Journal of Prime Research in Mathematics Volume **3**, *No. (2007): 101–110*

Towards classification of simple finite dimensional modular Lie superalgebras

Dimitry Leites [1)]

Characteristic p is for the time when we retire.

Sasha Beilinson, when we all were young.

1. Introduction

We use standard notations of [FH, S]; for a precise definition of the Cartan prolongation and its generalizations (Cartan-Tanaka-Shchepochkina or CTS-prolongations), see [Shch]; see also [BGL3]–[BGL5], [Leb1, Leb2]. Hereafter $\mathbb{K}$ is an algebraically closed (unless finite) field, $\operatorname{char}\mathbb{K} = p$.

The works of S. Lie, Killing and Cartan, now classical, completed classification over $\mathbb{C}$ of **simple Lie algebras**

of finite dimension and **certain infinite dimensional (of polynomial vector fields, or "vectorial" Lie algebras)**. (1)

In addition to the above two types, there are several more interesting types of simple Lie algebras but they do not contribute to the solution of our problem: **classification of simple finite dimensional modular Lie (super)algebras**, except one: the queer type described below (and, perhaps, examples, for $p = 2$, of the types described in [J, Sh] and their generalizations, if any). Observe that all finite dimensional simple Lie algebras are of the form $\mathfrak{g}(A)$; for their definition embracing the modular case and the classification, see [BGL5].

Lie algebras and Lie superalgebras over fields in characteristic $p > 0$, a.k.a. *modular* Lie (super)algebras, were distinguished in topology in the 1930s. The **simple** Lie algebras drew attention (over finite fields $\mathbb{K}$) as a byproduct of classification of simple finite groups, cf. [St]. Lie **super**algebras, even simple ones, did not draw much attention of mathematicians until their (outstanding) usefulness was observed by physicists in the 1970s. Researchers discovered more and more of new examples of simple modular Lie algebras for decades

[1)]MPIMiS, Inselstr. 22, DE-04103 Leipzig, Germany
on leave from Department of Mathematics, University of Stockholm, Roslagsv. 101, Kräftriket hus 6, SE-106 91 Stockholm, Sweden; mleites@math.su.se

I am thankful to S. Bouarroudj, P. Grozman, A. Lebedev, I. Shchepochkina, and also to V. Serganova and Yu. Kochetkov for help; MPIMiS, Leipzig, for financial support and most creative environment during 2004-06 when I was Sophus-Lie-Professor there.

until Kostrikin and Shafarevich ([KSh]) formulated a **conjecture** embracing all previously found examples for $p > 7$. The **generalized KSh–conjecture** states (for a detailed formulation, convenient to work with, see [Leb2]):

Select a $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$ of every $\mathfrak{g}$ of type[2] (1)*, take $\mathfrak{g}_{\mathbb{K}} := \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}$ and its simple subquotient $\mathfrak{si}(\mathfrak{g}_{\mathbb{K}})$ (for the Lie algebras of vector fields, there are several, depending on $\underline{N}$). Together with deformations*[3] *of these examples we get in this way all simple finite dimensional Lie algebras over algebraically closed fields if $p > 5$. If $p = 5$, Melikyan's examples*[4] *should be added to the examples obtained by the above method.*

After 30 years of work of several teams of researchers, Block, Wilson, Premet and Strade proved the generalized KSh conjecture for $p > 3$, see [S].

Even before the KSh conjecture was proved, its analog was offered in [KL] for $p = 2$. Although the KL conjecture was, as is clear now, a bit overoptimistic (in plain terms: wrong, as stated), it suggested a way to get such an abundance of examples (to verify which of them are really simple is one of the tasks still open) that Strade [S] cited [KL] as an indication that the case $p = 2$ is too far out of reach by modern means[5]. Still, [KL] made two interesting observations: It pointed at a striking similarity (especially for $p = 2$) between modular Lie algebras and Lie **super**algebras (even over $\mathbb{C}$), and it introduced totally new characters — *Volichenko algebras* (inhomogeneous with respect to parity subalgebras of Lie superalgebras); for the classification of simple Volichenko algebras (finite dimensional and infinite dimensional vectorial) over $\mathbb{C}$, see [LSer] (where one of the most interesting examples is missed, the version of the proof with repair will be put in `arXiv` soon).

Recently Strade had published a monograph [S] summarizing the description of newly classified simple finite dimensional Lie algebras over the algebraically closed fields $\mathbb{K}$ of characteristic $p > 3$, and also gave an overview of the "mysterious" examples (due to Brown, Frank, Ermolaev and Skryabin) of simple finite dimensional Lie algebras for $p = 3$ with no counterparts for $p > 3$. Several researchers started afresh to work on the cases where $p = 2$ and 3, and new examples of simple Lie algebras with no counterparts for $p \neq 2, 3$ started

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[2] Notice that the modular analog of the polynomial algebra—the algebra of divided powers—*and all prolongs* (vectorial Lie algebras) acquire for $p > 0$ one more (shearing) parameter $\underline{N}$.

[3] It is not clear, actually, if the conventional description of infinitesimal deformations in terms of $H^2(\mathfrak{g};\mathfrak{g})$ can always be applied if $p > 0$. This concerns both Lie algebras and Lie superalgebras (for the arguments, see [LL]); to give the correct (better say, universal) notion is an open problem, but we let it pass for the moment, besides, for $p \neq 2$ and $\mathfrak{g}$ with Cartan matrix, the conventional interpretation is applicable, see [BGL4].

[4] For their description as prolongs, and newly discovered super versions, see [GL4, BGL3].

[5] Contrarywise, **the "punch line" of this talk is: Cartan did not have the modern root technique, but got the complete list of simple Lie algebras; let's use his "old-fashioned" methods: they work!** Conjecture 2 expresses our hope in precise terms. How to *prove* the completeness of the list of examples we will have unearthed is another story.

to appear ([J, GL4, Leb1], observe that the examples of [GG, Lin1] are erroneous as observed in MathRevies and [Leb2], respectively). The "mysterious" examples of simple Lie algebras for $p = 3$ were interpreted as vectorial Lie algebras preserving certain distributions ([GL4]).

While writing [GL4] we realized, with considerable dismay, that there are reasons to put to doubt the universal applicability of the conventional definitions of the enveloping algebra $U(\mathfrak{g})$ (and its restricted version) of a given Lie algebra $\mathfrak{g}$, and hence doubt in applicability of the conventional definitions of Lie algebra representations and (co)homology to the modular case, cf. [LL]. But even accepting conventional definitions, there are plenty of problems to be solved before one will be able to start writing the proof of classification of simple modular Lie algebras, to wit: describe irreducible representations (as for vectorial Lie superalgebras, see [GLS]), decompose the tensor product of irreducible representations into indecomposables, cf. [Cla], and many more; for a review, see [GL2].

Classification of simple Lie **super**algebras for $p > 0$ and the study of their representations are of independent interest. A conjectural list of simple finite dimensional Lie superalgebras over an algebraically closed fields $\mathbb{K}$ for $p > 5$, known for some time, was recently cited in [BjL]:

Conjecture 1 (Super KSh, $p > 5$). *Apply the steps of the KSh conjecture to the simple complex Lie* **super***algebras* $\mathfrak{g}$ *of types* (1). *The examples thus obtained exhaust all simple finite dimensional Lie superalgebras over algebraically closed fields if* $p > 5$.

The examples obtained by this procedure will be referred to as *KSh-type Lie superalgebras*. The first step towards obtaining the list of KSh-type Lie superalgebras is classification of simple Lie superalgebras of types (1) over $\mathbb{C}$. This is done by I.Shchepochkina and me; for summaries with somewhat different emphases, and proof, see [K2, K3, LSh].

For a classification of finite dimensional simple modular Lie algebras with Cartan matrix, see [WK, KWK]. For a classification of finite dimensional simple modular Lie superalgebras with Cartan matrix, see [BGL5]. Not all finite dimensional Lie superalgebras over $\mathbb{C}$ are of the form $\mathfrak{g}(A)$; in addition to them, there are also queer types described below, and even simple vectorial.

I am sure that the same ideas of Block and Wilson that proved classification of simple *restricted* Lie algebras for $p > 5$ will work, if $p > 5$, *mutatis mutandis*, for Lie superalgebras and ideas of Premet and Strade will embrace the nonrestricted superalgebras as well; although the definition of *restrictedness* and even of the Lie superalgebra itself acquire more features, especially for $p = 2$.

Here I will describe the cases $p \leq 5$ where the situation is different and suggest *another*, different from KSh, way to get simple examples.

2. How to construct simple Lie algebras and superalgebras

2.1. How to construct simple Lie algebras if $p = 0$. Let us recall how Cartan used to construct simple $\mathbb{Z}$-graded Lie algebras over $\mathbb{C}$ of polynomial growth [C] and finite depth. Now that they are classified (for examples of infinite depth, see [K]), we know that, *all of them can be endowed with a $\mathbb{Z}$-grading $\mathfrak{g} = \mathop{\oplus}\limits_{-d\leq i} \mathfrak{g}_i$ of depth $d = 1$ or 2 so that $\mathfrak{g}_0$ is a simple Lie algebra $\mathfrak{s}$ or its trivial central extension $\mathfrak{cs} = \mathfrak{s} \oplus \mathfrak{c}$, where $\mathfrak{c}$ is a 1-dimensional center. Moreover, simplicity of $\mathfrak{g}$ requires $\mathfrak{g}_{-1}$ to be an irreducible $\mathfrak{g}_0$-module that generates $\mathfrak{g}_- := \mathop{\oplus}\limits_{i<0} \mathfrak{g}_i$ and $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$.*

Yamaguchi's theorem [Y], reproduced in [GL4, BjL], states that *for almost all simple finite dimensional Lie algebras $\mathfrak{g}$ over $\mathbb{C}$ and their $\mathbb{Z}$-gradings $\mathfrak{g} = \mathop{\oplus}\limits_{-d\leq i} \mathfrak{g}_i$, the generalized Cartan prolong of $\mathfrak{g}_{\leq} = \mathop{\oplus}\limits_{-d\leq i\leq 0} \mathfrak{g}_i$ is isomorphic to $\mathfrak{g}$, the rare exceptions being two of the four series of simple vectorial algebras; the other two series being* **partial** *prolongs (perhaps, after factorization modulo center).*

For illustration, we construct simple Lie algebras of type (1) over $\mathbb{C}$ by induction:

<u>**Depth** $d = 1$</u>. Here we use either usual or partial Cartan prolongations.

1) we start with 1-dimensional $\mathfrak{c}$, so $\dim \mathfrak{g}_{-1} = 1$ due to irreducibility. The complete prolong is isomorphic to $\mathfrak{vect}(1)$, the partial one to $\mathfrak{sl}(2)$.

2) Take $\mathfrak{g}_0 = \mathfrak{csl}(2) = \mathfrak{gl}(2)$ and its irreducible module $\mathfrak{g}_{-1}$. The component $\mathfrak{g}_1$ of the Cartan prolong is nontrivial only if $\mathfrak{g}_{-1}$ is $R(\varphi_1)$ or $R(2\varphi_1)$, where φ_i is the ith fundamental weight of the simple Lie algebra $\mathfrak{g}$ and $R(w)$ is the irreducible representation with highest weight w.

2a) If $\mathfrak{g}_{-1}$ is $R(\varphi_1)$, the component $\mathfrak{g}_1$ consists of two irreducible submodules, say $\mathfrak{g}_1'$ or $\mathfrak{g}_1''$. We can take any one of them or both; together with $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ this generates $\mathfrak{sl}(3)$ or $\mathfrak{svect}(2) \mathbin{\subset\!\!\!\!+} \mathfrak{d}$, where $\mathfrak{d}$ is spanned by an outer derivation, or $\mathfrak{vect}(2)$, respectively.

2b) If $\mathfrak{gl}(2) \simeq \mathfrak{co}(3) \simeq \mathfrak{csp}(2)$-module $\mathfrak{g}_{-1}$ is $R(2\varphi_1)$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{o}(5) \simeq \mathfrak{sp}(4)$.

3) Induction: Take $\mathfrak{g}_0 = \mathfrak{csl}(n) = \mathfrak{gl}(n)$ and its irreducible module $\mathfrak{g}_{-1}$. The component $\mathfrak{g}_1$ of the Cartan prolong is nontrivial only if $\mathfrak{g}_{-1}$ is $R(\varphi_1)$ or $R(2\varphi_1)$ or $R(\varphi_2)$.

3a) If $\mathfrak{g}_{-1} = R(\varphi_1)$, then $\mathfrak{g}_1$ consists of two irreducible submodules, $\mathfrak{g}_1'$ or $\mathfrak{g}_1''$. Take any of them or both; together with $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ this generates $\mathfrak{sl}(n+1)$ or $\mathfrak{svect}(n) \mathbin{\subset\!\!\!\!+} \mathfrak{d}$, where $\mathfrak{d}$ is spanned by an outer derivation, or $\mathfrak{vect}(n)$, respectively.

3b) If $\mathfrak{g}_{-1} = R(2\varphi_1)$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{sp}(2n)$.

3c) If $\mathfrak{g}_{-1} = R(\varphi_2)$, then $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{o}(2n)$.

4) The induction with $\mathfrak{g}_0 = \mathfrak{co}(2n-1)$-module $R(\varphi_1)$ returns $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{o}(2n+1)$. Observe that $\mathfrak{sl}(4) \simeq \mathfrak{o}(6)$. The induction with $\mathfrak{g}_0 = \mathfrak{co}(2n)$-module $R(\varphi_1)$ returns $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \simeq \mathfrak{o}(2n+2)$. (We have obtained $\mathfrak{o}(2n)$ twice; analogously, there many ways to obtain other simple Lie algebras as prolongs.)

5) The $\mathfrak{g}_0 = \mathfrak{sp}(2n)$-module $\mathfrak{g}_{-1} = R(\varphi_1)$ yields the Lie algebra $\mathfrak{h}(2n)$ of Hamiltonian vector fields.

<u>$\mathfrak{e}(6)$, $\mathfrak{e}(7)$</u>. The $\mathfrak{g}_0 = \mathfrak{co}(10)$-module $\mathfrak{g}_{-1} = R(\varphi_1)$ yields $\mathfrak{e}(6)$; the $\mathfrak{g}_0 = \mathfrak{ce}(6)$-module $\mathfrak{g}_{-1} = R(\varphi_1)$ yields $\mathfrak{e}(7)$.

<u>**Depth** $d = 2$</u>. Here we need **generalized** prolongations, see [Shch]. Again there are just a few algebras $\mathfrak{g}_0$ and $\mathfrak{g}_0$-modules $\mathfrak{g}_{-1}$ for which $\mathfrak{g}_1 \neq 0$ and $\mathfrak{g} = \oplus\mathfrak{g}_i$ is simple:

$\underline{\mathfrak{g}(2);\ \mathfrak{f}(4);\ \mathfrak{e}(8)}$. These Lie algebras correspond to the prolongations of their non-positive part (with $\mathfrak{g}_0$ being isomorphic to $\mathfrak{gl}(2)$; $\mathfrak{o}(6)$ or $\mathfrak{sp}(6)$; $\mathfrak{e}(7)$ or $\mathfrak{o}(14)$, respectively) in the following $\mathbb{Z}$-gradings. Let the nodes of the Dynkin graph of $\mathfrak{g}$ be rigged out with the coefficients of linear dependence of the maximal root with respect to simple ones. If any end node is rigged out with a 2, mark it (mark only one node even if several are rigged out with 2's) and set the degrees of the Chevalley generators to be:

$$\deg X_i^{\pm} = \begin{cases} \pm 1 & \text{if the } i\text{th node is marked} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

$\underline{\mathfrak{k}(2n+1)}$. The cases where $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is simple and of infinite dimension correspond to the prolongations of the non-positive part of $\mathfrak{sp}(2n+2)$ in the $\mathbb{Z}$-grading (2) with the last node marked. Then $\mathfrak{g}_0 = \mathfrak{csp}(2n)$, $\mathfrak{g}_{-1} = R(\varphi_1)$ and $\mathfrak{g}_{-1}$ is the trivial $\mathfrak{g}_0$-module. In these cases, $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \mathfrak{k}(2n+1)$.

2.2. Superization.

2.2.1. Queerification. This is the functor $Q : A \longrightarrow Q(A) := A[\varepsilon]$, where

$$p(\varepsilon) = \bar{1},\ \varepsilon^2 = -1 \ \text{ and } \varepsilon a = (-1)^{p(a)} a\varepsilon \text{ for any } a \in A.$$

We set $\mathfrak{q}(n) = Q(\mathfrak{gl}(n))$.

2.2.2. Definition of Lie superalgebras for $p = 2$. A Lie superalgebra for $p = 2$ as a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that $\mathfrak{g}_{\bar{0}}$ is a Lie algebra, $\mathfrak{g}_{\bar{1}}$ is an $\mathfrak{g}_{\bar{0}}$-module (made into the two-sided one by symmetry; more exactly, by anti-symmetry, but if $p = 2$, it is the same) and on $\mathfrak{g}_{\bar{1}}$ a *squaring* (roughly speaking, the halved bracket) is defined

$$\begin{gathered} x \mapsto x^2 \quad \text{such that } (ax)^2 = a^2 x^2 \text{ for any } x \in \mathfrak{g}_{\bar{1}} \text{ and } a \in \mathbb{K}, \text{ and} \\ (x+y)^2 - x^2 - y^2 \text{ is a bilinear form on } \mathfrak{g}_{\bar{1}} \text{ with values in } \mathfrak{g}_{\bar{0}}. \end{gathered} \tag{3}$$

For any $x, y \in \mathfrak{g}_{\bar{1}}$, we set

$$[x, y] := (x+y)^2 - x^2 - y^2. \tag{4}$$

We also assume, as usual, that

if $x, y \in \mathfrak{g}_{\bar{0}}$, then $[x, y]$ is the bracket on the Lie algebra;

if $x \in \mathfrak{g}_{\bar{0}}$ and $y \in \mathfrak{g}_{\bar{1}}$, then $[x, y] := l_x(y) = -[y, x] = -r_x(y)$, where l and r are the left and right $\mathfrak{g}_{\bar{0}}$-actions on $\mathfrak{g}_{\bar{1}}$, respectively.

The *Jacobi identity* involving odd element has now the following form:

$$[x^2, y] = [x, [x, y]] \text{ for any } x \in \mathfrak{g}_{\bar{1}}, y \in \mathfrak{g}. \tag{5}$$

Conjecture 2 (Amended KL = Super KSh, $p > 0$). *For $p > 0$, to get all* **$\mathbb{Z}$-graded** *simple finite dimensional examples of Lie algebras and Lie superalgebras:*

(a) apply the KSh procedure to every simple Lie algebra of type (1) over $\mathbb{C}$ (if $p = 2$, apply the KSh procedure also to every simple Lie superalgebra of of type (1) over $\mathbb{C}$ and their simple Volichenko subalgebras described in [LSer]*),*

(b) if $p = 2$, apply queerification (as in [Leb2]*) to the results of (a);*

(c) if $p = 2$, take Jurman's examples [J] *(and generalizations of the same construction, if any: It looks like a specific $p = 2$ non-super version of the queerification);*

(d) take the non-positive part of every simple (up to center) finite dimensional $\mathbb{Z}$-graded algebra obtained at steps (a)–(c) and (for $p = 5$, 3 and 2) the

exceptional ones of the form $\mathfrak{g}(A)$ *listed in* [BGL5]*, consider its complete and partial*[6)] *prolongs and distinguish their simple subquotients.*

To get non-graded examples, we have to take deformations of the simple algebras obtained at steps (a)–(d).

For preliminary results, see [GL4, BjL], [BGL3]–[BGL5], [ILL, Leb2]. (For $p = 3$ and Lie algebras, this is how Grozman and me got an interpretation of all the "mysterious" exceptional simple vectorial Lie algebras known before [GL4] was published; we also found two (if not three) series of new simple algebras.) Having obtained a supply of such examples, we can sit down to c o m p u t e c e r t a i n c o h o m o l o g y i n o r d e r t o d e s c r i b e t h e i r d e f o r m a t i o n s (provided we will be able to understand what we are computing, cf. footnote 2); for the already performed, see [KKCh, KuCh, Ch, BGL4]. In particular, Shens variations [Sh] are just good old deformations.

3. Further details

3.1. How to construct finite dimensional simple Lie algebras if $p \geq 5$. Observe that although in the modular case there is a wider variety of pairs $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ yielding nontrivial prolongs than for $p = 0$ (for the role of $\mathfrak{g}_0$ we can now take vectorial Lie algebras or their central extensions), *a posteriori* we know that we can always confine ourselves to the same pairs $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ as for $p = 0$. Melikyan's example looked as a deviation from the pattern, but Kuznetsov's observation [Ku1] elaborated in [GL4] shows that for $p \geq 5$ all is the same. Not so if $p \leq 3$:

3.2. New simple finite dimensional Lie algebras for $p = 3$. In [S], Strade listed known to him at that time examples of simple finite dimensional Lie algebras for $p = 3$. The construction of such algebras is usually subdivided into the following types and deformations of these types:

(1) algebras with Cartan matrix CM (sometimes encodable by Dynkin graphs, cf. [S, BGL5]),

(2) algebras of vectorial type (meaning that they have more roots of one sign than of the other with respect to a partition into positive and negative roots).

Case (1) was solved in [WK, KWK].

Conjecture 2 suggests to consider certain $\mathbb{Z}$-graded prolongs $\mathfrak{g}$. For Lie algebras and $p = 3$, Kuznetsov described various restrictions on the 0-th component of $\mathfrak{g}$ and the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ (for partial summary, see [GK, Ku1, Ku2], [BKK] and a correction in [GL4]). **What are these restrictions for Lie algebras for $p = 2$? What are they for Lie superalgebras for any $p > 0$?**

[6)] This term is too imprecise at the moment: it embraces Frank and Ermolaev algebras, Frank and other exceptional superalgebras ([BGL6, BGL7]).

3.3. Exceptional simple finite dimensional Lie superalgebras for $p \leq 5$. Elduque investigated which spinor modules over orthogonal algebras can serve as the odd part of a simple Lie superalgebra and discovered an exceptional simple Lie superalgebra for $p = 5$. Elduque also superized the Freudenthal Magic Square and expressed it in a new way, and his approach yielded nine new simple (exceptional as we know now thanks to the classification [BGL5]) finite dimensional Lie superalgebras for $p = 3$, cf. [CE, El1, CE2]. These Lie superalgebras possess Cartan matrices (CM's) and we described all CMs and presentations of these algebras in terms of Chevalley generators, see [BGL5] and references in it. In [BGL5] 12 more examples of exceptional simple Lie superalgebras are discovered; in [BGL3], we considered some of their "most promising" (in terms of prolongations) $\mathbb{Z}$-gradings and discovered several new series of simple vectorial Lie superalgebras.

3.4. New simple finite dimensional Lie algebras and Lie superalgebras for $p = 2$. Lebedev [Leb1, Leb2] offered a new series of examples of simple orthogonal Lie algebras without CM. Together with Iyer, we constructed their prolongations, missed in [Lin], see [ILL]; queerifications of these orthogonal algebras (and of several more serial and exceptional Lie algebras, provided they are restricted) are totally new types of examples of simple Lie superalgebras. CTS prolongs of **some** [7] of these superalgebras and examples found in [BGL5] are considered in [BGL7].

3.5. Conclusion. Passing to Lie superalgebras we see that even their definition, as well as that of their prolongations, are not quite straightforward for $p = 2$, but, having defined them ([LL, Leb2]), it remains to apply the above-described procedures to get at least a supply of examples. To prove the completeness of the stock of examples for any p is a much more difficult task that requires serious preliminary study of the representations of the examples known and to be obtained — more topics for Ph.D. theses.

The references currently in preparation are to be soon found in `arXiv`.

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