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A CYCLE OR JAHANGIR RAMSEY UNSATURATED GRAPHS

KASHIF ALI[1], SURAHMAT [1,2]

ABSTRACT. A graph is Ramsey unsaturated if there exists a proper supergraph of the same order with the same Ramsey number, and Ramsey saturated otherwise. We present some result concerning both Ramsey saturated and unsaturated graph. In particular, we show that a cycle C_n and a Jahangir J_m Ramsey unsaturated or saturated graphs of $R(C_n, W_m)$ and $R(P_n, J_m)$, respectively. We also suggest an open problems.

Key words : Ramsey number, cycle, wheel, Jahangir, unsaturated.
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1. **Introduction**

Throughout the paper, all graphs are finite and simple. Let G be such a graph. We write $V(G)$ or V for the vertex set of G and $E(G)$ or E for the edge set of G. For given graphs G and H, the *Ramsey number* $R(G, H)$ is the smallest positive integer N such that for every graph F of order N the following holds: either F contains G as a subgraph or the complement of F contains H as a subgraph. Chvátal and Harary [3] established a useful lower bound for finding the exact Ramsey numbers $R(G, H) \geq (c(G) - 1)(\chi(H) - 1) + 1$, where $c(G)$ is the number of vertices of the largest component of G and $\chi(H)$ is the chromatic number of H. Since then the Ramsey numbers $R(G, H)$ for many combination of graphs G and H have been extensively studied by various authors, see nice survey paper "*Small Ramsey Numbers*" in [4]. In particular, the Ramsey numbers for combination involving cycles and wheels have also been investigated.

Let P_n be a path with n vertices, C_n be a acycle with n vertices, W_m be a wheel of $m + 1$ vertices, i.e., a graph consisting of a cycle C_m with one

[1] School of Mathematical Science, GC University, Lahore, 68-B, New Muslim Town, Lahore, Pakistan, E-mail: akashif@sms.edu.pk.
[1,2] Department of Mathematics Education, Universitas Islam Malang, Jalan MT Haryono 193, Malang 65144, Indonesia, E-mail: caksurahmat@yahoo.com.

additional vertex adjacent to all vertices of C_m, and J_m be a Jahangir graph on $m+1$ vertices and m even; namely, a graph consisting of a cycle C_m with one additional vertex adjacent alternatively to $\frac{m}{2}$ vertices of C_m, as in Fig.1[1] In the Fig.1, a wheel W_{16} and a Jahangir graph J_{16} are drawn. In what follows we determine that a cycle C_n and Jahangir graphs J_m are Ramsey unsaturated or saturated.

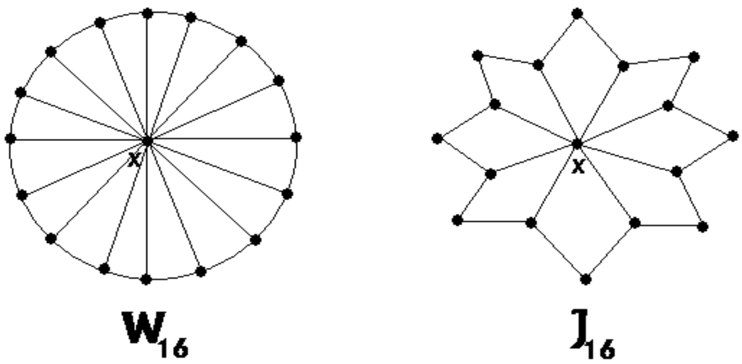

FIGURE 1. A wheel graph W_{16} and a Jahangir graph J_{16}.

Burr and Erdős [2] showed that $R(C_3, W_m) = 2m + 1$ for each $m \geq 5$. Ten years later Radziszowski and Xia [5] gave a simple and unified method to establish the Ramsey number $R(C_3, G)$, where G is either a path, a cycle or a wheel. Surahmat et al. [8] showed $R(C_4, W_m) = 9, 10$ and 9 for $m = 4, 5$ and 6 respectively. Independently, Tse [12] showed $R(C_4, W_m) = 9, 10, 9, 11, 12, 13, 14, 15$ and 17 for $m = 4, 5, 6, 7, 8, 9, 10, 11$ and 12, respectively. Recently, in [7] the Ramsey numbers of cycles versus small wheels were obtained, e.g., $R(C_n, W_4) = 2n - 1$ for $n \geq 5$ and $R(C_n, W_5) = 3n - 2$ for $n \geq 5$.

In this paper, another growth question is addressed but of a different nature. Given a graph G, is there a nontrivial supergraph of the same order with the same Ramsey number? This motivates the following definition.

Definition 1. *Given graphs G and H. The graph G is said to be Ramsey unsaturated of $R(G, H)$ if there exists an edge $e \in E(\overline{G})$ such that $R(G + e, H) = R(G, H)$; and the graph G is said to be Ramsey saturated of $R(G, H)$ if $R(G + e, H) > R(G, H)$ for all $e \in E(\overline{G})$.*

[1]The figure J_{16} appears on Jahangir's tomb in his mausoleum, it lies in 5 km north-west of Lahore, Pakistan across the River Ravi. His tomb was built by his Queen Noor Jehan and his son Shah-Jehan (This was emperor who constructed one of the wonder of world Taj Mahal in India) around 1637 A.D. It has a majestic structure made of red sand-stone and marble.

Definition 2. *Given graphs G and H. The graph H is said to be Ramsey unsaturated of $R(G,H)$ if there exists an edge $e \in E(\overline{H})$ such that $R(G,H+e) = R(G,H)$; and the graph H is said to be Ramsey saturated of $R(G,H)$ if $R(G,H+e) > R(G,H)$ for all $e \in E(\overline{H})$.*

A cycle is particulary interesting. Given a cycle C_n on n vertices call xy a $k-$chords if the distance between x and y on C_n is k. A Jahangir J_m call xy a $k-$chords if the distance between x and y on C_m of J_m is k. $J_m + k-$chords is obtained from cycle C_m of J_m adding one edge xy a $k-$chords. The aim of this paper is to determine that a cycle C_n and a Jahangir J_m are Ramsey unsaturated of $R(C_n, W_m)$ and $R(P_n, J_m)$ respectively by using $k-$chord.

In order to prove the theorems in the main results, we need the following known results.

Theorem 1. *(Surahmat, Baskoro and I. Tomescu* [9]*). $R(C_n, W_m) = 2n-1$ if even $m \geq 4$ and $n \geq \frac{5m}{2} - 1$.*

Theorem 2. *(Surahmat, Baskoro and I. Tomescu* [10]*). $R(C_n, W_m) = 3n-2$ if odd $m \geq 4$ and $n > \frac{5m-9}{2}$.*

Theorem 3. *(Surahmat and I. Tomescu* [11]*).*

$$R(P_n, J_m) = \begin{cases} 6 & \text{if } (n,m) = (4,4), \\ n+1 & \text{if } m = 4 \text{ and } n \geq 5, \\ n + \frac{m}{2} - 1 & \text{if } m \geq 6 \text{ is even and } n \geq (2m-1)(\frac{m}{2}-1)+1. \end{cases}$$

Theorem 4. *(E.T. Baskoro and Surahmat* [6, 1]*). $R(P_n, W_m) = 2n-1$ if $m = 4$ and $n \geq 4$ or $m \geq 6$ is even and $n \geq \frac{m}{2}(m-1)+1$.*

2. Main Results

The main results of this paper are the following.

Theorem 5. *A cycle C_n is Ramsey unsaturated of $R(C_n, W_m)$ for even $m \geq 4$ and even $n \geq \frac{5m}{2}$ or odd $n \geq \frac{5m}{2}+1$.*

Proof. Let $F = 2K_{n-1}$ of order $2n-2$. We have that F contains no $C_n + e$ and $\overline{F}$ contains no W_m for even $m \geq 4$. Thus, we have $R(C_n + e, W_m) \geq 2n-1$. Now we shall show $R(C_n + e, W_m) \leq 2n-1$.

Let G be a graph of order $2n-1$ where even $n \geq \frac{5m}{2}$ (or odd $n \geq \frac{5m}{2}+1$) for even $m \geq 4$ containing no $C_n + e$. We shall show that $\overline{G}$ contains W_m. By contradiction, suppose $\overline{G}$ contains no W_m. By Theorem 1, since $R(C_n, W_m) = 2n-1$ for even $m \geq 4$ and $n \geq \frac{5m}{2}-1$, then G contains C_n. Let $A = \{x_1, x_2, \ldots, x_n\}$ is a set of cycle C_n which $x_ix_{i+1}, x_nx_1 \in E(G)$ for each $i = 1, 2, \ldots, n-1$. If n is even, we get an independent set $A^1 = \{x_1, x_3, \ldots, x_{n-1}\}$ in G and also $|A^1| = \frac{n}{2} \geq \frac{\frac{5m}{2}}{2} = \frac{5m}{4} \geq m+1$ which implies $\overline{G} \supseteq K_{m+1} \supseteq W_m$, a contradiction. Nextly, if n is odd, we obtain an independent set $A^2 = \{x_1, x_3, \ldots, x_{n-2}\}$ in G and also $|A^2| = \frac{n-1}{2} \geq \frac{(\frac{5m}{2}+1)-1}{2} = \frac{5m}{4} \geq m+1$ which implies $\overline{G} \supseteq K_{m+1} \supseteq W_m$, a contradiction. Thus, we have $R(C_n+e, W_m) = R(C_n, W_m)$. This completes the proof. □

Theorem 6. *A cycle C_n is Ramsey unsaturated of $R(C_n, W_m)$ for odd $m \geq 5$ and even $n \geq \frac{5m-1}{2}$ or odd $n \geq \frac{5m+1}{2}$.*

Proof. Let $F = 3K_{n-1}$ of order $3n-3$. We have that F contains no C_n+e and $\overline{F}$ contains no W_m for odd $m \geq 5$. Thus, we have $R(C_n+e, W_m) \geq 3n-2$. Now we shall show $R(C_n+e, W_m) \leq 3n-2$.

Let G be a graph of order $3n-2$ where n is even and $n \geq \frac{5m-1}{2}$ (or n is odd and $n \geq \frac{5m+1}{2}$) for odd $m \geq 5$ containing no C_n+e. We shall show that $\overline{G}$ contains W_m. By contradiction, suppose $\overline{G}$ contains no W_m. By Theorem 2, since $R(C_n, W_m) = 3n-2$ for odd $m \geq 5$ and $n \geq \frac{5m-9}{2}$, then G contains C_n. Let $B = \{y_1, y_2, \ldots, y_n\}$ is a set of cycle C_n which $y_iy_{i+1}, y_ny_1 \in E(G)$ for each $i = 1, 2, \ldots, n-1$. If n is even, we get an independent set $B^1 = \{y_1, y_3, \ldots, y_{n-1}\}$ in G and also $|B^1| = \frac{n}{2} \geq \frac{\frac{5m-1}{2}}{2} = \frac{5m-1}{4} \geq m+1$ which implies $\overline{G} \supseteq K_{m+1} \supseteq W_m$, a contradiction. Nextly, if n is odd, we obtain an independent set $B^2 = \{y_1, y_3, \ldots, y_{n-2}\}$ in G and also $|B^2| = \frac{n-1}{2} \geq \frac{\frac{5m+1}{2}-1}{2} = \frac{5m-1}{4} \geq m+1$ which implies $\overline{G} \supseteq K_{m+1} \supseteq W_m$, a contradiction. Thus, we have $R(C_n+e, W_m) = R(C_n, W_m)$. This completes the proof. □

Theorem 7. *A Jahangir J_m is Ramsey saturated of $R(P_n, J_m)$ for $m = 4$ and $n \geq 4$.*

Proof. To proved this theorem, we consider the Fig. 2 as follows:

Since chromatic numbers of J_4+e are equal to 3, then by Chvátal and Harary [3] we get a lower bound $R(P_n, J_4+e) \geq (c(P_n)-1)(\chi(J_4+e)-1)+1 = (n-1)(3-1)+1 = 2n-1$. Thus by Theorem 3 we get $R(P_n, J_4+e) > R(P_n, J_4)$ for all $e \in E(\overline{J}_4)$. This implies that J_4 is Ramsey saturated of $R(P_n, J_4)$, this completes the proof. □

Theorem 8. *A Jahangir J_m is Ramsey unsaturated of $R(P_n, J_m)$ for $m \geq 6$ and $n \geq \frac{m}{2}(2m-1)+1$.*

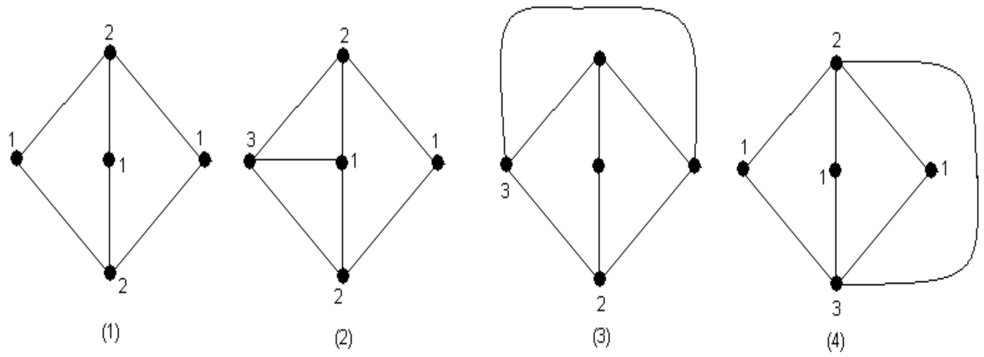

FIGURE 2. $\chi(J_4) = 2$ for (1), and $\chi(J_4 + e) = 3$ for (2), (3) and (4).

Proof. Consider now to proved a Jahangir graph J_m for m is even, $m \geq 6$ and $n \geq \frac{m}{2}(2m-1)+1$. We will show that the Jahangir graph J_m is Ramsey unsaturated of $R(P_n, J_m)$ for even $m \geq 6$. We have $R(P_n, J_m+3-chords) \geq n+\frac{m}{2}-1$ since $P_n \not\subseteq K_{n-1} \bigcup K_{\frac{m}{2}-1}$ and $J_m + e \not\subseteq \overline{K_{n-1} \bigcup K_{\frac{m}{2}-1}}$. It remains to prove that $R(P_n, J_m + 3 - chords) \leq n + \frac{m}{2} - 1$. Let F be a graph of order $n + \frac{m}{2} - 1$ and containing no path P_n. Let $L_1 = \{l_{1,1}, l_{1,2}, \ldots, l_{1,k_1}\}$ be the longest path in F. Then, $zl_{1,1}, zl_{1,k_1} \notin E(F)$ for each $z \in V_1 = V(F)\backslash V(L_1)$. We distinguish two cases:

Case 1. $k_1 \leq 2m - 1$. Since $n \geq \frac{m}{2}(2m - 1) + 1$, we can do the following process. For each $i = 2, \ldots, \frac{m}{2}-1$, let L_i be the longest path in $F[V_{i-1}]$, where $V_{i-1} = V(F)\backslash\bigcup_{j=1}^{i-1} V(L_j)$. By denoting the set of remaining vertices by B, we have $|B| \geq n + \frac{m}{2} - 1 - (\frac{m}{2}(2m-1)+1) \geq \frac{m}{2} \geq 3$ since $m \geq 6$. Let $x, y, z \in B$ be three distinct vertices which are not in any L_j for $j = 1, 2, \ldots, \frac{m}{2} - 1$. Clearly, x, y, z are not adjacent to all endpoints of these L_j. Without loose generalized, assume $|V(L_j)| \geq 2$ for each $j = 1, 2, \ldots, \frac{m}{2} - 1$, and let $l_{j,1}$ and l_{j,k_j} be endpoints of L_j. Thus, $l_{1,1}$ and a cycle C_m form J_m in $\overline{F}$ where $V(C_m) = \{l_{1,k_1}, x, l_{2,k_2}, y, l_{3,k_3}, z, l_{4,k_4}, \ldots, l_{2,1}\}$. Since l_{1,k_1} is not adjacent to any vertex in V_1 then we have $J_m + 2 - chords$ in $\overline{F}$.

Case 2. $k_1 > 2m - 1$. We will define *4−tuple* as below:

$$C_j = \{l_{1,i}, l_{1,i+1}, l_{1,i+2}, l_{1,i+3}\} \text{ for } i \equiv 2 \ mod\ 4 \text{ and } j = \tfrac{i+2}{4}. \qquad (1)$$

Since $k > 2m-1 = 4\frac{m}{2}-1$, we have

$$\begin{array}{lcl} C_1 & = & \{l_{1,2}, l_{1,3}, l_{1,4}, l_{1,5}\}, \\ C_2 & = & \{l_{1,6}, l_{1,7}, l_{1,8}, l_{1,9}\}, \\ & \vdots & \\ C_{\frac{m}{2}-1} & = & \{l_{1,2m-6}, l_{1,2m-5}, l_{1,2m-4}, l_{1,2m-3}\}. \end{array} \quad (2)$$

Let $Y = V(F) \setminus V(L_1)$. We have $|Y| = n + \frac{m}{2} - 1 - k \geq \frac{m}{2}$ since $k \leq n-1$. Hence we can consider $\frac{m}{2}$ distinct elements in $Y : y_1, y_2, \ldots, y_{\frac{m}{2}}$ and $\frac{m}{2}-1$ pairs of elements $Y_i = \{y_i, y_{i+1}\}$ for $i = 1, \ldots, \frac{m}{2}-1$. By the maximality of L_1 it follows that for each $i = 1, \ldots, \frac{m}{2}-1$ at least one vertex in C_i is not adjacent to any vertex in Y_i. Denote by c_i the vertex in C_i which is not adjacent to any vertex in Y_i for $i = 1, \ldots, \frac{m}{2}-1$. We have $\overline{F} \supseteq J_m$, where J_m consists of the cycle C_m having $V(C_m) = \{y_1, c_1, y_2, c_2, \ldots, y_{\frac{m}{2}-1}, c_{\frac{m}{2}-1}, y_{\frac{m}{2}}, l_{1,k_1}\}$ and the hub $l_{1,1}$. Since l_{1,k_1} is not adjacent to any vertex in V_1 then we have $J_m + 2 - chords$ in $\overline{F}$. This completes the proof. □

3. Open Problems

In this section we shall propose in the following some open problems:

(1) Determine the maximal of $r \geq 2$ such that $R(C_n + re, W_m) = R(C_n, W_m)$ for even $m \geq 4$ and even $n \geq \frac{5m}{2}$ or odd $n \geq \frac{5m}{2}+1$.
(2) Determine the maximal $r \geq 2$ such that $R(C_n + re, W_m) = R(C_n, W_m)$ for odd $m \geq 5$ and even $n \geq \frac{5m-1}{2}$ or odd $n \geq \frac{5m+1}{2}$.
(3) Determine the maximal $r \geq 2$ such that $R(P_n, J_m + re) = R(P_n, J_m)$.

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