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MONOTONIC SURFACES FOR COMPUTER GRAPHICS

MALIK ZAWWAR HUSSAIN, MARIA HUSSAIN

ABSTRACT. Computer graphics environment requires realistic visual models of data generated. These data can be either 2D or 3D and corresponding visual models are called curves and surfaces. In this paper, a piecewise rational cubic function [8] has been extended to rational bicubic function. Simple constraints are derived on the free parameters in the description of rational bicubic function to preserve the shape of monotonic data.

Key words : Rational Function, Monotonic Data, Monotonic Visual Model, Free Parameters.
AMS SUBJECT CLASSIFICATION: 68U05, 65D05, 65D07.

1. Introduction

Computer graphics is associated with creation of realistic scenes and animated images. Central objective of computer graphics is generation of images of environment. These environments may be fictional or real. Images are made by combining large number of curves and surfaces.

In most of computer graphics applications data are provided and corresponding real scenes are created. Data are observed from scientific phenomenons, mathematical description and real scenes. These data may have some special shape property e.g. positivity, monotonicity and convexity and of course it is required that visual model exhibits this shape property. The problem become critical when visual model fails to exhibit this shape property.

One of the shape property of data is monotonicity. Monotonic data arises in many phenomenons e.g. in determining the ultimate tensile strength of a material. Result is obtained by making observation of stress and strain. The generated data is always monotonic. Probability distribution data is another example.

Major contributions to shape preservation of curves and surfaces are [1-11] and references there in. In this paper, we extend the rational cubic function defined in [8] to rational bicubic function. Simple constraints are derived on

Department of Mathematics, University of the Punjab, Lahore, Pakistan,
E-mail: malikzawwar@math.pu.edu.pk.

free parameters in the description of rational bicubic function to preserve the shape of monotonic data.

2. Rational Cubic Function

In this section we introduce the piecewise rational cubic function [8] used in this paper. Let (x_i, f_i), $i = 0, 1, 2, \ldots, n$ be given set of data points where $x_0 < x_1 < x_2 < \cdots < x_n$. Piecewise rational cubic function is defined over each interval $I_i = [x_i, x_{i+1}]$, $i = 0, 1, 2, \ldots, n-1$ as:

$$S(x) = \frac{p_i(\theta)}{q_i(\theta)}, \tag{1}$$

with

$$\begin{aligned} p_i(\theta) &= f_i(1-\theta)^3 + (\alpha_i f_i + h_i d_i)(1-\theta)^2\theta + (\beta_i f_{i+1} - h_i d_{i+1})(1-\theta)\theta^2 \\ &\quad + f_{i+1}\theta^3, \\ q_i(\theta) &= (1-\theta)^3 + \alpha_i(1-\theta)^2\theta + \beta_i(1-\theta)\theta^2 + \theta^3, \end{aligned}$$

$$h_i = x_{i+1} - x_i, \quad \theta = \frac{x - x_i}{h_i}.$$

The rational cubic function (1) has the following properties:

$$\begin{aligned} S(x_i) = f_i, &\quad S(x_{i+1}) = f_{i+1}, \\ S^{(1)}(x_i) = d_i, &\quad S^{(1)}(x_{i+1}) = d_{i+1}. \end{aligned}$$

$S^{(1)}(x)$ denotes the derivative with respect to x and d_i denotes derivative values (given or estimated by some method) at knot x_i. $S(x) \in C^{(1)}[x_0, x_n]$ has α_i and β_i as free parameters in the interval $[x_i, x_{i+1}]$. We note that in each interval I_i, when we take $\alpha_i = 3$ and $\beta_i = 3$, the piecewise rational cubic function reduces to standard Cubic Hermite.

3. Monotonic Rational Cubic Function

In this section we explain the result of [8]. Let (x_i, f_i), $i = 0, 1, 2, \ldots, n$ be monotonic data defined over the interval $[x_0, x_n]$. The monotonic data satisfies the following conditions:

$$\begin{aligned} f_i &< f_{i+1}, \\ \Delta_i &= \frac{f_{i+1} - f_i}{h_i} > 0, \quad \forall\, i = 0, 1, 2, \ldots, n-1. \\ d_i &> 0, \quad \forall\, i = 0, 1, 2, \ldots, n. \end{aligned}$$

The rational cubic function defined in (1) preserves the shape of monotonic data if

$$S^{(1)}(x) > 0, \quad \forall\ x\ \in [x_0, x_n].$$

For each interval $[x_i, x_{i+1}]$, $S^{(1)}(x)$ can be expressed as:

$$S^{(1)}(x) = \frac{\sum_{i=0}^{5}(1-\theta)^{5-i}\theta^i A_i}{(q_i(\theta))^2}, \tag{2}$$

with

$$\begin{aligned}
A_0 &= d_i, \\
A_1 &= 2\beta_i\Delta_i + d_i - 2d_{i+1}, \\
A_2 &= (\alpha_i\beta_i + 2\beta_i + 3)\Delta_i - (\alpha_i + 2)d_{i+1} - \beta_i d_i, \\
A_3 &= (\alpha_i\beta_i + 2\alpha_i + 3)\Delta_i - \alpha_i d_{i+1} - (\beta_i + 2)d_i, \\
A_4 &= 2\alpha_i\Delta_i + d_{i+1} - 2d_i, \\
A_5 &= d_{i+1}.
\end{aligned}$$

$S^{(1)}(x) > 0$ if $A_i > 0,\ i = 0, 1, 2, \ldots, 5$.
$A_i > 0,\ i = 0, 1, 2, \ldots, 5$ if

$$\begin{aligned}
\alpha_i &> \frac{d_i}{\Delta_i}. \\
\beta_i &> \frac{d_{i+1}}{\Delta_i}.
\end{aligned}$$

This leads to the following theorem:

Theorem 3.1. The rational cubic function (1) generates pleasing monotonic curve through monotonic data if in each subinterval $I_i = [x_i, x_{i+1}], i = 0, 1, 2, \ldots, n-1$, shape parameters α_i and β_i satisfy the following conditions:

$$\begin{aligned}
\alpha_i &= l_i + \frac{d_i}{\Delta_i},\ l_i > 0. \\
\beta_i &= m_i + \frac{d_{i+1}}{\Delta_i},\ m_i > 0.
\end{aligned}$$

4. **Rational Bicubic Function**

The piecewise rational cubic function (1) is extended to rational bicubic function $S(x, y)$ over the rectangular domain $D = [x_0, x_m] \times [y_0, y_n]$. Let $\pi : a = x_0 < x_1 < \cdots < x_m = b$ be partition of $[a, b]$ and $\tilde{\pi} : c = y_0 < y_1 < \cdots < y_n = d$ be partition of $[c, d]$. Rational bicubic function is defined over each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, where $i = 0, 1, 2, \ldots, m-1;\ j = 0, 1, 2, \ldots, n-1$ as:

$$S(x, y) = S_{i,j}(x, y) = A_i(\theta)F(i, j)\hat{A}_j^T(\phi), \tag{3}$$

where

$$F = \begin{pmatrix} F_{i,j} & F_{i,j+1} & F^{y}_{i,j} & F^{y}_{i,j+1} \\ F_{i+1,j} & F_{i+1,j+1} & F^{y}_{i+1,j} & F^{y}_{i+1,j+1} \\ F^{x}_{i,j} & F^{x}_{i,j+1} & F^{xy}_{i,j} & F^{xy}_{i,j+1} \\ F^{x}_{i+1,j} & F^{x}_{i+1,j+1} & F^{xy}_{i+1,j} & F^{xy}_{i+1,j+1} \end{pmatrix},$$

$$A_i(\theta) = [a_0(\theta)\ a_1(\theta)\ a_2(\theta)\ a_3(\theta)],$$

$$\hat{A}_j(\phi) = [\hat{a}_0(\phi)\ \hat{a}_1(\phi)\ \hat{a}_2(\phi)\ \hat{a}_3(\phi)],$$

with

$$\begin{aligned} a_0(\theta) &= \frac{(1-\theta)^3 + \alpha_i(1-\theta)^2\theta}{q_i(\theta)}, \\ a_1(\theta) &= \frac{\theta^3 + \beta_i(1-\theta)\theta^2}{q_i(\theta)}, \\ a_2(\theta) &= \frac{h_i(1-\theta)^2\theta}{q_i(\theta)}, \\ a_3(\theta) &= \frac{-h_i(1-\theta)\theta^2}{q_i(\theta)}, \\ \hat{a}_0(\phi) &= \frac{(1-\phi)^3 + \hat{\alpha}_j(1-\phi)^2\phi}{q_j(\phi)}, \\ \hat{a}_1(\phi) &= \frac{\phi^3 + \hat{\beta}_j(1-\phi)\phi^2}{q_j(\phi)}, \\ \hat{a}_2(\phi) &= \frac{\hat{h}_j(1-\phi)^2\phi}{q_j(\phi)}, \\ \hat{a}_3(\phi) &= \frac{-\hat{h}_j(1-\phi)\phi^2}{q_j(\phi)}. \end{aligned}$$

Substituting the values of A, F and $\hat{A}$ in equation (3) the rational bicubic function $S(x,y)$ can be expressed as:

$$S(x,y) = \frac{(1-\theta)^3\gamma_{i,j} + (1-\theta)^2\theta\eta_{i,j} + (1-\theta)\theta^2\delta_{i,j} + \theta^3\omega_{i,j}}{(1-\theta)^3 + \alpha_i(1-\theta)^2\theta + \beta_i(1-\theta)\theta^2 + \theta^3}, \tag{4}$$

with

$$\begin{aligned} \gamma_{i,j} &= [(1-\phi)^3 F_{i,j} + (1-\phi)^2\phi(\hat{\alpha}_j F_{i,j} + \hat{h}_j F^{y}_{i,j}) + (1-\phi)\phi^2(\hat{\beta}_j F_{i,j+1} \\ &\quad - \hat{h}_j F^{y}_{i,j+1}) + \phi^3 F_{i,j+1}]/q_j(\phi), \end{aligned}$$

$$
\begin{aligned}
\eta_{i,j} &= [(1-\phi)^3(\alpha_i F_{i,j} + h_i F^x_{i,j}) + (1-\phi)^2\phi\{\hat{\alpha}_j(\alpha_i F_{i,j} + h_i F^x_{i,j}) + \hat{h}_j(\alpha_i F^y_{i,j} \\
&\quad + h_i F^{xy}_{i,j})\} + (1-\phi)\phi^2\{\hat{\beta}_j(\alpha_i F_{i,j+1} + h_i F^x_{i,j+1}) - \hat{h}_j(\alpha_i F^y_{i,j+1} \\
&\quad + h_i F^{xy}_{i,j+1})\} + \phi^3(\alpha_i F_{i,j+1} + h_i F^x_{i,j+1})]/q_j(\phi),
\end{aligned}
$$

$$
\begin{aligned}
\delta_{i,j} &= [(1-\phi)^3(\beta_i F_{i+1,j} - h_i F^x_{i+1,j}) + (1-\phi)^2\phi\{\hat{\alpha}_j(\beta_i F_{i+1,j} - h_i F^x_{i+1,j}) \\
&\quad + \hat{h}_j(\beta_i F^y_{i+1,j} - h_i F^{xy}_{i+1,j})\} + (1-\phi)\phi^2\{\hat{\beta}_j(\beta_i F_{i+1,j+1} - h_i F^x_{i+1,j+1}) \\
&\quad - \hat{h}_j(\beta_i F^y_{i+1,j+1} - h_i F^{xy}_{i+1,j+1})\} + \phi^3(\beta_i F_{i+1,j+1} - h_i F^x_{i+1,j+1})]/q_j(\phi),
\end{aligned}
$$

$$
\begin{aligned}
\omega_{i,j} &= [(1-\phi)^3 F_{i+1,j} + (1-\phi)^2\phi(\hat{\alpha}_j F_{i+1,j} + \hat{h}_j F^y_{i+1,j}) + (1-\phi)\phi^2 \\
&\quad (\hat{\beta}_j F_{i+1,j+1} - \hat{h}_j F^y_{i+1,j+1}) + \phi^3 F_{i+1,j+1}]/q_j(\phi),
\end{aligned}
$$

$$
\begin{aligned}
q_i(\theta) &= (1-\theta)^3 + \alpha_i(1-\theta)^2\theta + \beta_i(1-\theta)\theta^2 + \theta^3, \\
q_j(\phi) &= (1-\phi)^3 + \hat{\alpha}_j(1-\phi)^2\phi + \hat{\beta}_j(1-\phi)\phi^2 + \phi^3.
\end{aligned}
$$

The normalized variables θ and ϕ along x and y axes are defined as:

$$\theta = \frac{x - x_i}{h_i} \quad , \quad \phi = \frac{y - y_j}{\hat{h}_j},$$

with

$$h_i = x_{i+1} - x_i \quad , \quad \hat{h}_j = y_{j+1} - y_j.$$

Unfortunately, these rational functions are not very useful for surface design as any one of the free parameter α_i, β_i, $\hat{\alpha}_j$ and $\hat{\beta}_j$ applies to entire network of curve. Thus there is no local control on the surface model. This ambiguity is overcome by introducing variable weights and desired local control has been achieved. For this purpose new free parameters $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ are introduced such that:

$$\alpha_i(y_j) = \alpha_{i,j},\ \beta_i(y_j) = \beta_{i,j},\ \hat{\alpha}_j(x_i) = \hat{\alpha}_{i,j},\ \hat{\beta}_j(x_i) = \hat{\beta}_{i,j},$$
$$i = 0, 1, 2, \ldots, m-1;\ j = 0, 1, 2, \ldots, n-1.$$

The shape of the surface can be modified by assigning different values to these parameters. This property of free parameters will impose different constraints on $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$.

4.1. **Choice of Derivatives.** In most applications, the derivative parameters d_i, $F^x_{i,j}$, $F^y_{i,j}$ and $F^{xy}_{i,j}$ are not given and hence must be determined either from given data or by some other means. These methods are the approximation based on various mathematical theories. An obvious choice is mentioned here:

4.1.1. *Arithmetic Mean Method for 2D Data.*

$$\begin{aligned} d_0 &= \Delta_0 + (\Delta_0 - \Delta_1)\frac{h_0}{(h_0+h_1)}. \\ d_n &= \Delta_{n-1} + (\Delta_{n-1} - \Delta_{n-2})\frac{h_{n-1}}{(h_{n-1}+h_{n-2})}. \\ d_i &= \frac{\Delta_i + \Delta_{i-1}}{2}, \quad i = 1,2,3,\ldots,n-1. \end{aligned}$$

4.1.2. *Arithmetic Mean Method for 3D Data.*

$$\begin{aligned} F^x_{0,j} &= \Delta_{0,j} + (\Delta_{0,j} - \Delta_{1,j})\frac{h_0}{(h_0+h_1)}. \\ F^x_{m,j} &= \Delta_{m-1,j} + (\Delta_{m-1,j} - \Delta_{m-2,j})\frac{h_{m-1}}{(h_{m-1}+h_{m-2})}. \\ F^x_{i,j} &= \frac{\Delta_{i,j} + \Delta_{i-1,j}}{2}. \\ & \quad i = 1,2,3,\ldots,m-1; \quad j = 0,1,2,\ldots,n. \\ F^y_{i,0} &= \hat{\Delta}_{i,0} + (\hat{\Delta}_{i,0} - \hat{\Delta}_{i,1})\frac{\hat{h}_0}{(\hat{h}_0+\hat{h}_1)}. \\ F^y_{i,n} &= \hat{\Delta}_{i,n-1} + (\hat{\Delta}_{i,n-1} - \hat{\Delta}_{i,n-2})\frac{\hat{h}_{n-1}}{(\hat{h}_{n-1}+\hat{h}_{n-2})}. \\ F^y_{i,j} &= \frac{\hat{\Delta}_{i,j} + \hat{\Delta}_{i,j-1}}{2}. \\ & \quad i = 0,1,2,\ldots,m; \quad j = 1,2,3,\ldots,n-1. \\ F^{xy}_{i,j} &= \frac{1}{2}\left\{\frac{F^x_{i,j+1} - F^x_{i,j-1}}{\hat{h}_{j-1}+\hat{h}_j} + \frac{F^y_{i+1,j} - F^y_{i-1,j}}{h_{i-1}+h_i}\right\}. \\ & \quad i = 1,2,\ldots,m-1; \quad j = 1,2,\ldots,n-1. \end{aligned}$$

Where $\Delta_{i,j} = \frac{F_{i+1,j}-F_{i,j}}{h_i}$ and $\hat{\Delta}_{i,j} = \frac{F_{i,j+1}-F_{i,j}}{\hat{h}_j}$. These arithmetic mean methods are computationally economical and suitable for visualization of shaped data.

5. **Monotonic Rational Bicubic Function**

Let $(x_i, y_j, F_{i,j})$, $i = 0,1,2,\ldots,m$; $j = 0,1,2,\ldots,n$ be monotonic data defined over the rectangular domain $D = [x_0, x_m] \times [y_0, y_n]$ such that

$$F_{i,j} < F_{i+1,j}, \quad \Delta_{i,j} > 0, \quad F_{i,j} < F_{i,j+1}, \quad \hat{\Delta}_{i,j} > 0, \quad F^x_{i,j} > 0, \quad F^y_{i,j} > 0, \; \forall\, i,\; j.$$

The rational bicubic function $S(x, y)$ defined in (3) preserves the shape of monotonic data if

$$\frac{\partial S}{\partial x} = S_x(x, y) > 0, \quad \frac{\partial S}{\partial y} = S_y(x, y) > 0, \quad \forall (x, y) \in D.$$

$$S_x(x, y) = \frac{\sum_{i=0}^{5}(1-\theta)^{5-i}\theta^i A_i}{(q_i(\theta))^2 q_j(\phi)}, \tag{5}$$

where

$$A_0 = \sum_{j=0}^{3}(1-\phi)^{3-j}\phi^j A_{0,j}, \tag{6}$$

with

$$\begin{aligned} A_{0,0} &= F^x_{i,j}, \\ A_{0,3} &= F^x_{i,j+1}, \\ A_{0,1} &= \hat{\alpha}_{i,j} F^x_{i,j} + \hat{h}_j F^{xy}_{i,j}, \\ A_{0,2} &= \hat{\beta}_{i,j} F^x_{i,j+1} - \hat{h}_j F^{xy}_{i,j+1}. \end{aligned}$$

$$A_5 = \sum_{j=0}^{3}(1-\phi)^{3-j}\phi^j A_{5,j}, \tag{7}$$

with

$$\begin{aligned} A_{5,0} &= F^x_{i+1,j}, \\ A_{5,3} &= F^x_{i+1,j+1}, \\ A_{5,1} &= \hat{\alpha}_{i,j} F^x_{i+1,j} + \hat{h}_j F^{xy}_{i+1,j}, \\ A_{5,2} &= \hat{\beta}_{i,j} F^x_{i+1,j+1} - \hat{h}_j F^{xy}_{i+1,j+1}. \end{aligned}$$

$$A_1 = \sum_{j=0}^{3}(1-\phi)^{3-j}\phi^j A_{1,j}, \tag{8}$$

with

$$\begin{aligned}
A_{1,0} &= 2\beta_{i,j}\Delta_{i,j} - 2A_{5,0} + A_{0,0}, \\
A_{1,3} &= 2\beta_{i,j}\Delta_{i,j+1} - 2A_{5,3} + A_{0,3}, \\
A_{1,1} &= A_{0,1} + 2\hat{\alpha}_{i,j}\beta_{i,j}\Delta_{i,j} + 2\frac{\hat{h}_j}{h_i}\beta_{i,j}(F^y_{i+1,j} - F^y_{i,j}) \\
&\quad - 2(\hat{\alpha}_{i,j}F^x_{i+1,j} + \hat{h}_j F^{xy}_{i+1,j}), \\
A_{1,2} &= A_{0,2} + 2\hat{\beta}_{i,j}\beta_{i,j}\Delta_{i,j+1} - 2\frac{\hat{h}_j}{h_i}\beta_{i,j}(F^y_{i+1,j+1} - F^y_{i,j+1}) \\
&\quad - 2(\hat{\beta}_{i,j}F^x_{i+1,j+1} - \hat{h}_j F^{xy}_{i+1,j+1}).
\end{aligned}$$

$$A_4 = \sum_{j=0}^{3}(1-\phi)^{3-j}\phi^j A_{4,j}, \tag{9}$$

with

$$\begin{aligned}
A_{4,0} &= 2\alpha_{i,j}\Delta_{i,j} + A_{5,0} - 2A_{0,0}, \\
A_{4,3} &= 2\alpha_{i,j}\Delta_{i,j+1} + A_{5,3} - 2A_{0,3}, \\
A_{4,1} &= 2\hat{\alpha}_{i,j}\alpha_{i,j}\Delta_{i,j} + 2\frac{\hat{h}_j}{h_i}\alpha_{i,j}(F^y_{i+1,j} - F^y_{i,j}) - 2(\hat{\alpha}_{i,j}F^x_{i,j} + \hat{h}_j F^{xy}_{i,j}) + A_{5,1}, \\
A_{4,2} &= A_{5,2} + 2\alpha_{i,j}\hat{\beta}_{i,j}\Delta_{i,j+1} - 2\frac{\hat{h}_j}{h_i}\alpha_{i,j}(F^y_{i+1,j+1} - F^y_{i,j+1}) \\
&\quad - 2(\hat{\beta}_{i,j}F^x_{i,j+1} - \hat{h}_j F^{xy}_{i,j+1}).
\end{aligned}$$

$$A_2 = \sum_{j=0}^{3}(1-\phi)^{3-j}\phi^j A_{2,j}, \tag{10}$$

with

$$\begin{aligned}
A_{2,0} &= (\alpha_{i,j}\beta_{i,j} + 2\beta_{i,j} + 3)\Delta_{i,j} - (\alpha_{i,j} + 2)A_{5,0} - \beta_{i,j}A_{0,0}, \\
A_{2,3} &= (\alpha_{i,j}\beta_{i,j} + 2\beta_{i,j} + 3)\Delta_{i,j+1} - (\alpha_{i,j} + 2)A_{5,3} - \beta_{i,j}A_{0,3}, \\
A_{2,1} &= 0.5(\alpha_{i,j} + 2)(A_{1,1} - A_{0,1}) + 3\hat{\alpha}_{i,j}\Delta_{i,j} + 3\frac{\hat{h}_j}{h_i}(F^y_{i+1,j} - F^y_{i,j}) \\
&\quad - \beta_{i,j}(\hat{\alpha}_{i,j}F^x_{i,j} + \hat{h}_j F^{xy}_{i,j}), \\
A_{2,2} &= 0.5(\alpha_{i,j} + 2)(A_{1,2} - A_{0,2}) + 3\hat{\beta}_{i,j}\Delta_{i,j+1} - 3\frac{\hat{h}_j}{h_i}(F^y_{i+1,j+1} - F^y_{i,j+1}) \\
&\quad - \beta_{i,j}(\hat{\beta}_{i,j}F^x_{i,j+1} - \hat{h}_j F^{xy}_{i,j+1}).
\end{aligned}$$

$$A_3 = \sum_{j=0}^{3} (1-\phi)^{3-j} \phi^j A_{3,j}, \tag{11}$$

with

$$\begin{aligned}
A_{3,0} &= (\alpha_{i,j}\beta_{i,j} + 2\alpha_{i,j} + 3)\Delta_{i,j} - \alpha_{i,j}A_{5,0} - (\beta_{i,j}+2)A_{0,0}, \\
A_{3,3} &= (\alpha_{i,j}\beta_{i,j} + 2\alpha_{i,j} + 3)\Delta_{i,j+1} - \alpha_{i,j}A_{5,3} - (\beta_{i,j}+2)A_{0,3}, \\
A_{3,1} &= 0.5(\beta_{i,j}+2)(A_{4,1} - A_{5,1}) + \hat{\alpha}_{i,j}(3\Delta_{i,j} - \alpha_{i,j}F^x_{i+1,j}) \\
&\quad + 3\frac{\hat{h}_j}{h_i}(F^y_{i+1,j} - F^y_{i,j}) - \alpha_{i,j}\hat{h}_j F^{xy}_{i+1,j}, \\
A_{3,2} &= 0.5(\beta_{i,j}+2)(A_{4,2} - A_{5,2}) + \hat{\beta}_{i,j}(3\Delta_{i,j+1} - \alpha_{i,j}F^x_{i+1,j+1}) \\
&\quad - 3\frac{\hat{h}_j}{h_i}(F^y_{i+1,j+1} - F^y_{i,j+1}) + \alpha_{i,j}\hat{h}_j F^{xy}_{i+1,j+1}.
\end{aligned}$$

$$S_y(x,y) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i B_i}{q_i(\theta)(q_j(\phi))^2}, \tag{12}$$

where

$$B_0 = \sum_{j=0}^{5} (1-\phi)^{5-j} \phi^j B_{0,j}, \tag{13}$$

with

$$\begin{aligned}
B_{0,0} &= F^y_{i,j}, \\
B_{0,5} &= F^y_{i,j+1}, \\
B_{0,1} &= 2\hat{\beta}_{i,j}\hat{\Delta}_{i,j} - 2F^y_{i,j+1} + F^y_{i,j}, \\
B_{0,4} &= 2\hat{\alpha}_{i,j}\hat{\Delta}_{i,j} + F^y_{i,j+1} - 2F^y_{i,j}, \\
B_{0,2} &= (\hat{\alpha}_{i,j}\hat{\beta}_{i,j} + 2\hat{\beta}_{i,j} + 3)\hat{\Delta}_{i,j} - (\hat{\alpha}_{i,j}+2)F^y_{i,j+1} - \hat{\beta}_{i,j}F^y_{i,j}, \\
B_{0,3} &= (\hat{\alpha}_{i,j}\hat{\beta}_{i,j} + 2\hat{\alpha}_{i,j} + 3)\hat{\Delta}_{i,j} - \hat{\alpha}_{i,j}F^y_{i,j+1} - (\hat{\beta}_{i,j}+2)F^y_{i,j}.
\end{aligned}$$

$$B_3 = \sum_{j=0}^{5} (1-\phi)^{5-j} \phi^j B_{3,j}, \tag{14}$$

with

$$\begin{aligned}
B_{3,0} &= F^y_{i+1,j}, \\
B_{3,5} &= F^y_{i+1,j+1}, \\
B_{3,1} &= 2\hat{\beta}_{i,j}\hat{\Delta}_{i+1,j} - 2F^y_{i+1,j+1} + F^y_{i+1,j}, \\
B_{3,4} &= 2\hat{\alpha}_{i,j}\hat{\Delta}_{i+1,j} + F^y_{i+1,j+1} - 2F^y_{i+1,j}, \\
B_{3,2} &= (\hat{\alpha}_{i,j}\hat{\beta}_{i,j} + 2\hat{\beta}_{i,j} + 3)\hat{\Delta}_{i+1,j} - (\hat{\alpha}_{i,j} + 2)F^y_{i+1,j+1} - \hat{\beta}_{i,j}F^y_{i+1,j}, \\
B_{3,3} &= (\hat{\alpha}_{i,j}\hat{\beta}_{i,j} + 2\hat{\alpha}_{i,j} + 3)\hat{\Delta}_{i+1,j} - \hat{\alpha}_{i,j}F^y_{i+1,j+1} - (\hat{\beta}_{i,j} + 2)F^y_{i+1,j}.
\end{aligned}$$

$$B_1 = \sum_{j=0}^{5}(1-\phi)^{5-j}\phi^j B_{1,j}, \tag{15}$$

with

$$\begin{aligned}
B_{1,0} &= \alpha_{i,j}F^y_{i,j} + h_iF^{xy}_{i,j}, \\
B_{1,5} &= \alpha_{i,j}F^y_{i,j+1} + h_iF^{xy}_{i,j+1}, \\
B_{1,1} &= B_{1,0} + 2\hat{\beta}_{i,j}\left\{\alpha_{i,j}\hat{\Delta}_{i,j} + \frac{h_i}{\hat{h}_j}(F^x_{i,j+1} - F^x_{i,j})\right\} \\
&\quad - 2(\alpha_{i,j}F^y_{i,j+1} + h_iF^{xy}_{i,j+1}), \\
B_{1,4} &= B_{1,5} + 2\hat{\alpha}_{i,j}\left\{\alpha_{i,j}\hat{\Delta}_{i,j} + \frac{h_i}{\hat{h}_j}(F^x_{i,j+1} - F^x_{i,j})\right\} \\
&\quad - 2(\alpha_{i,j}F^y_{i,j} + h_iF^{xy}_{i,j}), \\
B_{1,2} &= 0.5(\hat{\alpha}_{i,j} + 2)(B_{1,1} - B_{1,0}) - \hat{\beta}_{i,j}(\alpha_{i,j}F^y_{i,j} + h_iF^{xy}_{i,j}) \\
&\quad + 3\left\{\alpha_{i,j}\hat{\Delta}_{i,j} + \frac{h_i}{\hat{h}_j}(F^x_{i,j+1} - F^x_{i,j})\right\}, \\
B_{1,3} &= 0.5(\hat{\beta}_{i,j} + 2)(B_{1,4} - B_{1,5}) - \hat{\alpha}_{i,j}(\alpha_{i,j}F^y_{i,j+1} + h_iF^{xy}_{i,j+1}) \\
&\quad + 3\left\{\alpha_{i,j}\hat{\Delta}_{i,j} + \frac{h_i}{\hat{h}_j}(F^x_{i,j+1} - F^x_{i,j})\right\}.
\end{aligned}$$

$$B_2 = \sum_{j=0}^{5}(1-\phi)^{5-j}\phi^j B_{2,j}, \tag{16}$$

with

$$
\begin{aligned}
B_{2,0} &= \beta_{i,j}F^{y}_{i+1,j} - h_iF^{xy}_{i+1,j},\\
B_{2,5} &= \beta_{i,j}F^{y}_{i+1,j+1} - h_iF^{xy}_{i+1,j+1},\\
B_{2,1} &= B_{2,0} + 2\hat{\beta}_{i,j}\left\{\beta_{i,j}\hat{\Delta}_{i+1,j} - \frac{h_i}{\hat{h}_j}(F^{x}_{i+1,j+1} - F^{x}_{i+1,j})\right\}\\
&\quad - 2(\beta_{i,j}F^{y}_{i+1,j+1} - h_iF^{xy}_{i+1,j+1}),\\
B_{2,4} &= B_{2,5} + 2\hat{\alpha}_{i,j}\left\{\beta_{i,j}\hat{\Delta}_{i+1,j} - \frac{h_i}{\hat{h}_j}(F^{x}_{i+1,j+1} - F^{x}_{i+1,j})\right\}\\
&\quad - 2(\beta_{i,j}F^{y}_{i+1,j} - h_iF^{xy}_{i+1,j}),\\
B_{2,2} &= 0.5(\hat{\alpha}_{i,j}+2)(B_{2,1} - B_{2,0}) - \hat{\beta}_{i,j}(\beta_{i,j}F^{y}_{i+1,j} - h_iF^{xy}_{i+1,j})\\
&\quad + 3\left\{\beta_{i,j}\hat{\Delta}_{i+1,j} - \frac{h_i}{\hat{h}_j}(F^{x}_{i+1,j+1} - F^{x}_{i+1,j})\right\},\\
B_{2,3} &= 0.5(\hat{\beta}_{i,j}+2)(B_{2,4} - B_{2,5}) - \hat{\alpha}_{i,j}(\beta_{i,j}F^{y}_{i+1,j+1} - h_iF^{xy}_{i+1,j+1})\\
&\quad + 3\left\{\beta_{i,j}\hat{\Delta}_{i+1,j} - \frac{h_i}{\hat{h}_j}(F^{x}_{i+1,j+1} - F^{x}_{i+1,j})\right\}.
\end{aligned}
$$

$S_x(x,y) > 0$ if

$$\sum_{i=0}^{5}(1-\theta)^{5-i}\theta^iA_i > 0 \ and \ (q_i(\theta))^2q_j(\phi) > 0.$$

$(q_i(\theta))^2q_j(\phi) > 0$ if

$$\hat{\alpha}_{i,j} > 0 \ and \ \hat{\beta}_{i,j} > 0.$$

$\sum_{i=0}^{5}(1-\theta)^{5-i}\theta^iA_i > 0$ if $A_i > 0,\ i = 0,1,2,\ldots,5.$

$A_i > 0,\ i = 0, 1, 2, \ldots, 5$ if

$$\begin{aligned}
\hat{\alpha}_{i,j} &> Max\left\{\frac{-\hat{h}_j F^{xy}_{i,j}}{F^x_{i,j}}, \frac{-\hat{h}_j F^{xy}_{i+1,j}}{F^x_{i+1,j}}\right\}.\\
\hat{\beta}_{i,j} &> Max\left\{\frac{\hat{h}_j F^{xy}_{i,j+1}}{F^x_{i,j+1}}, \frac{\hat{h}_j F^{xy}_{i+1,j+1}}{F^x_{i+1,j+1}}\right\}.\\
\alpha_{i,j} &> Max\left\{\frac{F^x_{i,j}}{\Delta_{i,j}}, \frac{F^x_{i,j+1}}{\Delta_{i,j+1}}\right\}.\\
\beta_{i,j} &> Max\left\{\frac{F^x_{i+1,j}}{\Delta_{i,j}}, \frac{F^x_{i+1,j+1}}{\Delta_{i,j+1}}\right\}.
\end{aligned}$$

$S_y(x, y) > 0$ if

$$\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i B_i \ and \ q_i(\theta)(q_j(\phi))^2.$$

$q_i(\theta)(q_j(\phi))^2 > 0$ if

$$\alpha_{i,j} > 0 \ and \ \beta_{i,j} > 0.$$

$\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i B_i > 0$ if $B_i > 0,\ i = 0, 1, 2, 3.$
$B_i > 0,\ i = 0, 1, 2, 3$ if

$$\begin{aligned}
\hat{\alpha}_{i,j} &> Max\left\{\frac{F^y_{i,j}}{\hat{\Delta}_{i,j}}, \frac{F^y_{i+1,j}}{\hat{\Delta}_{i+1,j}}\right\}.\\
\hat{\beta}_{i,j} &> Max\left\{\frac{F^y_{i,j+1}}{\hat{\Delta}_{i,j}}, \frac{F^y_{i+1,j+1}}{\hat{\Delta}_{i+1,j}}\right\}.\\
\alpha_{i,j} &> Max\left\{\frac{-h_i F^{xy}_{i,j}}{F^y_{i,j}}, \frac{-h_i F^{xy}_{i,j+1}}{F^y_{i,j+1}}\right\},\\
\beta_{i,j} &> Max\left\{\frac{h_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \frac{h_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}\right\}.
\end{aligned}$$

Therefore, we can conclude the above discussion in the following theorem:

Theorem 5.1. The rational bicubic function defined in (3) preserves the shape of monotonic data if in each rectangular patch $I_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, free parameters $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ satisfy the following conditions:

$$\begin{aligned}
\hat{\alpha}_{i,j} &= l_{i,j} + Max\left\{0, \frac{-\hat{h}_j F^{xy}_{i,j}}{F^x_{i,j}}, \frac{-\hat{h}_j F^{xy}_{i+1,j}}{F^x_{i+1,j}}, \frac{F^y_{i,j}}{\hat{\Delta}_{i,j}}, \frac{F^y_{i+1,j}}{\hat{\Delta}_{i+1,j}}\right\},\ l_{i,j} > 0.\\
\hat{\beta}_{i,j} &= m_{i,j} + Max\left\{0, \frac{\hat{h}_j F^{xy}_{i,j+1}}{F^x_{i,j+1}}, \frac{\hat{h}_j F^{xy}_{i+1,j+1}}{F^x_{i+1,j+1}}, \frac{F^y_{i,j+1}}{\hat{\Delta}_{i,j}}, \frac{F^y_{i+1,j+1}}{\hat{\Delta}_{i+1,j}}\right\},\ m_{i,j} > 0.\\
\alpha_{i,j} &= n_{i,j} + Max\left\{0, \frac{F^x_{i,j}}{\Delta_{i,j}}, \frac{F^x_{i,j+1}}{\Delta_{i,j+1}}, \frac{-h_i F^{xy}_{i,j}}{F^y_{i,j}}, \frac{-h_i F^{xy}_{i,j+1}}{F^y_{i,j+1}}\right\},\ n_{i,j} > 0.\\
\beta_{i,j} &= k_{i,j} + Max\left\{0, \frac{F^x_{i+1,j}}{\Delta_{i,j}}, \frac{F^x_{i+1,j+1}}{\Delta_{i,j+1}}, \frac{h_i F^{xy}_{i+1,j}}{F^y_{i+1,j}}, \frac{h_i F^{xy}_{i+1,j+1}}{F^y_{i+1,j+1}}\right\},\ k_{i,j} > 0.
\end{aligned}$$

6. **Demonstration**

The monotonic data set for first example is shown in Table 1.

Table 1

x	1	1.5	4	4.5	5
y	5	10	17	20	30

The Figure 1 is produced by the default setting of the parameters α_i and β_i satisfying the monotonicity conditions derived in section 3 with $l_i = m_i = 0.6$. The monotonic data for second example is in Table 2.

Table 2

x	0.1	0.2	0.3	2	3	3.5
y	1.1052	1.2214	1.3499	7.3891	20.0855	33.1155

The Figure 2 is produced by the default setting of the parameters α_i and β_i satisfying the monotonicity conditions derived in section 3 with $l_i = m_i = 0.4$.

Data set in Table 3 is generated from the following function:

$$F = x^3 + y^3.$$

Figure 3 is produced by default setting of free parameters $\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ satisfying the monotonicity conditions derived in section 5 with $l_{i,j} = m_{i,j} = n_{i,j} = k_{i,j} = 0.5$.

The data set in Table 4 is generated by Cobb-Douglas production function defined as:

$$P(L,K) = 1.01L^{0.75}K^{0.25}.$$

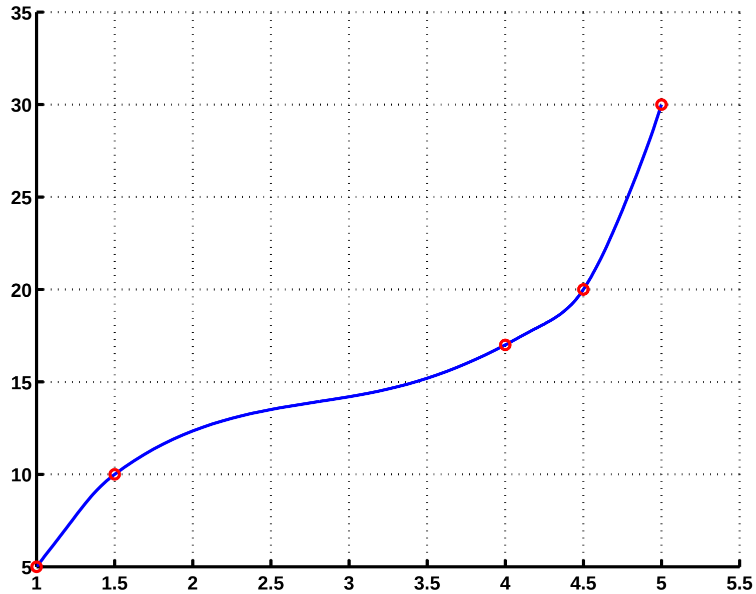

FIGURE 1

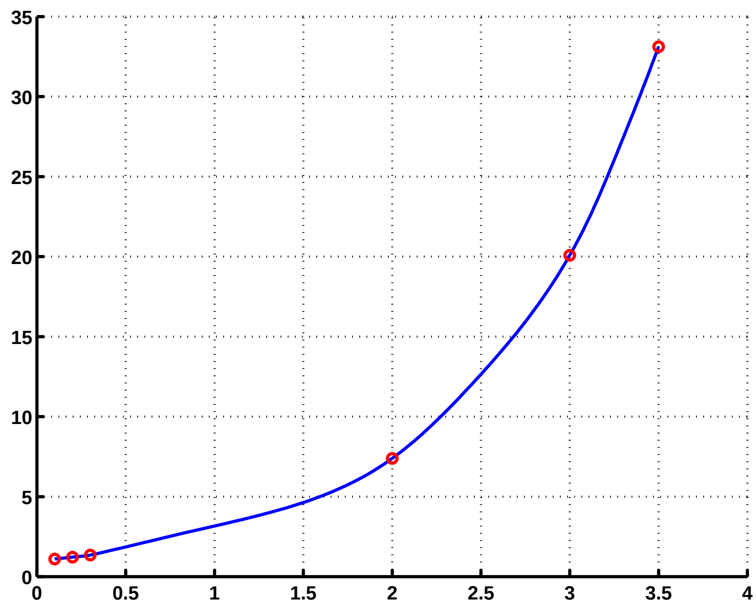

FIGURE 2

TABLE 3

y/x	-3	-2	-1	1	2	3
-3	-54	-35	-28	-26	-19	0
-2	-35	-16	-9	-7	0	19
-1	-28	-9	-2	0	7	26
1	-26	-7	0	2	9	28
2	-19	0	7	9	16	35
3	0	19	26	28	35	54

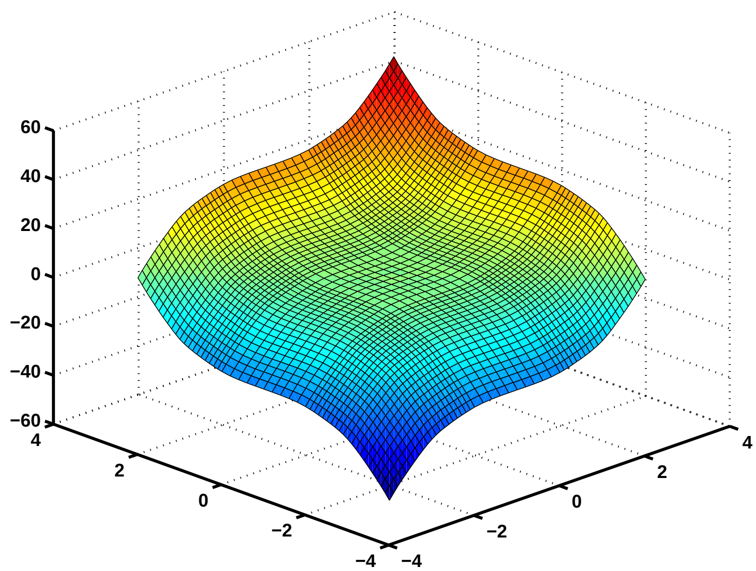

FIGURE 3

This function represents the growth in American economy where L is the amount of labour, K is the amount of capital invested and P is the total production. It was observed that production increases as either L or K increases.

Visual model in Figure 4 is produced by default setting of free parameters

TABLE 4

L/K	1	100	200	300
1	1.0100	3.1939	3.7982	4.2034
100	31.9390	101.0000	120.1099	132.9235
200	53.7148	169.8611	202.0000	223.5497
300	72.8052	230.2302	273.7914	303.0000

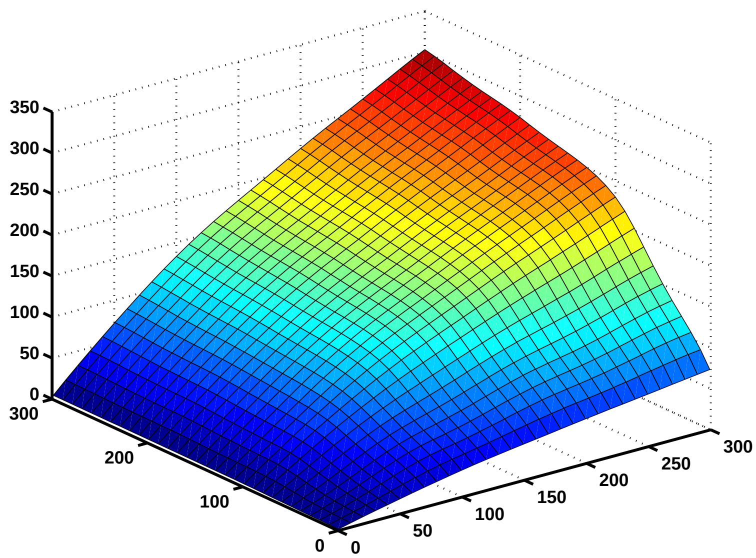

Figure 4

$\alpha_{i,j}$, $\beta_{i,j}$, $\hat{\alpha}_{i,j}$ and $\hat{\beta}_{i,j}$ satisfying the monotonicity conditions derived in section 5 with $l_{i,j} = m_{i,j} = n_{i,j} = k_{i,j} = 1$.

7. Conclusion

In this paper, the problem of monotonicity preservation of surfaces is discussed. Simple constraints are developed on free parameters in the description of rational bicubic function to preserve the shape of monotonic data. Choice of the derivative parameters is left at the wish of the user as well. The method is local, computationally economical and visually pleasing.

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