Journal of Prime Research in Mathematics Vol. **3***(2007), 120-128*

ON NEWTON INTERPOLATING SERIES AND THEIR APPLICATIONS

GHIOCEL GROZA

Abstract. Newton interpolating series are constructed by means of Newton interpolating polynomials with coefficients in an arbitrary field K (see Section 1). If $K = \mathbb{C}$ is the field of complex numbers with the ordinary absolute value, particular convergent series of this form were used in number theory to prove the transcendence of some values of exponential series (see Theorem 1). Moreover, if $K = \mathbb{R}$, by means of these series it can be obtained solutions of a multipoint boundary value problem for a linear ordinary differential equation (see Theorem 2). If $K = \mathbb{C}_p$, some particular convergent series of this type (so-called Mahler series) are used to represent all continuous functions from $\mathbb{Z}_p$ in $\mathbb{C}_p$ (see [4]).

For an arbitrary field K, with respect to suitable addition and multiplication of two elements the set of Newton interpolating series becomes a commutative K-algebra $K_S[[X]]$ which generalizes the canonical K-algebra of formal power series. If we consider K a local field, we construct a subalgebra of $K_S[[X]]$, even for more variables, which is a generalization of Tate algebra used in rigid analytic geometry (see Section 3).

Key words : Newton interpolating series, noetherian ring, Tate algebras, two-point boundary value problem.
AMS SUBJECT:13J05, 32B05, 32P05, 34B10, 13A18, 13F30.

1. Basic definitions

Consider K a field, R a commutative $K-$algebra and $S = \{\alpha_n\}_{n\geq 1}$ a fixed sequence of elements of K. By means of Newton interpolating polynomials we construct formal series of the form

$$f = \sum_{i=0}^{\infty} a_i u_i, \quad a_i \in R, \tag{1}$$

Corresponding author. * School of Mathematical Sciences, GC University, Lahore and Department of Mathematics and Computer Science, Technical University of Civil Engineering, Bucharest, Romania, E-mail: grozag@mail.utcb.ro .

where

$$u_0 = 1,\ u_i = \prod_{j=1}^{i} (X - \alpha_j)\,,\ i \geq 1, \tag{2}$$

which are called *Newton interpolating series at* $\alpha_1, \alpha_2, \ldots$. We denote by $R_S[[X]]$ the set of all Newton interpolating series having the form (1). Then there exist the elements $d_k(i,j)$ in K uniquely defined such that

$$u_i u_j = \sum_{k=\max\{i,j\}}^{i+j} d_k(i,j) u_k. \tag{3}$$

We define addition and multiplication of two elements $f = \sum_{i=0}^{\infty} a_i u_i, g = \sum_{i=0}^{\infty} b_i u_i, \in R_S[[X]]$ as follows

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i) u_i \tag{4}$$

and

$$fg = \sum_{k=0}^{\infty} c_k u_k \tag{5}$$

with

$$c_k = \sum_{(\alpha,\beta)\in I(k)} d_k(\alpha,\beta) a_\alpha b_\beta, \tag{6}$$

where $I(k) = \{(\alpha,\beta) \in \mathbb{N}\times\mathbb{N}; \max\{\alpha,\beta\} \leq k,\ \alpha+\beta \geq k\}$ and $d_k(\alpha,\beta)$ are given in (3). It is easily seen that with these definitions of addition and multiplication $R_S[[X]]$ becomes a commutative K-algebra which contains $R[X]$.

2. Newton interpolating series with complex coefficients

Consider $K = R = \mathbb{C}$ the field of complex numbers. In the first case we take m a positive integer and the sequence $S = \{\alpha_n\}_{n\geq 1}$, where for every $i = 1, 2, \ldots, m,\ \alpha_i = i = \alpha_{i+mj}, j = 1, 2, \ldots$. By means of these type of convergent Newton interpolating series can be proved the following result of Lindeman (see [5], Theorem 6, Ch. 2, Sec. 3):

Theorem 1. *If γ is an algebraic number different from zero, then e^γ is a transcendental number.*

In second case we take $K = R = \mathbb{R}$ the field of real numbers, $\alpha_1 = 0,\ \alpha_2 = 1,\ \alpha_3 = \frac{1}{2}$ and for $k \geq 4$

$$\alpha_k = \frac{2s+1}{2^{m+1}},\ \text{where } 2^m + 1 < k \leq 2^{m+1} + 1,\ \ s = k - 2^m - 2. \tag{7}$$

Consider a function $f : [0,1] \to \mathbb{R}$ and $\alpha_k \in [0,1]$, $k = 1,2,...$. We say that f *can be represented into Newton interpolating series at* $\{\alpha_k\}_{k\geq 1}$ if there exists a series of the form (1) which converges uniformly to f on [0,1]. Taking into account the importance of power series in the theory of initial value problems for differential equations, it seems to be very useful to study Newton interpolating series in order to find the solution of the multipoint boundary value problem for differential equations. Now, we consider a linear differential equation with analytic function coefficients

$$y^{(n)}(x) = c(x) + b_0(x)y(x) + b_1(x)y'(x) + ... + b_{n-1}(x)y^{(n-1)}(x), \qquad (8)$$

with $x \in [0,1]$, c, $b_i \in C^\infty([0,1])$, $i \in \{0,1,...,n-1\}$ and there exists $C_0 > 0$ such that

$$\max\left\{ \left\|c^{(j)}\right\|_\infty, \left\|b_i^{(j)}\right\|_\infty, i \in \{0,1,...,n-1\}\right\} < C_0^{j+1}, \ j = 0,1,... . \qquad (9)$$

By means of Newton interpolating series given by (7) can be proved the following result (see [3], Theorem 4.1):

Theorem 2. *If* $\{\alpha_k\}_{k\geq 1}$ *is a sequence of real numbers from* $[0,1]$ *of the form (7), then every solution* $y \in C^n([0,1])$ *of the equation (8) and its derivatives* $y^{(k)}$, $k = 1,2,...,n$ *are represented into Newton interpolating series at* $\{\alpha_k\}_{k\geq 1}$.

This theorem is a useful tool to find approximate solutions of two-point boundary value problems for differential equations of the form (8).

Example. Consider the two-point boundary value problem ([3])

$$y''(x) - 2500y(x) = 2500\cos^2(\pi x), \ \ x \in [0,1], \ \ y(0) = y(1) = 0. \qquad (10)$$

We know that this two-point boundary value problem has a solution y which can be represented by a Newton interpolating series with α_k given by (7). Moreover by Theorem 2 the derivatives y' and y'' of y can be represented by a Newton interpolating series which are the derivative series of Newton interpolating series of y. We approximate the solution by taking the partial sums $S_{33}(x) = \sum_{i=0}^{33} a_i u_i(x)$. The boundary conditions imply $a_0 = a_1 = 0$. Using, for example, 32 equidistant collocation nodes $z_{j+1} = \frac{j}{32}$, $j = 0,1,...,31$ we obtain the coefficients a_i.

Table 1 lists the absolute errors in y. The computations were performed on a computer with a 14-hexadecimal-digit mantissa.

Table 1

x	simple shooting	Newton series (7) with $n = 33$
0.1	$.19 \cdot 10^{-7}$	$.13 \cdot 10^{-10}$
0.2	$.28 \cdot 10^{-5}$	$.25 \cdot 10^{-12}$
0.3	$.41 \cdot 10^{-3}$	$.14 \cdot 10^{-11}$
0.4	$.61 \cdot 10^{-1}$	$.11 \cdot 10^{-11}$
0.5	$.90 \cdot 10$	$.64 \cdot 10^{-9}$
0.6	$.13 \cdot 10^{4}$	$.48 \cdot 10^{-8}$
0.7	$.20 \cdot 10^{6}$	$.41 \cdot 10^{-7}$
0.8	$.29 \cdot 10^{8}$	$.22 \cdot 10^{-5}$
0.9	$.44 \cdot 10^{10}$	$.27 \cdot 10^{-5}$
1.0	$.65 \cdot 10^{12}$	0

Errors associated with the example

3. Newton interpolating series with coefficients in a field

Let K be a valued field with respect to a non-trivial non-archimedean absolute value $|\,|$. For $x, y \in K$ we put $d(x,y) = |x-y|$. Then (K,d) is a metric space and we can introduce the customary topological concepts into such of space in terms of the metric. If K is a locally compact field, then it is called a *local field.*

If R is a commutative ring with identity and $\|\,\|$ is a non-archimedean norm on R, we consider the sets : $\overset{\circ}{R} = \{x \in R; \|x\| \leq 1\}$, $\overset{\vee}{R} = \{x \in R; \|x\| < 1\}$ (see [2], Chapter 1). Then $\overset{\circ}{R}$ is a commutative ring with identity and $\overset{\vee}{R}$ is an ideal in $\overset{\circ}{R}$. We denote the residue ring $\overset{\circ}{R} \,/\, \overset{\vee}{R}$ by $\widetilde{R}$. If $R = K$, then $\overset{\circ}{K}$ is a local ring called *the valuation ring* of $|\,|$ and $\overset{\vee}{K}$ is the maximal ideal of $\overset{\circ}{K}$. If K is a local field, then it is a complete field and the residue field $\widetilde{K}$ is a finite field of order $m = p^s$, where p is the characteristic of $\widetilde{K}$ ([1], Proposition 2.3.3, p. 51).

Let $S = \{\alpha_n\}_{n \geq 1}$ be a fixed sequence of elements of $\overset{\circ}{K}$. If r is a positive integer, $\nu = (\nu_1, \nu_2, ..., \nu_r) \in \mathbf{N}^r$, we put $N(\nu) = \nu_1 + \nu_2 + ... + \nu_r$ and $\mathbf{X} = (X_1, X_2, ..., X_r)$. We order $\mathbf{N}^r$ in the following manner $\nu < \mu$ if either $N(\nu) < N(\mu)$ or $N(\nu) = N(\mu)$ and ν is less than μ with respect to the lexicographical order. The symbol ∞^r will be an element such that $\nu < \infty^r$, for every $\nu \in \mathbf{N}^r$. We denote by $K_S[[\mathbf{X}]]$ the set of formal series of the form

$$f = \sum_{\nu=\mathbf{0}}^{\infty^r} a_\nu U_\nu, \tag{11}$$

where $a_\nu \in R$, $U_\nu = \prod_{j=1}^{r} u_{\nu_j} \in K[\mathbf{X}]$ and $u_{\nu_j} \in K[X_j]$ are given by (2). If $\tau = (\tau_1, \tau_2, ..., \tau_r) \in \mathbf{N}^r$, $j \in \mathbf{N}$, we define $\nu + \tau = (\nu_1 + \tau_1, \nu_2 + \tau_2, ..., \nu_r + \tau_r)$ and $j\nu = (j\nu_1, j\nu_2, ..., j\nu_r)$. If $f, g = \sum_{\nu=\mathbf{0}}^{\infty^r} b_\nu U_\nu \in K_S[[\mathbf{X}]]$, we define addition and multiplication of f and g as follows:

$$f + g = \sum_{\nu=\mathbf{0}}^{\infty^r} (a_\nu + b_\nu) U_\nu, \tag{12}$$

$$fg = \sum_{\nu=\mathbf{0}}^{\infty^r} p_\nu U_\nu, \tag{13}$$

where

$$p_\nu = \sum_{\mu,\theta \in I(\nu)} D_\nu(\mu, \theta) a_\mu b_\theta, \tag{14}$$

$D_\nu(\mu, \theta) = d_{\nu_1}(\mu_1, \theta_1)...d_{\nu_r}(\mu_r, \theta_r)$, $d_i(s, t)$ are defined in (3) and $I(\nu) = \{(\mu, \theta) \in \mathbb{N}^r \times \mathbb{N}^r; \max\{\mu, \theta\} \leq \nu,\ \mu + \theta \geq \nu\}$.

Consider $f = \sum_{\nu=\mathbf{0}}^{\infty^r} a_\nu U_\nu \in K_S[\mathbf{X}]$ a non-zero series. If τ is the smallest index ν for which a_ν is different from zero, then τ will be called *the order* of f and will denoted $o(f)$. We agree to attach the order ∞^r to the element 0 of $K_S[\mathbf{X}]$. It follows in the usual way that $o(f + g) \geq \min\{o(f), o(g)\}$, but $o(fg) \geq o(f) + o(g)$ does not hold for every $f, g \in K_S[[\mathbf{X}]]$.

Remark 1. It is easy to see that there exists an injective map $\varphi : K[\mathbf{X}] \to K_S[[\mathbf{X}]]$ such that for all P, $Q \in K[\mathbf{X}]$,

$$\varphi(P + Q) = \varphi(P) + \varphi(Q)$$

and

$$\varphi(PQ) = \varphi(P)\varphi(Q)$$

where the addition and the multiplication in $K_S[[\mathbf{X}]]$ are defined by (12) and (13).

We consider the K-algebra $K_S[[\mathbf{X}]]$ and we take $T = \{\gamma_\nu\}_{\nu \in \mathbf{N}^r}$ a fixed sequence of elements of $K^\times$ such that

$$|\gamma_{\mu+\nu}| \leq |\gamma_\mu| \, |\gamma_\nu|, \tag{15}$$

for every μ and ν.

We denote

$$\mathcal{H}_T K_S[[\mathbf{X}]] = \left\{ f = \sum_{\nu=\mathbf{0}}^{\infty^r} a_\nu U_\nu \in K_S[[\mathbf{X}]];\ \lim_{N(\nu)\to\infty} a_\nu \gamma_\nu = 0 \right\}. \tag{16}$$

If $f = \sum_{\nu=\mathbf{0}}^{\infty^r} a_\nu U_\nu \in \mathcal{H}_T K_S[[\mathbf{X}]]$, the real number

$$\|f\| = \sup_\nu |a_\nu \gamma_\nu| \tag{17}$$

is well defined. As usual we call $\| \, \|$, given in (17), *the Gauss norm* on $\mathcal{H}_T K_S[[\mathbf{X}]]$. Many properties of $\mathcal{H}_T K_S[[\mathbf{X}]]$ are analogues of those of Tate algebras.

Theorem 3. *If K is a complete valued field, then $\mathcal{H}_T K_S[[\mathbf{X}]]$ is a subalgebra of the K- algebra $K_S[[\mathbf{X}]]$ and the Gauss norm is a K-algebra non-archimedean norm on $\mathcal{H}_T K_S[[\mathbf{X}]]$ making it into a K-Banach algebra.*

Proof. Let $f, g = \sum_{\nu=\mathbf{0}}^{\infty^r} b_\nu U_\nu$ be elements of $\mathcal{H}_T K_S[[\mathbf{X}]]$. Then, by (12) and (17), we obtain $\|f \pm g\| = \sup_\nu |(a_\nu \pm b_\nu)\gamma_\nu| \leq \max\{\|f\|, \|g\|\}$. Similarly, since $u_{\nu_j} \in \overset{\circ}{K}[X_j]$, it follows that $d_{\nu_j}(s,t) \in \overset{\circ}{K}$ and (13), (14), (15) and (17) imply

$$\|fg\| \leq \sup_\nu \left\{ \max_{(\mu,\theta)\in I(\nu)} |a_\mu \gamma_\mu b_\theta \gamma_\theta| \right\} \leq \|f\| \, \|g\| .$$

Thus $\mathcal{H}_T K_S[[\mathbf{X}]]$ is a subalgebra of $K_S[[\mathbf{X}]]$ and the Gauss norm is a K-algebra norm on $\mathcal{H}_T K_S[[\mathbf{X}]]$.

We want to show that $\mathcal{H}_T K_S[[\mathbf{X}]]$ is complete. We take $f^{[t]} = \sum_{\nu=\mathbf{0}}^{\infty^r} a_{\nu,t} U_\nu$, $t \geq 1$, a Cauchy sequence of elements from $\mathcal{H}_T K_S[[\mathbf{X}]]$. Since

$$|a_{\nu,t+1} - a_{\nu,t}| \, |\gamma_\nu| \leq \left\| f^{[t+1]} - f^{[t]} \right\| , \tag{18}$$

for a fixed ν, each sequence $a_{\nu,t}$, $t = 0, 1, 2, ...$ is a Cauchy sequence in K. For $\nu \in \mathbf{N}^r$, let $a_\nu \in K$ be the limit of this sequence. Set $f = \sum_{\nu=0}^{\infty^r} a_\nu U_\nu \in K_S[[\mathbf{X}]]$. We have to prove that f is an element of $\mathcal{H}_T K_S[[\mathbf{X}]]$ and $\lim_{t\to\infty} \left\| f - f^{[t]} \right\| = 0$. We may assume $\left\| f^{[s]} - f^{[t]} \right\| \leq \frac{1}{t}$ for all $s \geq t$, $t = 1, 2, ...$. By (18) we obtain $|a_{\nu,s} - a_{\nu,t}| \, |\gamma_\nu| \leq \frac{1}{t}$, $s = t, t+1, \ldots$. Now the continuity of $| \, |$ implies that

$$|a_\nu - a_{\nu,t}| \, |\gamma_\nu| \leq \frac{1}{t}, \tag{19}$$

for all $\nu \in \mathbf{N}^r, t \in \mathbf{N},\ t \neq 0$. Since $f^{[t]} \in \mathcal{H}_T K_S[[\mathbf{X}]]$, $\lim_{N(\nu)\to\infty} a_{\nu,t}\gamma_\nu = 0$ and by (19) $\lim_{N(\nu)\to\infty} a_\nu \gamma_\nu = 0$. Hence $f \in \mathcal{H}_T K_S[[\mathbf{X}]]$. Furthermore, we have $\left\| f - f^{[t]} \right\| = \sup_\nu |a_\nu - a_{\nu,t}| \, |\gamma_\nu| \leq \frac{1}{t}$ and this implies $\lim_{t\to\infty} \left\| f - f^{[t]} \right\| = 0$. $\square$

Consider K a local field having the residue field a finite field of order m. Let $\beta_1, \beta_2, ..., \beta_m$ be elements of $\overset{\circ}{K}$ such that the cosets $\beta_j + \overset{\vee}{K}$, $j \in \{1, 2, ..., m\}$ are distinct. We take the sequence $S = \{\alpha_n\}_{n \geq 1}$, where

$$\alpha_i = \beta_i, \ \ \alpha_{i+mj} = \alpha_i, \ i = 1, 2, ..., m, \ \ j = 1, 2, ... \tag{20}$$

and $T = \{\gamma_\nu\}_{\nu \in \mathbf{N}^r}$ a fixed sequence of elements of $K^\times$ such that (15) holds and for every $\nu \in \mathbb{N}^r$

$$\gamma_\nu \notin \overset{\vee}{K}^{\left[\frac{N(\nu)}{m}\right]}, \ where \ \overset{\vee}{K}^0 = \overset{\circ}{K}. \tag{21}$$

Proposition 4. *Each series* $f = \sum_{\nu=\mathbf{0}}^{\infty^r} a_\nu U_\nu$ *of* $\mathcal{H}_T K_S[[\mathbf{X}]]$ *defines a map, denoted also by* f, $\overset{\circ}{K}^r \to K$, $\mathbf{y} \to f(\mathbf{y}) = \sum_{\nu=\mathbf{0}}^{\infty^r} a_\nu U_\nu(\mathbf{y})$. *Moreover* $\sup_{\mathbf{y} \in \overset{\circ}{K}^r} |f(\mathbf{y})| \leq \|f\|$ *and* f *gives rise to a continuous function on* $\overset{\circ}{K}^r$, *where for* $\mathbf{y} = (y_1, ..., y_r)$, $\|\mathbf{y}\| = \max_{1 \leq i \leq r} \{|y_i|\}$.

Proof. If $\mathbf{y} = (y_1, ..., y_r) \in \overset{\circ}{K}^r$, then there exists $x_{i,j} \in \{\alpha_1, \alpha_2, ..., \alpha_m\}$ such that $\widetilde{y_i} = \widetilde{x}_{i,j}$ in $\widetilde{K}$. Then $U_\nu(\mathbf{y}) \in \overset{\vee}{K}^{\left[\frac{N(\nu)}{m}\right]}$, and since $\overset{\vee}{K}$ is a principal ideal from (21) it follows that $\lim_{N(\nu) \to \infty} a_\nu U_\nu(\mathbf{y}) = 0$. Thus the series $\sum_{\nu=0}^{\infty^r} a_\nu U_\nu(\mathbf{y})$ converges to some element of K, whence the first assertion follows. Furthermore $|a_\nu U_\nu(\mathbf{y})| \leq |a_\nu \gamma_\nu| \leq \|f\|$ and hence it follows that $|f(\mathbf{y})| \leq \|f\|$. Now we consider a real number $\varepsilon > 0$. If we consider the partial sums $S_\mu = \sum_{\nu=\mathbf{0}}^{\mu} a_\nu U_\nu$, then there exists μ_0 such that for every $\mu \geq \mu_0$ and $\mathbf{y} \in \overset{\circ}{K}^r$, $|f(\mathbf{y}) - S_\mu(\mathbf{y})| \leq \sup_{\mu \geq \mu_0} |a_\mu \gamma_\mu| \leq \epsilon$. Hence S_μ converges uniformly to f and f gives rise to a continuous function on $\overset{\circ}{K}^r$. □

Proposition 5. *(Identity Theorem). If* $f \in \mathcal{H}_T K_S[[\mathbf{X}]]$ *vanishes for all* $\mathbf{y} \in \overset{\circ}{K}^r$, *then* $f = 0$.

Proof. Since the K-algebras $K_S[[X_1, ..., X_r]]$ and $K_S[[X_1, ..., X_{r-1}]]_S[[X_r]]$ are isomorphic it is enough to prove the theorem for $n = 1$. If $f = \sum_{i=0}^{\infty} a_i u_i$ and $y = \alpha_1$, then it follows that $a_0 = 0$. Then by putting $y = \alpha_2$ we obtain $a_1 = 0$ and by induction we find that $a_i = 0$, $i \in \{0, 1, ..., m-1\}$. Now we consider $y \in \overset{\circ}{K} \setminus \{\alpha_1\}$ such that $|y - \alpha_1|$ is small enough. Then $f(y) = 0$ implies

$a_m = 0$. By taking now $y \in \overset{\circ}{K} \setminus \{\alpha_2\}$ such that $|y - \alpha_2|$ is small enough we obtain that $a_{m+1} = 0$ and the proposition follows by induction on i. □

Theorem 6. *If K is a local field, S is given by (20), $\gamma_{\nu+\mu} = \gamma_\nu \gamma_\mu$ for every ν and μ, and (21) holds, then the K-Banach algebra $\mathcal{H}_T K_S[[\mathbf{X}]]$ is a noetherian domain.*

Proof. Since the K-algebras $K_S[[X_1, ..., X_r]]$ and $K_S[[X_1, ..., X_{r-1}]]_S[[X_r]]$ are isomorphic and $\gamma_{\nu+\mu} = \gamma_\nu \gamma_\mu$ we can consider $r = 1$. Let $\mathcal{I}$ be an ideal in $\mathcal{H}_T R_S[[X]]$. Thus we take $M = \{j;\ j \leq m$ *there exists* $h \in \mathcal{I}$ *with* $o(h) \equiv j(\bmod\ m)\}$. We can choose a finite number of elements $g_i = \sum_{\nu=o(g_i)}^{\infty} b_{\nu,i} u_\nu \in \mathcal{I}$, with $b_{o(g_i),i} = 1$, $i = 1, 2, ..., s$, such that:

a) for each i there exists $j \in M$ such that $o(g_i) \equiv j \pmod m$ and $o(g_i) \neq o(g_k)$, for every $i \neq k$;

b) for every $h \in \mathcal{I}$ there exists g_i such that $o(h) = o(g_i) + m\tau$ with $\tau \in \mathbb{N}$.

We shall prove that the ideal $\mathcal{I}$ is generated by the elements g_i, $i = 1, 2, ..., s$. Let $f = \sum_{\nu=o(f)}^{\infty} a_\nu u_\nu$ an element of $\mathcal{I}$. Then by b) there exists g_i and $v_{i,o(f)} \in K_S[[X]]$ such that $f_{o(f)} = f - a_{o(f)} v_{i,o(f)} g_i$ belongs to $\mathcal{I}$ and has the order greater than $o(f)$. By successive applications of this method we get, for every θ the elements $v_{i,\theta} \in K_S[[X]]$ such that

$$f_\theta = f - \sum_{i=1}^{s} \sum_{\nu=0}^{\theta} v_{i,\nu} g_i$$

is an element of $\mathcal{I}$ of order greater than θ. Since for fixed i $v_i = \sum_{\theta=0}^{\infty} v_{i,\theta} \in \mathcal{H}_T K_S[[X]]$ it follows that $\mathcal{H}_T R_S[[X]]$ is a noetherian domain. □

4. Acknowledgement

The author gratefully acknowledges the hospitality of School of Mathematical Sciences, Government College University, Lahore, Pakistan which provided an excellent research environment during my stay in Lahore. The research of the author was supported by a grant of Higher Education Commission of Pakistan.

References

[1] Y. Amice, *Les nombres p-adiques*, Presses Universitaires de France, Paris, 1975.

[2] S. Bosch, U. Güntzer and R. Remmert, *Non-Archimedean Analysis*, Grundlehren der mathematischen Wissenschaften **261**, Springer Verlag, Berlin, 1984.

[3] G. Groza, N. Pop, *Numerical solutions of two-point boudary value problems for ordinary differential equations using particular Newton interpolating series*, Proceedings ICAADE 2006, in Applied Analysis and Differential Equations, World Scientific 2007, p. 165-176.
[4] A. Robert, *A course in p-adic Analysis*, Springer-Verlag, New York, 2000.
[5] A. B. Shidlovskii, *Transcendental Numbers*, de Gruyter, 1989.
[6] O. Zariski and P. Samuel, *Commutative Algebra*, (vol.2), Springer-Verlag, New York, 1960.