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NEW SUBCLASS OF STARLIKE FUNCTIONS OF COMPLEX ORDER

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Abstract. The aim of the present paper is to investigate a new subclass of starlike functions of complex order, $b \neq 0$. Let $f(z) = z + a_2 z^2 + ...$ be an analytic function in the unit disc $D = \{z| \ \ |z| < 1\}$ which satisfies

$1 + \frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right) = \frac{1+A\omega(z)}{1+B\omega(z)}$,for some $\omega \in \Omega$ and for all $z \in D$.

Then f is called a Janowski starlike function of complex order b, where A and B are complex numbers such that $Re(1 - A\bar{B}) \geq |A - B|$, $im(1 - A\bar{B}) < |A - B|$, $|B| < 1$, and $\omega(z)$ is a Schwarz function in the unit disc D [1], [10], [12]. The class of these functions is denoted by $S^*(A, B, b)$.

In this paper we will give the representation theorem, distortion theorem, two point distortion theorem, Koebe domain under the montel normalization, and coefficient inequality for this class.

Key words : Starlike, distortion, Koebe, Montel normalization, coefficient.
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1. **Introduction**

Let Ω be the family of functions $\omega(z)$ regular in the unit disc $D = \{z| \ \ |z| < 1\}$ and satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in D$.

Next, for arbitrary fixed complex numbers A and B such that $Re(1-A\bar{B}) \geq |A - B|$, $im(1 - A\bar{B}) < |A - B|$ and $|B| < 1$, denote by $P_C(A, B)$ the family of functions

$$p(z) = 1 + b_1 z + ... \tag{1}$$

regular in D such that $p(z)$ is in $P_C(A, B)$ if and only if

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

for some function $\omega(z) \in \Omega$ and every $z \in D$.

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Moreover, let $S^*(A,B,b)$ denote the family of functions

$$f(z) = z + a_2 z^2 + ... \tag{2}$$

regular in D and such that f(z) is in $S^*(A,B,b)$ if and only if

$$1 + \frac{1}{b}(z\frac{f'(z)}{f(z)} - 1) = p(z) \tag{3}$$

for some p(z) in $P_C(A,B)$ and all z in D.

Let f and F be analytic in the unit disc D. The function f is subordinate to F, written f $\prec F, if F\ is\ univalent, f(0) = F(0), and\ f(U) \subset F(U)$.

The following lemma is due to I. S. Jack's [2] and plays a very important role in our proofs.

Lemma 1.1. Let $\omega(z)$ be regular in the unit disc with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, one has $z_1\omega'(z_1) = k\omega(z_1)$, for some real $k \geq 1$.

Definition 1.1. The Koebe domain $K(F)$ for a family F of regular functions $f(z)$ in F is the set of all points ω contained in $f(D)$ for every function $f(z)$in F. In symbols

$$K(F) = \bigcap_{f(z)\in D} f(D).$$

Supposing that the set F is invariant under rotations, so that $e^{i\alpha}f(e^{-i\alpha}z)$ is in F, are concludes that the Koebe domain will be either the single point $\omega = 0$ or an open disc $|\omega| < R$in which case R is often easy to find. Indeed, supposing that we have a sharp lower bound $M(r)$ for $\left|f(re^{i\theta})\right|$ for all functions in F and F contains only univalent functions, then

$$R = \lim_{R\to 1^-} M(r)$$

gives the disc $|\omega| < R$ as the Koebe domain for the set F.

We can also impose a Montel-type normalization. This means that for some fixed r_0, $0 < r_0 < 1$, we consider the family of normalized functions $f(z)$ regular and univalent in D with $f(0) = 0$, $f'(0) = 1$, $f(r_0) = r_0$ [1].

2. Main Results

In this section, we will give new results for the class $S^*(A,B,b)$.

Lemma 2.1. The function

$$\omega = \begin{cases} \frac{b(A-B)z}{1+Bz}, & B \neq 0 \\ bAz \quad , & B = 0 \end{cases}$$

maps $|z| < r$ onto the disc centred at C(r), and having radius $\rho(r)$, where

$$\begin{cases} C(r) = \left(\frac{-Re(c\bar{B})r^2}{1-|B|^2r^2}, \frac{-im(c\bar{B})}{1-|B|^2r^2}\right), & \rho(r) = \frac{|c|r}{1-|B|^2r^2}, \quad B \neq 0 \\ C(r) = (0,0), & \rho(r) = |A|\,|b|\,r, \quad B = 0 \end{cases}$$

and c=b(A-B).

Lemma 2.2. The function

$$\omega = \begin{cases} \frac{1+Az}{1+Bz}, & B \neq 0 \\ 1 + Az, & B = 0 \end{cases}$$

maps $|z| < r$ onto the disc centred at C(r) and having radius $\rho(r)$ where

$$\begin{cases} C(r) = \left(\frac{1-Re(AB)r^2}{1-|B|^2r^2}, \frac{im(AB)}{1-|B|^2r^2}\right), & \rho(r) = \frac{|A-B|r}{1-|B|^2r^2}, \quad B \neq 0 \\ C(r) = (1,0), & \rho(r) = |A|\,r, \quad B = 0 \end{cases}$$

Theorem 2.1. Let $f(z) = z + a_2z^2 + ...$ be an analytic function in the unit disc D. If f(z) satisfies the condition

$$\left(z\frac{f'(z)}{f(z)} - 1\right) \prec \begin{cases} \frac{b(A-B)z}{1+Bz}, & B \neq 0 \\ bAz, & B = 0 \end{cases}, \tag{4}$$

then $f(z) \in S^*(A, B, b)$, and this result is sharp on the function

$$\begin{cases} z(1+Bz)^{\frac{b(A-B)}{B}}, & B \neq 0 \\ ze^{bAz} & B = 0 \end{cases}.$$

Proof

We define the function $\omega(z)$ by

$$\frac{f(z)}{z} = \begin{cases} (1 + B\omega(z))^{\frac{b(A-B)}{B}} & B \neq 0; \\ e^{bA\omega(z)} & B = 0, \end{cases} \tag{5}$$

where $(1 + B\omega(z))^{\frac{b(A-B)}{B}}$ and $e^{bA\omega(z)}$ have the value 1 at z=0. Then $\omega(z)$ is analytic and $\omega(0) = 0$. If we take the logarithmic derivative of (5) and after brief calculations, we get

$$z\frac{f'(z)}{f(z)} - 1 \prec \begin{cases} \frac{b(A-B)z\omega'(z)}{1+B\omega(z)} & B \neq 0 \\ bAz\omega'(z) & B = 0 \end{cases}. \tag{6}$$

Now it is easy to realize that the subordination (4) is equivalent to $|\omega(z)| < 1$ for all $z \in D$. Indeed, assume the contrary: then, there exist a $z_1 \in D$ such that $|\omega(z_1)| = 1$. So by Lemma 2.1., $z_1\omega'(z_1) = k\omega(z_1)$ for some $k \geq 1$, and for such $z_1 \in D$(by using Lemma 2.1), we have

$$\left(z_1\frac{f'(z_1)}{f(z_1)}-1\right)=\begin{cases}\frac{b(A-B)k\omega(z_1)}{1+B\omega(z_1)}=F_1(\omega(z_1))\notin F_1(D) & , \quad B\neq 0\\ bAk\omega(z_1)=F_2(\omega(z_1))\notin F_2(D) & , \quad B=0\end{cases}. \tag{7}$$

But this contradicts (4); so our assumption is wrong, i. e., $|\omega(z)|<1$ for all $z\in D$. By using condition (6), we get

$$1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)=\begin{cases}\frac{1+A\omega(z)}{1+B\omega(z)} & , \quad B\neq 0\\ 1+A\omega(z) & , \quad B=0\end{cases} \tag{8}$$

Sharpness follows from the fact that for

$$f(z)=\begin{cases} z(1+Bz)^{\frac{b(A-B)}{B}} & , \quad B\neq 0\\ ze^{bAz} & , \quad B=0\end{cases}$$

one has

$$1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)=\begin{cases}\frac{1+Az}{1+Bz} & , \quad B\neq 0\\ 1+Az & , \quad B=0\end{cases}$$

and then we have $f(z)\in S^*(A,B,b)$.

Corollary 2.1. Let $f(z)\in S^*(A,B,b)$. Then f(z) can be written in the form

$$f_*(z)=\begin{cases} z(1+B\omega(z))^{\frac{b(A-B)}{B}} & , \quad B\neq 0;\\ ze^{bA\omega(z)} & , \quad B=0.\end{cases}$$

Theorem 2.2. If $f(z)\in S^*(A,B,b)$, then, for $|z|=r<1$

$$\frac{r(1-|B|\,r)^{\frac{|Bb(A-B)|-|B|^2b+Re(A\bar{B})b}{2|B|^2}}}{(1+|B|\,r)^{\frac{|Bb(A-B)|+|B|^2b-Re(A\bar{B})b}{2|B|^2}}}\leq|f(z)|\leq\frac{r(1+|B|\,r)^{\frac{|Bb(A-B)|-|B|^2b+Re(A\bar{B})b}{2|B|^2}}}{(1-|B|\,r)^{\frac{|Bb(A-B)|+|B|^2b-Re(A\bar{B})b}{2|B|^2}}}\ ,\ B\neq 0$$

(*)

$$re^{-|b||A|r}\leq|f(z)|\leq re^{|b||A|r}\ ,\quad B=0$$

These bounds are sharp because the extremal function is

$$f_*(z)=\begin{cases} z(1+Bz)^{\frac{b(A-B)}{B}} & , \quad B\neq 0\\ ze^{bAz} & , \quad B=0\end{cases}. \tag{9}$$

Proof

From the definition of the class $S^*(A,B,b)$ and Lemma 2.2, we have

$$\begin{cases} \left| z\frac{f'(z)}{f(z)} - \frac{1-(|B|^2+(|B|^2-A\bar{B})b)r^2}{1-|B|^2r^2} \right| \leq \frac{|b(A-B)|}{1-|B|^2r^2} & , \quad B \neq 0 \\ \left| z\frac{f'(z)}{f(z)} - 1 \right| \leq |b|\,|A|\,r & , \quad B = 0 \end{cases} . \tag{10}$$

After brief calculations from (10), we get

$$\begin{cases} \frac{1-|b(A-B)|r+(-|B|^2+Re(|B|^2-A\bar{B})b)r^2}{1-|B|^2r^2} \leq Rez\frac{f'(z)}{f(z)} \leq \frac{1+|b(A-B)|+(-|B|^2+Re(|B|^2-A\bar{B})b)r^2}{1-|B|^2r^2}, & B \neq 0; \\ 1-|b|\,|A|\,r \leq Rez\frac{f'(z)}{f(z)} \leq 1+|b|\,|A|\,r & , \ B = 0, \end{cases} \tag{11}$$

on the other hand, we have

$$Rez\frac{f'(z)}{f(z)} = r\frac{\partial}{\partial r}\log|f(z)|. \tag{12}$$

If we substitute (12) into (11) and after simple calculations, we get

$$\begin{cases} \frac{1}{r} - \frac{|b(A-B)|}{1-|B|^2r^2} + \frac{Re(|B|^2-A\bar{B})br}{1-|B|^2r^2} \leq \frac{\partial}{\partial r}\log|f(z)| \leq \frac{1}{r} - \frac{|b(A-B)|}{1-|B|^2r^2} + \frac{Re(|B|^2-A\bar{B})b)r}{1-|B|^2r^2} & , \ B \neq 0 \\ \frac{1}{r} - |b|\,|A| \leq \frac{\partial}{\partial r}\log|f(z)| \leq \frac{1}{r} + |b|\,|A| & , \ B = 0 \end{cases} \tag{13}$$

Integrating both sides of (13) we obtain (*).

Corollary 2.2. The class $S^*(A, B, b)$ is compact.

Proof

If we use the inequalities (*), (10) and after simple calculations, we get

$$|f'(z)| \leq \begin{cases} \frac{|1+|B|r|^{\frac{|Bb(A-B)|-|B|^2b+Re(A\bar{B})-2|B|^2}{2|B|^2}}}{|1-|B|r|^{\frac{|Bb(A-B)|+|B|^2b-Re(A\bar{B})+2|B|^2}{2|B|^2}}} & , \ B \neq 0 \\ e^{|b||A|r}(1+|b|\,|A|\,r) & , B = 0 \end{cases}$$

Therefore, $S^*(A, B, b)$ is normal and compact.

Theorem 2.3. The radius of starlikeness of the class $S^*(A, B, b)$ is

$$r_S = \begin{cases} \frac{2}{|b(A-B)|-\sqrt{|b(A-B)|^2-4\left[Re(|B|^2-A\bar{B})b-|B|^2\right]}} & , \quad B \neq 0 \\ \frac{1}{|b||A|} & B = 0 \end{cases} \tag{14}$$

The radius is sharp because the extremal function is given in (9).

Proof

Using Lemma 2.2, the set of values $(1 + \frac{1}{b}(z\frac{f'(z)}{f(z)} - 1))$ are obtained, which comprises the closed disc with centre C(r) and the radius $\rho(r)$, where

$$\begin{cases} C(r) = \left(\frac{1-Re(A\bar{B})r^2}{1-|B|^2r^2}, \frac{-im(A\bar{B})}{1-|B|^2r^2}\right), & \rho(r) = \frac{|A-B|r}{1-|B|^2r^2}, \ B \neq 0 \\ C(r) = (1,0) & , \ \rho(r) = |A|\, r \quad , \ B = 0 \end{cases} . \tag{15}$$

Therefore, by using the definition of the class $S^*(A,B,b)$, we have

$$\left| 1 + \frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right) - C(r) \right| \leq \rho(r).$$

This gives

$$\begin{cases} Re\left(z\frac{f'(z)}{f(z)}\right) \geq \frac{1-|b(A-B)|r+(-|B|^2+Re(|B|^2-A\bar{B})b)r^2}{1-|B|^2r^2} & , \ B \neq 0 \\ Re\left(z\frac{f'(z)}{f(z)}\right) \geq 1 - |b|\,|A|\,r & , \ B = 0 \end{cases} \tag{16}$$

Hence for $r < r_S$, the first hand side of the preceeding inequality is positive, implying that

$$r_S = \begin{cases} \frac{2}{|b(A-B)| - \sqrt{|b(A-B)|^2 - 4\left[Re(|B|^2-A\bar{B})b - |B|^2\right]}} & , \quad B \neq 0; \\ \frac{1}{|b||A|} & , \quad B = 0. \end{cases}$$

Also note that the inequality (14) becomes an equality for the function $f_*(z)$ from which it follows that

$$r_S = \begin{cases} \frac{2}{|b(A-B)| - \sqrt{|b(A-B)|^2 - 4\left[Re(|B|^2-A\bar{B})b - |B|^2\right]}} & , \quad B \neq 0; \\ \frac{1}{|b||A|} & , \quad B = 0. \end{cases}$$

Theorem 2.4. If f(z) belongs to $S^*(A,B,b)$, and the function

$$f_0(z) = \begin{cases} \frac{azf\left(\frac{z+a}{1+\bar{a}z}\right)}{f(a)(z+a)(1+\bar{a}z)^{-\frac{Ab}{B}}} , & B \neq 0, \quad a \in D \\ \frac{azf\left(\frac{z+a}{1+\bar{a}z}\right)}{f(a)(z+a)(1+\bar{a}z)^{2A-1}} , & B = 0, \quad a \in D \end{cases} \tag{17}$$

satisfies the condition

$$\left(z\frac{f_0'(z)}{f_0(z)} - 1\right) \prec \begin{cases} \frac{b(A-B)z}{1+Bz} & , \quad B \neq 0; \\ bAz & , \quad B = 0, \end{cases} \tag{18}$$

then $f_0(z) \in S^*(A,B,b)$.

Proof

We define the function $\omega(z)$ by

$$f_{0\rho}(z) = \begin{cases} \frac{azf\left(\rho\left(\frac{z+a}{1+\bar{a}z}\right)\right)}{f(a)(z+a)(1+\bar{a}z)^{-\frac{Ab}{B}}} = (1+B\omega(z))^{\frac{b(A-B)}{B}}, & B \neq 0, \quad a \in D, \quad 0<\rho<1 \\ \frac{azf\left(\rho\left(\frac{z+a}{1+\bar{a}z}\right)\right)}{f(a)(z+a)(1+\bar{a}z)^{2A-1}} = e^{bA\omega(z)}, & B=0, \quad a \in D, \quad 0<\rho<1 \end{cases} \tag{19}$$

where $(1+B\omega(z))^{\frac{b(A-B)}{B}}$ and $e^{bA\omega(z)}$ have the value 1 at z=0. Then $\omega(z)$is analytic in D, $\omega(0)=0$, and

$$\left(z\frac{f'_{0\rho}(z)}{f_{0\rho}(z)}-1\right) = \begin{cases} z\frac{(1-|a|^2)}{(1+\bar{a}z)}\left[\rho\left(\frac{z+a}{1+\bar{a}z}\right)\frac{f'\left(\frac{z+a}{1+\bar{a}z}\right)}{f\left(\frac{z+a}{1+\bar{a}z}\right)}\right] - \frac{z}{(z+a)} + \frac{Ab}{B}\frac{\bar{a}z}{1+\bar{a}z} = \frac{b(A-B)z\omega'(z)}{1+B\omega(z)}, B\neq 0, a \in D \\ z\frac{(1-|a|^2)}{(1+\bar{a}z)}\left[\rho\left(\frac{z+a}{1+\bar{a}z}\right)\frac{f'\left(\frac{z+a}{1+\bar{a}z}\right)}{f\left(\frac{z+a}{1+\bar{a}z}\right)}\right] - \frac{z}{(z+a)} - (2A-1)\frac{\bar{a}z}{1+\bar{a}z} = bAz\omega'(z), B=0, a\in D \end{cases} \tag{20}$$

We can easily conclude that this subordination is equivalent to $|\omega(z)| < 1$ for all $z \in D$. On the contrary let's assume that there exists $z_1 \in D$, $Max_{|z|=|z_1|}$, such that $|\omega(z)|$ attains its maximum value on the circle $|z|=r$, that is $|\omega(z_1)|=1$. Then when the conditions $z_1\omega'(z_1^) = k\omega(z_1)$, $k \geq 1$ are satisfied for such $z_1 \in D$(using I. S. Jack's Lemma), we obtain

$$\left(z_1\frac{f'_{0\rho}(z_1)}{f_{0\rho}(z_1)}-1\right) = \begin{cases} \frac{(A-B)k\omega(z_1)}{1+B\omega(z_1)} = F_1(\omega(z_1)) \notin F_1(D), & B \neq 0 \\ bAk\omega(z_1) = F_2(\omega(z_1)) \notin F_2(D), & B=0 \end{cases}, \tag{21}$$

which contradicts (18) implying that the assumption is wrong, i. e., $|\omega(z)|<1$ for all $z \in D$. On the other hand, we have

$$\left(z\frac{f'_{0\rho}(z)}{f_{0\rho}(z)}\right) = \begin{cases} \frac{1+A\omega(z)}{1+B\omega(z)}, & B\neq 0 \\ 1+A\omega(z), & B=0 \end{cases}$$

Using also the property of compactness of $S^*(A,B,b)$, we conclude that

$$f_0(z) = \lim_{\rho\to 1} f_{0\rho}(z)$$

is in $S^*(A,B,b)$.

Theorem 2.5. Let $f(z) \in S^*(A,B,b)$. Then

$$\begin{cases} F_2(k,|u|) \leq \left|\frac{kf(u)}{f(ku)}\right| \leq F_1(k,|u|), & B\neq 0; \\ F_4(k,|u|) \leq \left|\frac{kf(u)}{f(ku)}\right| \leq F_3(k,|u|), & B=0, \end{cases} \tag{22}$$

where

$$F_1(k,|u|)=\frac{\left(\frac{1+|B|(1-k)|u|-k|u|^2}{1-k|u|^2}\right)^{\frac{|Bb(A-B)|-|B|^2b+Re(A\bar{B})b}{2|B|^2}}\left(\frac{1-k^2|u|^2}{1-k|u|^2}\right)^{1-Re\left(\frac{Ab}{B}\right)}}{\left(\frac{1-|B|(1-k)|u|-k|u|^2}{1-k|u|^2}\right)^{\frac{|Bb(A-B)|+|B|^2b-Re(A\bar{B})b}{2|B|^2}}},$$

$$F_2(k,|u|)=\frac{\left(\frac{1-|B|(1-k)|u|-k|u|^2}{1-k|u|^2}\right)^{\frac{|Bb(A-B)|-|B|^2b+Re(A\bar{B})b}{2|B|^2}}\left(\frac{1-k^2|u|^2}{1-k|u|^2}\right)^{1-Re\left(\frac{Ab}{B}\right)}}{\left(\frac{1-|B|(1-k)|u|-k|u|^2}{1-k|u|^2}\right)^{\frac{|Bb(A-B)|+|B|^2b-Re(A\bar{B})b}{2|B|^2}}},$$

$$F_3(k,|u|)=e^{|b||A|\frac{(1-k)|u|}{1-k|u|^2}}\left(\frac{1-k^2\,|u|^2}{1-k\,|u|^2}\right)^{2ReA},$$

and

$$F_4(k,|u|)=e^{-|b||A|\frac{(1-k)|u|}{1-k|u|^2}}\left(\frac{1-k^2\,|u|^2}{1-k\,|u|^2}\right)^{2ReA}$$

for $-1<k<1$.

Proof

Using Theorem 2.2, Theorem 2.4 and taking $a=ku$, $u=\frac{z+a}{1+\bar{a}z}\Leftrightarrow z=\frac{u-a}{1-\bar{a}z}$, $-1<k<1$ we get (22) after simple calculations.

Theorem 2.6. The Koebe domain of the class $S^*(A,B,b)$ under the Montel normalization is

$$R=\begin{cases}\frac{(1-|B|)^{\frac{|Bb(A-B)|-|B|^2b+Re(A\bar{B})b}{2|B|^2}}}{(1+|B|)^{\frac{|Bb(A-B)|+|B|^2b-Re(A\bar{B})b}{2|B|^2}}}(1-r_0^2)^{1-Re\left(\frac{Ab}{B}\right)}\left(1-2r_0Cos\theta+r_0^2\right)^{\frac{1}{2}\left(Re\left(\frac{Ab}{B}-1\right)\right)} & ,B\neq 0\\ e^{-|b||A|}\left(1-r_0^2\right)^{2ReA}\left(1-2r_0Cos\theta+r_0^2\right)^{-2ReA} & ,\ B=0\end{cases}\tag{23}$$

Proof

Using the definition of the Koebe domain and Theorem 2.5, we get

$$\left|f\left(\frac{z+a}{1+\bar{a}z}\right)\right|\geq\begin{cases}\frac{|1-|B|r|^{\frac{|Bb(A-B)|-|B|^2b+Re(A\bar{B})b}{2|B|^2}}}{|1+|B|r|^{\frac{|Bb(A-B)|+|B|^2b-Re(A\bar{B})b}{2|B|^2}}}\frac{|z+a|\left|(1+\bar{a}z)^{-\frac{Ab}{B}}\right|}{|a|}\,|f(a)|\ , & B\neq 0;\\ e^{-|b||A||z|}\frac{|z+a|\left|(1+\bar{a}z)^{2A-1}\right|}{|a|}\,|f(a)| & ,\ B=0.\end{cases}\tag{24}$$

If we take $a = r_0$, $f(a) = f(r_0) = r_0$, $u = \frac{z+a}{1+\bar{a}z} \Leftrightarrow z = \frac{u-a}{1-\bar{a}z}$, $u = re^{i\theta}$ in (24) and pass to limit $r \to 1^-$, we obtain (23).

Theorem 2.7. If $f(z) = z + a_2z^2 + ... + a_nz^n + ...$ belongs to $S^*(A, B, b)$, then

$$|a_n| \leq \begin{cases} \prod_{k=0}^{n-2} \frac{|b(A-B)+kB|}{k+1} & , \ B \neq 0 \\ \prod_{k=0}^{n-2} \frac{|bA|}{k+1} & , \ B = 0 \end{cases} \tag{25}$$

These bounds are sharp because the extremal function is

$$f_*(z) = \begin{cases} \frac{z}{(1-B\delta z)^{\frac{-b(A-B)}{B}}} & , |\delta| = 1 \, , \ B \neq 0; \\ ze^{bAz} & , B = 0. \end{cases} \tag{26}$$

Proof

Let $B \neq 0$. If we use the definition of the class $S^*(A, B, b)$, then we write

$$1 + \frac{1}{b}\left(z\frac{f'(z)}{f(z)} - 1\right) = p(z). \tag{27}$$

Equality (27) can be written by using the Taylor expansion of $f(z)$ and $p(z)$ in the form

$$z+2a_2z^2+3a_3z^3+...+na_nz^n+... = (z+a_2z^2+a_3z^3+...+a_nz^n+...)(1+bp_1z+bp_2z^2+...+bp_nz^n+...) \tag{28}$$

Evaluating the coefficient of z^n in both sides of (28), we get

$$na_n = a_n + bp_1a_{n-1} + bp_2a_{n-2} + ... + bp_{n-1}. \tag{29}$$

on the other hand, if we use the same tecnique of [11], we obtain (because $\left|\frac{c}{d}\right|^n \frac{1}{|c.d|} = \left|\frac{B}{1}\right|^n \left|\frac{1}{B}\right| = |B|^{n-1} < 1$)

$$|p_n| \leq |A - B| \, . \tag{30}$$

If we consider the relations (29) and (30) together, then we obtain

$$(n-1)\,|a_n| \leq |b|\,|A-B|\,(1 + |a_2| + |a_3| + ... + |a_{n-1}|), \tag{31}$$

which can be written in the form

$$|a_n| \leq \frac{1}{(n-1)} \sum_{k=1}^{n-1} |b|\,|A-B|\,|a_k| \quad , \quad |a_1| = 1 \tag{32}$$

To prove (25), we will use the induction principle.

Now, observe that

(i) for $n = 2$,

$$|a_n| \leq \frac{|b|\,|A-B|}{(n-1)} \sum_{k=1}^{n-1} |a_k| \quad , \quad |a_1| = 1 \Rightarrow |a_2| \leq \frac{|b|\,|A-B|}{(2-1)} \sum_{k=1}^{(2-1)} |a_k|,$$

$$\Rightarrow |a_2| \leq |b|\,|A-B|\,|a_1| \Rightarrow |a_2| \leq |b|\,|A-B|$$

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|b(A-B)+kB|}{k+1} \Rightarrow |a_2| \leq \prod_{k=0}^{(2-2)} \frac{|b(A-B)+kB|}{k+1} \Rightarrow |a_2| \leq |b|\,|A-B|$$

Therefore the right hand side of these inequalities one same for n=2.
(ii) for $n = 3$,

$$|a_n| \leq \frac{|b|\,|A-B|}{(n-1)} \sum_{k=1}^{n-1} |a_k| \ , \ |a_1| = 1 \Rightarrow |a_3| \leq \frac{|b|\,|A-B|}{(3-1)} \sum_{k=1}^{(3-1)} |a_k| = \frac{1}{2} |b|\,|A-B|\,(1+|a_2|)$$

$$\Rightarrow |a_3| \leq \frac{1}{2} |b|^2 |A-B|^2 + \frac{1}{2} |b|\,|A-B|\,,$$

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|b(A-B)+kB|}{k+1} \Rightarrow |a_3| \leq \prod_{k=0}^{3-2} \frac{|b(A-B)+kB|}{k+1} = |b|\,|A-B|\,\frac{|b(A-B)+B|}{2}$$

$$\Rightarrow |a_3| \leq \frac{1}{2} |b|\,|A-B|\,[|b|\,|A-B| + |B|] \leq \frac{1}{2} |b|\,|A-B|\,[|b|\,|A-B| + 1]$$

$$\Rightarrow |a_3| \leq \frac{1}{2} |b|^2 |A-B|^2 + \frac{1}{2} |b|\,|A-B|$$

Therefore the right hand side of these inequalities one same for n=3.
Suppose that this result is true for $n = p$. Then we have

$$|a_n| \leq \frac{|b|\,|A-B|}{(n-1)} \sum_{k=1}^{n-1} |a_k|, \tag{33}$$

$$|a_1| = 1 \Rightarrow |a_p| \leq \frac{|b|\,|A-B|}{(p-1)} (1 + |a_2| + |a_3| + ... + |a_{p-1}|),$$

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|b(A-B)+kB|}{k+1} \Rightarrow |a_p| \leq \prod_{k=0}^{p-2} \frac{|b(A-B)+kB|}{k+1} \tag{34}$$

$$\Rightarrow |a_p| \leq \frac{1}{(p-1)!} |b| |A-B| (|b| |A-B|+1)(|b| |A-B|+2)...(|b| |A-B|+(p-2))$$

From (33), (34), and the induction hypothesis, we have

$$\frac{|b| |A-B|}{(p-1)}(1+|a_2|+|a_3|+...+|a_{p-1}|)$$

$$= \frac{1}{(p-1)!} |b| |A-B| (|b| |A-B|+1)(|b| |A-B|+2)...(|b| |A-B|+(p-2))$$

If we write $x = |b| |A-B| > 0$, equality (34) can be written in the form

$$\frac{x}{(p-1)}(1+|a_2|+|a_3|+...+|a_{p-1}|) \tag{35}$$
$$= \frac{1}{(p-1)!} x(x+1)(x+2)...(x+(p-2)).$$

After simple calculations from (35), we get

$$\frac{1}{p}(x+(p-1))\frac{x}{(p-1)}(1+|a_2|+|a_3|+...+|a_{p-1}|)$$
$$= \frac{1}{p!}(x+1)(x+2)(x+3)...(x+(p-2))(x+(p-1))$$
$$\Rightarrow \frac{1}{p}\left[\frac{x}{(p-1)}(1+|a_2|+|a_3|+...+|a_{p-1}|)\right] +$$
$$\left[\frac{1}{p}(1+|a_2|+|a_3|+...+|a_{p-1}|)\right]$$
$$= \frac{1}{p!}(x+1)(x+2)(x+3)...(x+(p-2))(x+(p-1)) \tag{36}$$
$$\Rightarrow \frac{1}{p}|a_p| + \left[\frac{1}{p}(1+|a_2|+|a_3|+...+|a_{p-1}|)\right]$$
$$= \frac{1}{p!}(x+1)(x+2)(x+3)...(x+(p-2))(x+(p-1))$$
$$\Rightarrow \frac{x}{p}(1+|a_2|+|a_3|+...+|a_{p-1}|+|a_p|)$$
$$= \frac{1}{p!}x(x+1)(x+2)(x+3)...(x+(p-2))(x+(p-1)).$$

Equality (36) shows that the results is valid for $n = p + 1$.

Therefore, we have (25).

We also note that by giving specific values to b we obtain the following inequalities

(i) For $b = 1$

$$|a_n| \leq \begin{cases} \prod_{k=0}^{n-2} \frac{|(A-B)+kB|}{k+1} & , \quad B \neq 0 \\ \prod_{k=0}^{n-2} \frac{|A|}{k+1} & , \quad B = 0 \end{cases}$$

This is the coefficient inequality for the class $S^*(A, B, 1)$. The class $S^*(A, B, 1)$ is a generalization of the W. Janowski class [10].

Because A, B are arbitrary fixed complex numbers, $Re(1 - A\bar{B}) \geq |A - B|$, $im(1 - A\bar{B}) < |A - B|$and $|B| < 1$.

(ii) For $b = (1 - \alpha)$, $0 \leq \alpha < 1$, we get

$$|a_n| \leq \begin{cases} \prod_{k=0}^{n-2} \frac{(1-\alpha)|(A-B)+kB|}{k+1} & , \quad B \neq 0 \\ \prod_{k=0}^{n-2} \frac{|(1-\alpha)A|}{k+1} & , \quad B = 0 \end{cases}$$

This is the coefficient inequality for the class $S^*(A, B, 1 - \alpha)$. The class $S^*(A, B, 1 - \alpha)$ is a subclass of starlike functions of order α[2], [6].

(iii) Similarly if we take $b = Cos\lambda e^{-i\lambda}$ and $b = (1 - \alpha)Cos\lambda e^{-i\lambda}$ we obtain the coefficient inequality for the subclasses of $\lambda-$spirallike functions and $\lambda-$spirallike functions of order α respectively [5], [8], [9], [11].

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