end{aligned}$$

• for $i = n$

$$\begin{aligned} w_\lambda(v_n^j) &= (2n(j-1)+2n)+(2nt-2nj+1)+ \\ &\quad (2nt+2n(j-1)+2n-3)+(2nt-2n(j-1)-2(n-1)+1) \\ &= (2nj-2n+2n)+(2nt-2nj+1)+ \\ &\quad (2nt+2nj-2n+2n-3)+(2nt-2nj+2n-2n+2+1) \\ &= 6nt+1. \end{aligned}$$

and the weights of vertices a_i^j can be determined as follows:

• for $i = 1$

$$\begin{aligned} w_\lambda(a_1^j) &= (4nt-2n(j-1)-2(n-1))+(2nt+2nj-1) \\ &= (4nt-2nj+2n-2n+2)+(2nt+2nj-1) \\ &= 6nt+1. \end{aligned}$$

• for $i = 2, 3, ..., n$

$$\begin{aligned} w_\lambda(a_i^j) &= (4nt-2n(j-1)-2(i-2))+(2nt+2n(j-1)+2i-3) \\ &= (4nt-2nj+2n-2i+4)+(2nt+2nj-2n+2i-3) \\ &= 6nt+1. \end{aligned}$$

Since $w_\lambda(v_i^j) = 6nt+1$ and $w_\lambda(a_i^j) = 6nt+1$ for all $i = 1, 2, ..., n$ and $j = 1, 2, ..., t$, then λ is the vertex-magic total labeling of tS_n with the magic constant $k = 6nt+1$. □

Figure 3 shows an example of vertex-magic total labeling of 3 copies of S_5 with the magic constant $k = 91$.

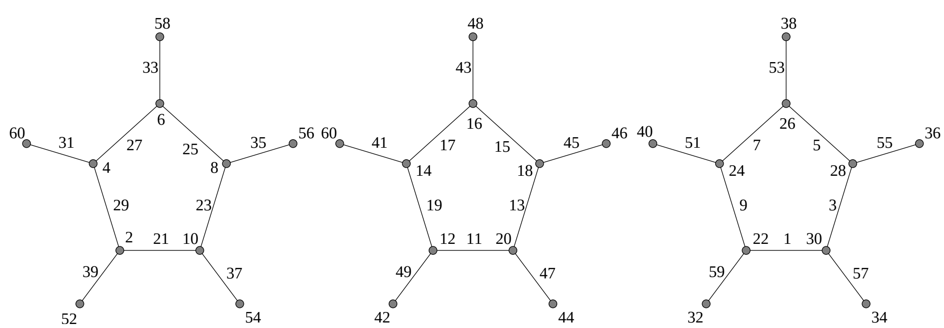

FIGURE 3. Vertex-magic total labeling of $3S_5$ with $k = 91$

Unlike labeling of a regular graph which always has dual, the vertex magic total labeling of t copies of suns tS_n does not have dual as tS_n is not regular.

3. Conclusion

We conclude this paper with a conjecture and an open problem for the direction of further research in this area.

Corollary 1 and 2 provide construction of vertex-magic total labeling of disjoint union of two isomorphic generalized Petersen graphs. For the case of non-isomorphic, it is not known. So we pose the following open problem.

Open problem 1. *Find a vertex-magic total labeling of the disjoint union of two non-isomorphic generalized Petersen graphs*

Theorem 3 gives the vertex magic total labeling of t copies of suns tS_n where all components are isomorphic. For the case where the components are not isomorphic, the labeling is given in Theorem 2. However, the labeling holds only for 2 non-isomorphic suns. It is possible to extend the labeling to the disjoint union of t non-isomorphic suns for any integer $t \geq 3$. The labeling of $S_4 \cup S_5 \cup S_6$ shown in Figure 4 is an evidence that such labeling exists. Thus we present a conjecture as follows.

Conjecture 1. *There is a vertex-magic total labeling of the disjoint union of t non-isomorphic suns, for any positive integer $t \geq 3$.*

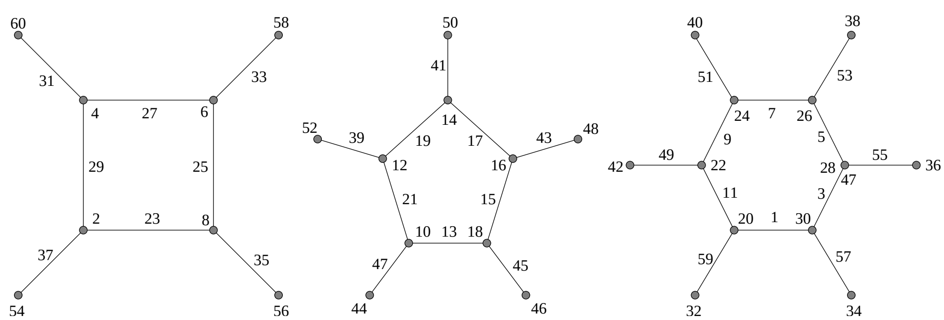

FIGURE 4. Vertex-magic total labeling of $S_4 \cup S_5 \cup S_6$ with $k = 91$

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