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MATRIX LIE RINGS THAT CONTAIN A ONE-DIMENSIONAL LIE ALGEBRA OF SEMI-SIMPLE MATRICES

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Abstract. Let k be a field and $\bar{k}$ an algebraic closure of k. Suppose that k contains more than five elements if char $k \neq 2$. Let $\mathfrak{h}$ be a one-dimensional subalgebra of the Lie k-algebra $sl_2(\bar{k})$ consisting of semi-simple matrices. In this paper, it is proved that if $\mathfrak{g}$ is a subring of the Lie ring $sl_2(\bar{k})$ containing $\mathfrak{h}$, then $\mathfrak{g}$ is either solvable or there exists a quaternion algebra A over a subfield F of $\bar{k}$ such that $F \supseteq k$ and $\mathfrak{g}$ is isomorphic to the Lie F-algebra of all elements in A that are skew-symmetric with respect to a symplectic type involution defined on A.

Key words : Lie rings, Lie algebras, Semi-simple matrices.
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This paper addresses a task which has its roots in the theory of linear (matrix) groups. Within the framework of this theory the attention of many authors was for long attracted to the study of linear groups over various associative rings containing the subgroup of diagonal matrices. One should remark that the most complete survey devoted to the subject is presented in 1. In the course of such an investigation the question about the description of subgroups of the special linear group of degree 2 over a field that contain the subgroup of diagonal matrices (with determinant 1) turned out to be unexpectedly tough (2., 3., 4., 5.). Such situation is favored by linear groups of small degrees which are not subject to many of rules that apply to linear groups of larger degrees. Accordingly, while small linear groups sound straightforward, many of very real problems concerning them are far from being settled. For example, the solution of the above question given in 6. has extremely calculating character and, therefore, can not be extended to groups which contain not the whole group of diagonal matrices but only a part (a subgroup) of this

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group, for instance, the subgroup of diagonal matrices over a proper subfield. In connection with this, it seems appropriate to consider a comparable question about matrix Lie rings, that is, the question about the examination of Lie rings consisting of matrices of degree 2 with trace zero provided these rings contain the abelian Lie algebra of diagonal matrices. Under enough general assumptions, such rings turned out to admit a simple, clear and uniform description, namely, each Lie ring of this kind is a Lie algebra of elements of a suitable quaternion algebra that are skew-symmetric with respect to an involutary anti-automorphism defined on this quaternion algebra.

In order to formulate our main result, we first remind some standard facts concerning the structure of associative division algebras with involutions.

Let A be an associative division algebra of finite dimension m^2 $(m > 1)$ over its center F. Assume that A is an algebra with an involution σ the restriction of which to F is identical. Then the dimension (over F) of the space of elements in A, which are symmetric with respect to σ, is equal to either $\frac{m(m+1)}{2}$ or $\frac{m(m-1)}{2}$. In the first case we say that σ is of orthogonal type and in the second that σ is of symplectic type.

For the reader convenience we recall now the definition of quaternion algebras. Let a field F of characteristic $\neq 2$ and two non-zero elements $a, b \in F$ be given. Denote by A a 4-dimensional F-vector space and take a base $1, u, v, w$ for A. We define an associative multiplication on these basis elements by the following conditions: the element 1 satisfies the identity relation, $u^2 = a, v^2 = b, uv = -vu = w$. Then we extend this multiplication by linearity to a multiplication on A. The algebra over F obtained by this construction is called a quaternion algebra and is denoted by

$$A = \left(\frac{a, b}{F}\right).$$

The algebra A admits a unique symplectic type involution, namely, the involution defined by the following conditions: $1 \to 1, u \to -u, v \to -v, w \to -w$. One can associate to each element $c = c_0 + c_1 u + c_2 v + c_3 w$ $(c_i \in F)$ in A the matrix

$$g_c = \begin{pmatrix} c_0 + c_1\sqrt{a} & (c_2 + c_3\sqrt{a})b \\ c_2 - c_3\sqrt{a} & c_0 - c_1\sqrt{a} \end{pmatrix}$$

such that the equalities $g_c + g_d = g_{c+d}, g_c g_d = g_{cd}$ hold for all $c, d \in A$. Thus we construct an exact representation of the quaternion algebra A by matrices of degree 2 either over the field F (if a is a square in F) or over the field $F(\sqrt{a})$ (if a is not a square in F). Besides, it is well known that each quaternion algebra is either a division algebra or is isomorphic to the algebra of 2×2 matrices with entries lying in its center.

If A is an associative ring and $a, b \in A$, then we write $[ab]$ to denote the Lie product $ab - ba$. If P is a field, n is an integer such that $n \geq 2$, then the set $sl_n(P)$ of $n \times n$ matrices over P with trace zero is a Lie P-algebra with respect to $[ab]$. Now we are in a position to formulate our main result.

Theorem 1. *Let k be a field and $\bar{k}$ an algebraic closure of k. Suppose that k contains more than five elements if* char $k \neq 2$. *Let $\mathfrak{h}$ be a one-dimensional subalgebra of the Lie k-algebra $sl_2(\bar{k})$ consisting of semi-simple matrices. If $\mathfrak{g}$ is a subring of the Lie ring $sl_2(\bar{k})$ containing $\mathfrak{h}$, then $\mathfrak{g}$ is either solvable, or there exists a quaternion algebra A over a subfield F of $\bar{k}$ such that $F \supseteq k$ and $\mathfrak{g}$ is isomorphic to the Lie F-algebra of all elements in A that are skew-symmetric with respect to a symplectic type involution defined on A.*

Crucial here, in the proof of Theorem 1, is the fact that the Lie ring $\mathfrak{g}$, which appears in Theorem 1, is in reality, a Lie algebra over a suitable subfield of the field $\bar{k}$. To prove Theorem 1 we begin with giving notations which will be used hereafter in this paper.

Now let n be again a natural number not less than 2. Let δ denote the "Kronecker delta map": $\delta_{ij} = 0$ unless $i = j$ and $\delta_{ii} = 1$. Then we define the set of matric units $\{e_{ij} \mid 1 \leq i, j \leq n\}$ where e_{ij} is the matrix whose entry in the (i,j) position is δ_{ij}.

Let P be a field. By $M_n(P)$ we denote the set of all $n \times n$ matrices over P. Suppose P admits a non-trivial automorphism J. If $x = \sum_{i,j=1}^{n} x_{ij} e_{ij} \in M_n(P)$ $(x_{ij} \in P)$, then $x^J = \sum_{i,j=1}^{n} x_{ij}^J e_{ij}$, and ${}^t x$ is the transpose of x. Fix $\Phi \in M_n(P)$. Let us designate by $su_n(P, \Phi, J)$ the set of all $x \in M_n(P)$ such that the trace of x is zero and the equation $x\Phi + \Phi\ {}^t x^J = 0$ is satisfied. If P_0 is a subfield of P consisting of elements fixed by J, then $su_n(P, \Phi, J)$ is a subalgebra of the Lie P_0-algebra $sl_n(P)$.

If X is a subset of an additive abelian group, then $X^{\#}$ denotes the set of all non-zero elements in X.

If the field P is a quadratic extension of a subfield k, J is a unique non-trivial automorphism of P over k, $b \in k^{\#}$ and $\Phi = \begin{pmatrix} b & 0 \\ 0 & -1 \end{pmatrix}$, then, in what follows, we write $su_2(P, k, b)$ instead of $su_2(P, \Phi, J)$.

When $n = 2$, set $X = e_{12}, Y = e_{21}, H = e_{11} - e_{22}$. If k is a field, $b \in k^{\#}$ and θ is a fixed non-zero element in an algebraic closure $\bar{k}$ of k such that $\theta^2 \in k$, then the set $k\theta H + k(bX + Y) + k\theta(bX - Y)$ is denoted by $wsu_2(k(\theta), b)$. Clearly, $wsu_2(k(\theta), b)$ is a subalgebra of the Lie k-algebra $sl_2(\bar{k})$. Assume char $k \neq 2$. Then it is evident that $wsu_2(k(\theta), b) = su_2(k(\theta), k, b)$ if $\theta \notin k$ and $wsu_2(k(\theta), b) = sl_2(k)$ if $\theta \in k$. In addition, if we used the above representation of the quaternion algebra $\left(\frac{\theta^2, b}{k}\right) := A$ by matrices of degree 2, then $wsu_2(k(\theta),\ \ b)$ is the set of all elements in A that are skew symmetric with respect to a unique symplectic type involution defined on A. Therefore, to prove Theorem 1 it is enough to predict the validity of the following statement.

Proposition 2. *Let $k, \bar{k}, \mathfrak{g}, \mathfrak{h}$ be as in Theorem 1. Suppose also that the ring $\mathfrak{g}$ is not solvable. Then $\mathfrak{g}$ is conjugate by an appropriate matrix in the general linear group $GL_2(\bar{k})$ to the Lie ring $wsu_2(R(\theta), b)$ where R is a subfield of $\bar{k}$ containing k.*

The proof of Proposition 2 requires beforehand establishing several auxiliary results. The first of these results, which concerns the description of subrings of the ring $sl_2(\bar{k})$ containing $sl_2(k)$, will be proved in the more general setting than it is required within the framework of this paper. This will serve to highlight the lack, in full measure, of parallelism between linear groups and matrix Lie rings. Notice that for fields of characteristic zero, Proposition 3 below follows from 7. In what follows, we denote by $\operatorname{diag}(a_1, \dots, a_n)$ the diagonal matrix of order n, which contains elements $a_1, \dots, a_n$ on the diagonal.

Proposition 3. *Let k be a field of characteristic p, $\bar{k}$ an algebraic closure of k, n an integer, $n \geq 2$. Suppose $p \neq 2$. Let $\mathfrak{g}$ be a subring of the Lie ring $sl_n(\bar{k})$ containing $sl_n(k)$. Then there exists a subfield L of $\bar{k}$ such that $\mathfrak{g}$ normalizes $sl_n(L)$. More precisely, either $\mathfrak{g} = sl_n(L)$ where L is a subfield of $\bar{k}$ containing k, or $p > 0$, n is divisible by p, and $\mathfrak{g}$ is generated by the ring $sl_n(L)$, where L is a subfield of $\bar{k}$ containing k, and by a set of diagonal matrices* $\operatorname{diag}(c_1, c_2, \dots, c_n) \in sl_n(\bar{k})$ *such that the elements $c_1 - c_2, c_1 - c_3, \dots, c_1 - c_n$ belong to L.*

Proposition 3 is an immediate consequence of the following statement.

Lemma 4. *Let $k, \bar{k}, n, p$ be such as in Proposition 3. If $\mathfrak{g}$ is a subring of the Lie ring $sl_n(\bar{k})$ generated by $sl_n(k)$ and by a matrix $a = \sum_{i,j=1}^{n} a_{ij} e_{ij} \in sl_n(k)$, then there exists a subfield P of $\bar{k}$ such that $\mathfrak{g}$ normalizes $sl_n(P)$. More precisely, either $\mathfrak{g} = sl_n(P)$ where P is a subfield of the field $\bar{k}$ obtained by adjunction to k of the elements a_{ij} $(1 \leq i \neq j \leq n), a_{11} - a_{jj}$ $(2 \leq j \leq n)$, or $p > 0$, n is divisible by p, and $\mathfrak{g}$ is generated by the ring $sl_n(P)$ where the field P is as above, and by the matrix* $\operatorname{diag}(a_{11}, a_{22}, \dots, a_{nn})$.

Proof. We divide the proof of the lemma into several steps.

1) If i_0, j_0 are integers such that $1 \leq i_0 \neq j_0 \leq n$, then $k a_{j_0 i_0} e_{i_0 j_0} \subseteq \mathfrak{g}$ for

$$[-\frac{r}{2} e_{i_0 j_0}, [e_{i_0 j_0}, a]] = r a_{j_0 i_0} e_{i_0 j_0}$$

where $r \in k$.

2) If i_0, j_0 are integers such that $1 \leq i_0 \neq j_0 \leq n$ and $\alpha e_{i_0 j_0} \in \mathfrak{g}$ for a non-zero $\alpha \in \bar{k}$, then $k \alpha e_{j_0 i_0} \subseteq \mathfrak{g}$ for $[\frac{r}{2} e_{j_0 i_0}, [\alpha e_{i_0 j_0}, e_{j_0 i_0}]] = r \alpha e_{j_0 i_0}$ where $r \in k$.

3) Let i_0, j_0, q_0 be pairwise distinct integers belonging to the set $\{1, 2, \dots, n\}$ and α a non-zero element in $\bar{k}$ such that $k \alpha e_{i_0 j_0} \subseteq \mathfrak{g}$. Then $k \alpha e_{i_0 q_0} \subseteq \mathfrak{g}$ and $k \alpha e_{j_0 q_0} \subseteq \mathfrak{g}$.

Indeed, the first inclusion follows from the equality $[r\alpha e_{i_0 j_0}, e_{j_0 q_0}] = r\alpha e_{i_0 q_0}$ with $r \in k$. To prove the second we recall that by 2) $k\alpha e_{j_0 i_0} \subseteq \mathfrak{g}$, and so, according to the first inclusion of the present step, $k\alpha e_{j_0 q_0} \subseteq \mathfrak{g}$.

4) Let i_0, j_0, q_0, p_0 be pairwise distinct integers in the set $\{1, 2, \dots, n\}$ and $k\alpha e_{i_0 j_0} \subseteq \mathfrak{g}$ for some non-zero $\alpha \in \bar{k}$. Let us make certain of that $k\alpha e_{q_0 p_0} \subseteq \mathfrak{g}$.

Indeed, we apply 3) to the numbers i_0, j_0, q_0 to get the inclusion $k\alpha e_{j_0 q_0} \subseteq \mathfrak{g}$. So, making use of 3) to the triplet j_0, q_0, p_0, we obtain $k\alpha e_{q_0 p_0} \subseteq \mathfrak{g}$, as required.

5) Steps 1)–4) imply $ka_{i_0 j_0} e_{i_1 j_1} \subseteq \mathfrak{g}$ for any quadruple $\{i_0, j_0, i_1, j_1\} \subseteq \{1, 2, \dots, n\}$ such that $i_0 \neq j_0, i_1 \neq j_1$. In particular, $a_{ij} e_{ij} \in \mathfrak{g}$ for all $i, j, 1 \leq i \neq j \leq n$, and so $a' = a_{11}e_{11} + a_{22}e_{22} + \dots + a_{nn}e_{nn} \in \mathfrak{g}$. Therefore, $[a', ke_{1j}] = k(a_{11} - a_{jj})e_{1j} \in \mathfrak{g}$ for all $j = 2, 3, \dots, n$.

6) Again, let i_0, j_0 be integers such that $1 \leq i_0 \neq j_0 \leq n$. If α, β are non-zero in $\bar{k}$ and $\alpha e_{i_0 j_0}, \beta e_{i_0 j_0} \in \mathfrak{g}$, then $k\alpha\beta e_{i_0 j_0} \subseteq \mathfrak{g}$ by virtue of the equality $[[\beta e_{i_0 j_0}, \alpha e_{j_0 i_0}], \frac{r}{2} e_{i_0 j_0}] = r\alpha\beta e_{i_0 j_0}$ $(r \in k)$. Since all the elements a_{ij} $(1 \leq i \neq j \leq n), a_{11} - a_{jj}$ $(2 \leq j \leq n)$ are algebraic over k, we obtain from step 5) that $sl_n(P) \subseteq \mathfrak{g}$. Further, as $a_{11} - a_{22} \in P, a_{11} - a_{33} \in P, \dots, a_{11} - a_{nn} \in P$, we have $(n-1)a_{11} - a_{22} - a_{33} - \dots - a_{nn} = na_{11} \in P$. Suppose first that n is not divisible by p (this always takes place when $p = 0$). Then $a_{11} \in P$, and so $a_{22}, a_{33}, \dots, a_{nn} \in P$. With this in mind, we have $a \in sl_n(P)$, and therefore $\mathfrak{g} = sl_n(P)$. Suppose now that n is divisible by p (in particular, $p > 0$). By 5), $a' = \sum_{i=1}^{n} a_{ii}e_{ii} \in \mathfrak{g}$, and if $b = \sum_{u,q=1}^{n} b_{uq}e_{uq} \in sl_n(P)$, then $[a'b] = \sum_{u,q=1}^{n} e_{uq}b_{uq}(a_{uu} - a_{qq}) \in sl_n(P)$, i. e., a' normalizes $sl_n(P)$. The lemma is proved. □

When $n = 2$, Lemma 4 may be formulated as follows.

Lemma 5. *Let k be a field and $\bar{k}$ be an algebraic closure of k. Assume* char $k \neq 2$. *Let $\mathfrak{g}$ be a subring of the Lie ring $sl_2(\bar{k})$ such that $\mathfrak{g} \supseteq sl_2(k)$. Then $\mathfrak{g} = sl_2(L)$ where L is a subfield of $\bar{k}$ such that $L \supseteq k$.*

It is the last lemma that is the statement which will be used in the proof of Proposition 2.

Now let K be a field with characteristic different from 2 and θ be an element in an algebraic closure of K such that $\theta \notin K, \theta^2 \in K$. Let $b \in K^{\#}$. The following lemma features the more meaningful cases of the description of Lie rings which are intermediate between $su_2(K(\theta), K, b)$ and $sl_2(K(\theta))$.

Lemma 6. *Any subring $\mathfrak{g}$ of the Lie ring $sl_2(K(\theta))$ containing the subring $su_2(K(\theta), K, b)$ and the matrix $U = d\theta H + eX + fY$ with $d \in K, e, f \in K(\theta)$ and $ef \in K$, coincides either with $su_2(K(\theta), K, b)$ or with $sl_2(K(\theta))$.*

Proof. We distinguish three cases for U: 1) $e = f = 0$; 2) $e \neq 0, f \neq 0$; 3) exactly one of e, f does not vanish.

If 1) takes place, then $U \in su_2(K(\theta), K, b)$, and so $\mathfrak{g} = su_2(K(\theta), K, b)$. Let 2) occurs. Denote by J a unique non-trivial automorphism of the field

$K(\theta)$ over K. From the condition of the lemma, we can write $e = rf^J$ with $r \in K^{\#}$. If $r = b$, then $U \in su_2(K(\theta), K, b)$, and so $\mathfrak{g} = su_2(K(\theta), K, b)$. Assume $r \neq b$. Then $V = d\theta H + bf^J X + fY \in su_2(K(\theta), K, b)$, and hence $V - U = (b - r)f^J X \in \mathfrak{g}$. Thus $\mathfrak{g}$ contains a matrix $W = sX$ with $s \in K(\theta)^{\#}$. Employing this fact, we find successively $[K\theta H, W] = K\theta sX \subseteq \mathfrak{g}$ and $[K\theta H, \theta sX] = KsX \subseteq \mathfrak{g}$. Since the elements $\theta s, s$ form a basis of $K(\theta)$ over K, we have $K(\theta)X \subseteq \mathfrak{g}$. Therefore, $[K(\theta)X, bX + Y] = K(\theta)H \subseteq \mathfrak{g}$, and so $[K(\theta)H, bX + Y] = K(\theta)(bX - Y) \subseteq \mathfrak{g}$. Now, letting t to be an arbitrary element in $K(\theta)$, we see that tbX and $t(bX - Y)$ lie in $\mathfrak{g}$ whence $K(\theta)Y \subseteq \mathfrak{g}$. Thus, $K(\theta)H, K(\theta)X, K(\theta)Y \subseteq \mathfrak{g}$, and we conclude that $\mathfrak{g} = sl_2(K(\theta))$.

If 3) takes place, then it is easily seen that $\mathfrak{g}$ contains an element sX with $s \in K(\theta)^{\#}$. So proceeding as in 2), we show that $\mathfrak{g} = sl_2(K(\theta))$ completing the proof of the lemma. □

Our next lemma gives, in fact, the description of Lie rings containing an abelian Lie algebra of diagonal matrices.

Lemma 7. *Let k be a field, $\bar{k}$ an algebraic closure of k; let θ, a, b be nonzero elements in $\bar{k}$, and $\mathfrak{h}$ the subring $k\theta H$ of the Lie ring $sl_2(\bar{k})$. Assume* char $k \neq 2$. *If $\mathfrak{g}_1$ is the subring of the Lie ring $sl_2(\bar{k})$ generated by $\mathfrak{h}$ and by the matrix $U = aH + bX + Y$, then $\mathfrak{g}_1 = wsu_2(k(b, a^2, \theta^2, a\theta)(\theta), b)$.*

Proof. If $r \in k$, then $[\frac{r}{2}\theta H, U] = r\theta(bX - Y)$, and by induction on m,

$$k\theta^m(bX + (-1)^m Y) \subseteq \mathfrak{g}_1 \tag{1}$$

for any integer $m \geq 1$. The element θ is algebraic over k. So taking in (1) m first to be even and then to be odd, we obtain

$$k(\theta^2)(bX + Y) \subseteq \mathfrak{g}_1, \tag{2}$$

$$k(\theta^2)\theta(bX - Y) \subseteq \mathfrak{g}_1. \tag{3}$$

Therefore,

$$[k(\theta^2)(bX + Y), \theta(bX - Y)] = k(\theta^2)\theta bH \subseteq \mathfrak{g}_1.$$

In particular, $k\theta bH \subseteq \mathfrak{g}_1$, i. e., the inclusion

$$k\theta b^m H \subseteq \mathfrak{g}_1 \tag{4}$$

is valid for $m = 1$. Suppose now that m is a fixed integer greater than 1 for which (4) holds. We have already shown that the following implication is true: $k\theta H \subseteq \mathfrak{g}_1 \Rightarrow k\theta bH \subseteq \mathfrak{g}_1$. In view of (4), one may replace here θ by θb^m to get $k\theta b^{m+1}H \subseteq \mathfrak{g}_1$. Hence, (4) holds for any integer $m \geq 1$. Since b is algebraic over k, this yields

$$k(b)\theta H \subseteq \mathfrak{g}_1. \tag{5}$$

But as it has already been established, the inclusion $k\theta H \subseteq \mathfrak{g}_1$ implies (2) and (3). So by virtue of (5), in (2), (3) the field k may be replaced by $k(b)$ to obtain

$$P(bX+Y) \subseteq \mathfrak{g}_1, \quad P\theta(bX-Y) \subseteq \mathfrak{g}_1$$

with $P = k(b, \theta^2)$. Further, $aH = U - (bX+Y) \in \mathfrak{g}_1$. Therefore, easy induction on m shows that

$$Pa^m(bX + (-1)^m Y) \subseteq \mathfrak{g}_1 \tag{6}$$

for any integer $m \geq 0$. If, further, Q to be the field $P(a^2)$, then proceeding as while obtaining (2), (3), we derive $Q(bX+Y) \subseteq \mathfrak{g}_1, Qa(bX-Y) \subseteq \mathfrak{g}_1$. Consequently,

$$\begin{aligned}
[Q(bX+Y), a(bX-Y)] &= QaH \subseteq \mathfrak{g}_1,\\
[Q(bX+Y), \theta(bX-Y)] &= Q\theta H \subseteq \mathfrak{g}_1,\\
[Q\theta H, bX+Y] &= Q\theta(bX-Y) \subseteq \mathfrak{g}_1,\\
[Q\theta H, a(bX-Y)] &= Q\theta a(bX+Y) \subseteq \mathfrak{g}_1,
\end{aligned}$$

So $\mathfrak{g}_1$ contains the linear Q-hull M of the elements $aH, \theta H, a(bX-Y), \theta(bX-Y), bX+Y, \theta a(bX+Y)$. But

$$M = Q(a\theta)\theta H + Q(a\theta)(bX+Y) + Q(a\theta)\theta(bX-Y) = wsu_2(Q(a\theta)(\theta), b),$$

and so $\mathfrak{g}_1 = wsu_2(Q(a\theta)(\theta), b)$ since $U \in M$. To complete the proof it suffices to notice that $Q(a\theta) = k(b, \theta^2, a^2, a\theta)$. The lemma is proved. □

The next step in the proof of Proposition 2 is to describe subrings of the Lie ring $sl_2(\bar{k})$ over an algebraic closure $\bar{k}$ of the field k that contain the subring $su_2(k(\theta), k, b)$ where $\theta \in \bar{k} \setminus k, \theta^2 \in k, b \in k$.

Lemma 8. *Let k be a field of characteristic different from 2. Let b be an element in $k^{\#}$ and θ be an element in an algebraic closure $\bar{k}$ of k. Assume that $\theta \notin k, \theta^2 \in k$ and that k contains more than five elements. Let $\mathfrak{g}$ be a subring of the Lie ring $sl_2(\bar{k})$ generated by the subring $su_2(k(\theta), k, b)$ and by the matrix $U = dH + eX + fY \in sl_2(\bar{k})$ with non-zero d, e, f. Denote by R the field $k(fe, d^2, d\theta, \theta(bf-e), bf+e)$. Then $\mathfrak{g} = wsu_2(R(\theta), b)$.*

Proof. We set $\mathfrak{g}_1$ to be the subring of $\mathfrak{g}$ generated by the subring

$$k\theta H \subseteq su_2(k(\theta), k, b)$$

and by U. If $D = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}$, then by Lemma 7, $D\mathfrak{g}_1 D^{-1}$ coincides with $wsu_2(L(\theta), fe)$ where $L = k(fe, d^2, d\theta)$. Consequently, $[L\theta H, bX+Y] = L\theta(bX-Y) \subseteq \mathfrak{g}$ and $[L\theta H, \theta(bX-Y)] = L(bX+Y) \subseteq \mathfrak{g}$. For any $x_1, x_2 \in k$, the matrix $U + x_1(bX+Y) + x_2\theta(bX-Y)$ is contained in $\mathfrak{g}$. Thus, the above reasoning shows that $\mathfrak{g}$ contains the subring $wsu_2(R(\theta), b)$ where R is a subfield of $\bar{k}$ obtained by adjunction to L of all elements $x_1(bf+e) + x_2\theta(bf-e)$

such that the matrix $U+x_1(bX+Y)+x_2\theta(bX-Y)$ has non-zero entries only. Pick an element x_1 in k. Since k contains at least seven elements, one may choose $x_2 \in k$ so that all the entries of the matrices

$$U + x_1(bX + Y) + x_2\theta(bX - Y),$$
$$U + (x_1 + 1)(bX + Y) + x_2\theta(bX - Y),$$
$$U + x_1(bX + Y) + (x_2 + 1)\theta(bX - Y)$$

are non zero. This means that the elements

$$x_1(bf + e) + x_2\theta(bf - e), \quad (x_1 + 1)(bf + e) + x_2\theta(bf - e),$$
$$x_1(bf + e) + (x_2 + 1)\theta(bf - e)$$

are contained in R whence $bf + e \in R, \theta(bf - e) \in R$. Therefore, $R = L(bf + e, \theta(bf - e))$, and so $f, e \in R(\theta)$. Since $d\theta \in R, d \in R\theta$. Thus $U = d_1\theta H + eX + fY$ where $d_1 \in R, ef \in R$, and applying Lemma 6 completes the proof of the lemma. □

Now we are able without any difficulties to prove Proposition 2 thereby completing the proof of Theorem 1.

Proof of Proposition 2. First we observe that char $k \neq 2$ for $\mathfrak{h}$ consists of semi-simple matrices only. Further, replacing $\mathfrak{g}$ by $T\mathfrak{g}T^{-1}$ where T is a suitable element in the group $GL_2(\bar{k})$, one may regard $\mathfrak{h}$ as consisting of diagonal matrices only. From this it is easy to infer that $\mathfrak{g}$ contains a matrix $U = aH + bX + cY$ with $a \neq 0, b \neq 0, c \neq 0$. Then $C = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \in GL_2(\bar{k})$ and replacing $\mathfrak{g}$ by $C\mathfrak{g}C^{-1}$ one may suppose that $c = 1$. Let $\mathfrak{h} = k\theta H$ where θ is non zero element in $\bar{k}$. If $\mathfrak{g}_1$ denotes a subring of $\mathfrak{g}$ generated by $\mathfrak{h}$ and U, then by Lemma 7, $\mathfrak{g}_1 = wsu_2(K(\theta), b)$ with $K = k(b, a^2, \theta^2, a\theta)$. If $\theta \in k$, then $\mathfrak{g}_1 = sl_2(K)$ and, by Lemma 5, $\mathfrak{g} = sl_2(L)$ where L is a subfield of $\bar{k}$ containing k. If $\theta \notin k$, then $wsu_2(K(\theta), b) = su_2(K(\theta), K, b)$ and invoking Lemmas 8 and 5 completes the proof of the proposition. □

References

1. N. A. Vavilov, Intermediate subgroups in Chevalley groups, in: Proc. Conf. Groups of Lie Type and their Geometries (Como-93), Cambridge Univ. Press (1995), 233–280.
2. O. H.King, On subgroups of the special linear group containing the special orthogonal group, J. Algebra **96** (1985), 178–193.
3. . Shang Zhi Li, The maximality of monomial subgroups of linear groups over division rings, J. Algebra **127** (1989), 22–39.
4. N. A. Vavilov, On subgroups of the split classical groups, Proc. Math. Institit Steklov, **1991**, no. 4, 27–41.

5. N. A. Vavilov, E. V. Dybkova, Subgroups of the general symplectic groups containing the group of diagonal matrices, I, II, J. Sov. Math. **24** (1984), 406–416; **30** (1985), 1823–1832.
6. O. H. King, On subgroups of the special linear group containing the diagonal subgroup, J. Algebra **132** (1990), 198–204.
7. Den Yin Wang, Extension of Lie algebras according to the extension of fields, J. Math. Res. Exposition **25** (2005),no. 3, 543–547.