Journal of Prime Research in Mathematics Vol. **3**(2007), 140-153

A DEGENERATE HYDRODYNAMIC DISPERSION MODEL

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Abstract. A Cauchy problem for a two-dimensional ultra-parabolic model of filtration through a porous ground of a viscous incompressible fluid containing a solute (tracer) is considered. The fluid is driven by the buoyancy force. The phenomenon of molecular diffusion of the tracer into the porous ground is taken into account. The porous ground consists of one-dimensional filaments oriented along some smooth non-degenerate vector field. Two cases are distinguished depending on spatial orientation of the filaments, and existence of generalized entropy solutions is proved for the both. In the first case, all filaments are parallel to the buoyancy (gravitational) force and, except for this, the equations of the model have rather general forms. In the second case, the filaments can be nonparallel to the buoyancy force and to each other, in general, but their geometric structure must be genuinely nonlinear. The proofs rely on the method of kinetic equation and the theory of Young measures and H-measures.

Key words : Ultraparabolic Equation, Genuine Nonlinearity, Non-Isotropic Porous Medium, Nonlinear Diffusion-Convection.
AMS SUBJECT: 35K65, 76S05, 76R50, 35D05.

1. Problem formulation and main results

In a space-time layer $\Pi := \mathbb{R}^2_x \times (0,T)$, $T = \text{const} > 0$, we consider the Cauchy problem for the non-isotropic model of filtration, which consists of the ultra-parabolic equation of balance of mass

$$u_t + \text{div}_x(a(u)\boldsymbol{v} + \mathbb{A}\boldsymbol{c}(u)) = \partial_n^*\partial_n b(u), \quad (\boldsymbol{x},t) \in \Pi, \tag{1a}$$

Darcy's law

$$\boldsymbol{v} = -\nabla_x p_* + g(u)\boldsymbol{e}_1, \quad (\boldsymbol{x},t) \in \Pi, \tag{1b}$$

and the continuity equation

$$\text{div}_x \boldsymbol{v} = 0, \quad (\boldsymbol{x},t) \in \Pi, \tag{1c}$$

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endowed with 1-periodic initial data belonging to $L^\infty(\mathbb{R}^2)$,

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad 0 \le u_0(\boldsymbol{x}) \le 1, \quad \boldsymbol{x} \in \mathbb{R}^2, \tag{1d}$$

and periodicity conditions

$$u(\boldsymbol{x}+\boldsymbol{e}_i,t) = u(\boldsymbol{x},t), \quad \nabla_x p_*(\boldsymbol{x}+\boldsymbol{e}_i,t) = \nabla_x p_*(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \Pi. \tag{1e}$$

Here $\boldsymbol{e}_i$ $(i = 1,2)$ are standard basis vectors in $\mathbb{R}^2$. Functions $u(\boldsymbol{x},t)$, $p_*(\boldsymbol{x},t)$ and $\boldsymbol{v}(\boldsymbol{x},t) = (v_1(\boldsymbol{x},t), v_2(\boldsymbol{x},t))$ are unknown. Functions $a(u)$, $b(u)$, $\boldsymbol{c}(u) = (c_1(u), c_2(u))$, $g(u)$, and initial data u_0 are given such that $a, b, c_1, c_2, g \in C^2(\mathbb{R})$, $b'(u) > 0\ \forall\, u \in \mathbb{R}$, $u_0(\boldsymbol{x}+\boldsymbol{e}_i) = u_0(\boldsymbol{x})\ \forall\, \boldsymbol{x} \in \mathbb{R}^2$, $i = 1,2$. $\mathbb{A}$ is a given 2×2-matrix with constant entries A_{ij}. The differential operators ∂_n and ∂_n^* are defined by the formulas

$$\partial_n = n_1(\boldsymbol{x})\partial_{x_1} + n_2(\boldsymbol{x})\partial_{x_2}, \quad \partial_n^* = \partial_{x_1}(n_1(\boldsymbol{x})\cdot) + \partial_{x_2}(n_2(\boldsymbol{x})\cdot),$$

where the given vector field $\boldsymbol{n}(\boldsymbol{x})$ is smooth, 1-periodic, and non-degenerate, i.e., $|\boldsymbol{n}(\boldsymbol{x})|^2 \neq 0$.

Model (1a)–(1c) describes a process of filtration through a porous ground of a viscous incompressible fluid containing a solute (tracer), with taking into account the phenomenon of molecular diffusion of the tracer into the porous ground, and under assumption that the porous ground consists of one-dimensional filaments (threads) oriented along the direction of the vector field $\boldsymbol{n}^\perp = (-n_2, n_1)$ [2, chapters 4, 10], [11]. In (1a)–(1c) $u(\boldsymbol{x},t)$ is the mass concentration of the tracer in the fluid, $\boldsymbol{v}(\boldsymbol{x},t)$ is the velocity of filtration, $p_*(\boldsymbol{x},t)$ is the pressure, $a(u)\boldsymbol{v}(\boldsymbol{x},t) + \mathbb{A}\boldsymbol{c}(u)$ is the instantaneous mass flux of the tracer, $g(u)\boldsymbol{e}_1$ is the density of the buoyancy (gravitational) force, and $b(u)$ is the diffusion function. The term $\partial_n^*\partial_n b(u)$ may be equivalently represented as $\mathrm{div}_x(\mathbb{B}\nabla_x u)$, where $\mathbb{B} = \boldsymbol{n} \otimes \boldsymbol{n}$ (or, equivalently, $\mathbb{B} = (n_i n_j)$) is the diffusion matrix. Such form of the diffusion matrix implies that the diffusion phenomenon is absent in the direction perpendicular to $\boldsymbol{n}$.

Note that equations (1b)–(1c) yield the second order elliptic equation

$$\Delta_x p_* = g(u)_{x_1}, \quad (\boldsymbol{x},t) \in \Pi. \tag{2}$$

Hence the model under consideration may be represented as the coupled system of ultra-parabolic equation (1a) for the concentration and elliptic equation (2) for the pressure. In this case Darcy's law (1b) is inserted into (1a). Thus the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ does not appear in the system of equations and is recovered a posteriori from Darcy's law.

The following notation for the linear spaces of periodic functions is used throughout this work. By Q we denote $\Omega \times (0,T)$, where $\Omega := [0,1) \times [0,1)$. By $L^p \subset L^p_{loc}(\mathbb{R}^2)$ and $H^{s,p} \subset H^{s,p}_{loc}(\mathbb{R}^2)$ we denote the Banach spaces, which consist of 1-periodic functions and are supplemented with the norms $\|u\|_{L^p} = \|u\|_{L^p(\Omega)}$, $\|u\|_{H^{s,p}} = \|u\|_{H^{s,p}(\Omega)}$. For $l \ge 0$ let C^l be the closed subspace of $u \in C^l(\mathbb{R}^2)$ such that u is 1-periodic in x_1 and x_2.

The differential operator $\boldsymbol{B} = \partial_n^* \partial_n$: $C^\infty \mapsto L^2$ is symmetric and non-negative in the Hilbert space L^2. By the Friedrichs theorem, it has the self-adjoint extension $\boldsymbol{B}$: $D(\boldsymbol{B}) \mapsto L^2$, where $D(\boldsymbol{B})$ consists of all functions $u \in L^2$ such that $\partial_n u \in L^2$. Being supplemented with the norm

$$\|u\|_{\mathfrak{H}}^2 := \|u\|_{L^2}^2 + \|\partial_n u\|_{L^2}^2,$$

$D(\boldsymbol{B})$ becomes the Hilbert space, which will be denoted by $\mathfrak{H}$.

We are now in a position to define an entropy solution of problem (1).

Definition 1. *A pair of functions $u \in L^\infty(0,T;L^\infty) \cap L^2(0,T;\mathfrak{H})$ and $p_* \in L^r(0,T;H^{1,r})$ ($\forall\, r \in [1,+\infty)$) is an entropy solution of problem* (1) *if the bound $0 \le u(\boldsymbol{x},t) \le 1$ holds a.e. in Q, the integral equality*

$$\int_Q (\nabla_x p_* \cdot \nabla_x \zeta - g(u)\zeta_{x_1}) d\boldsymbol{x}dt = 0 \tag{3}$$

holds for all 1-periodic in $\boldsymbol{x}$ test functions $\zeta \in C^1_{loc}(\Pi)$, and the integral inequality

$$\int_Q \{\varphi(u)\eta_t + (\psi_1(u)\boldsymbol{v} + \mathbb{A}\boldsymbol{\psi}_2(u)) \cdot \nabla_x \eta + w(u)\partial_n^* \partial_n \eta$$
$$-\varphi''(u)b'(u)|\partial_n u|^2 \eta\Big\}\, d\boldsymbol{x}dt + \int_\Omega \eta(u_0)\beta(x,0)d\boldsymbol{x} \ge 0, \tag{4}$$

where $\boldsymbol{v}$ is given by Darcy's law (1b), *holds for all functions φ, ψ_1, ψ_2, and w such that $\varphi \in C^2_{loc}(\mathbb{R})$, $\varphi''(u) \ge 0$, $\psi_1'(u) = a'(u)\varphi'(u)$, $\boldsymbol{\psi}_2'(u) = \boldsymbol{c}'(u)\varphi'(u)$, and $w'(u) = b'(u)\varphi'(u)$, and for all non-negative 1-periodic in $\boldsymbol{x}$ functions $\eta \in C^2_{loc}(\Pi)$ such that $\eta|_{t=T} = 0$.*

We say that model (1a)–(1c) is *genuinely nonlinear*, if the equation of balance of mass (1a) is genuinely nonlinear, i.e., if its coefficients satisfy the following condition.

Condition G. *For a.e. $\boldsymbol{x} \in \mathbb{R}^2$ the function*

$$\lambda \mapsto \sum_{k=1}^{2} \big(A_{1k} c_k(\lambda) n_2(\boldsymbol{x})$$
$$-A_{2k} c_k(\lambda) n_1(\boldsymbol{x})\big) + (1/2) b(\lambda)\big(n_2(\boldsymbol{x})\partial_n n_1(\boldsymbol{x}) - n_1(\boldsymbol{x})\partial_n n_2(\boldsymbol{x})\big)$$

is nonlinear on any nondegenerate subinterval of the interval $0 \le \lambda \le 1$, and the function $a''(\lambda)$ is linearly independent of the function

$$\lambda \mapsto \sum_{k=1}^{2} \big(A_{1k} c_k''(\lambda) n_2(\boldsymbol{x})$$
$$-A_{2k} c_k''(\lambda) n_1(\boldsymbol{x})\big) + (1/2) b''(\lambda)\big(n_2(\boldsymbol{x})\partial_n n_1(\boldsymbol{x}) - n_1(\boldsymbol{x})\partial_n n_2(\boldsymbol{x})\big)$$

on any nondegenerate subinterval inside the set $\{\lambda \in [0,1] \,|\, a''(\lambda) \ne 0\}$.

The following theorems are the main results of this article.

Theorem 1. *Let $\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{e}_1 = 0$ for all $\boldsymbol{x}\in\mathbb{R}^2$. Whenever $u_0\in L^\infty$ and $0\le u_0\le 1$ for a.e. $\boldsymbol{x}\in\mathbb{R}^2$, problem* (1) *has at least one entropy solution in the sense of definition 1.*

Theorem 2. *Assume that $\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{e}_1\neq 0$, in general, and condition G holds. Whenever $u_0\in L^\infty$ and $0\le u_0(\boldsymbol{x})\le 1$ for a.e. $\boldsymbol{x}\in\mathbb{R}^2$, problem* (1) *has at least one entropy solution in the sense of definition 1.*

Peculiarity of problem (1) consists of the spatially degenerate character of the diffusion process. This feature is not observed in the well-known classical models of two-phase filtration, for whom the extended theory of classical and generalized solutions has been constructed [1, 2, 11]. So, the above stated results complement this vast theory. Proofs of Theorems 1 and 2 rely on the method of kinetic equation, which allows to reduce quasilinear equations and systems to linear scalar equations on 'distribution' functions involving additional 'kinetic' variables, and on the theory of Young measures and H-measures.

2. Parabolic approximate problem. Partial compactness of the family of approximate velocity fields

For a fixed $\varepsilon>0$ we consider the approximate system

$$\Delta_x p_{*\varepsilon} = g(u_\varepsilon)_{x_1}, \quad (\boldsymbol{x},t)\in\Pi, \tag{5}$$

$$\partial_t u_\varepsilon + \mathrm{div}_x(a(u_\varepsilon)\boldsymbol{v}_\varepsilon + \mathbb{A}\boldsymbol{c}(u_\varepsilon)) = \partial_n^*\partial_n b(u_\varepsilon) + \varepsilon\Delta_x u_\varepsilon, \quad (\boldsymbol{x},t)\in\Pi, \tag{6}$$

$$\boldsymbol{v}_\varepsilon = -\nabla_x p_{*\varepsilon} + g(u_\varepsilon)\boldsymbol{e}_1, \quad (\boldsymbol{x},t)\in\Pi, \tag{7}$$

endowed with boundary conditions (1d) and (1e). From the well-known results in the theory of the Muskat–Leverett model [1] it follows that problem (5)–(7), (1d), (1e) has a unique smooth solution for any initial data $u_0\in L^\infty$ (the pressure $p_{*\varepsilon}$ is defined up to an arbitrary additional constant, which can be fixed zero by imposing that $\int_\Omega p_{*\varepsilon}(\boldsymbol{x},t)d\boldsymbol{x}=0$). Maximum principle and energy estimates imply the inequalities

$$0\le u_\varepsilon\le 1, \quad \|u_\varepsilon\|_{L^2(0,T;\mathfrak{H})} + \varepsilon\|\nabla_x u_\varepsilon\|_{L^2(Q)} + \|\nabla_x p_{*\varepsilon}\|_{L^r(Q)} \le C_*(Q), \tag{8}$$

where the exponent $r\in[1,\infty)$ is arbitrary and the constant C_* does not depend on ε. From these inequalities it follows that there exist a sequence of solutions u_ε, p_ε, $\boldsymbol{v}_\varepsilon$ to problem (5)–(7), (1d), (1e), and functions u, p, and $\boldsymbol{v}$ such that

$$\begin{aligned} u_\varepsilon \to u \qquad & \text{weakly* in } L^\infty(Q), \text{ weakly in } L^2(0,T;\mathfrak{H}),\\ \boldsymbol{v}_\varepsilon\to\boldsymbol{v},\ \nabla_x p_{*\varepsilon}\to\nabla_x p_* \qquad & \text{weakly in } L^r(Q),\ \forall\, r\in[1,\infty),\ \text{as } \varepsilon\searrow 0. \end{aligned} \tag{9}$$

Since problem (1) is nonlinear then it is necessary to prove strong convergence of a sequence of solutions of problem (5)–(7), (1d), (1e). We start with establishing a partial compactness property for approximate velocity fields.

Introduce

$$\boldsymbol{m}_\varepsilon(\boldsymbol{x},t) = -\boldsymbol{v}_\varepsilon(\boldsymbol{x},t) + \frac{g(u_\varepsilon(\boldsymbol{x},t))n_1(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})}{|\boldsymbol{n}(\boldsymbol{x})|^2}. \tag{10}$$

Proposition 3. *For any bounded set $\mathcal{K} \subset \mathbb{R}^2_x$ there is a constant $C_1(\mathcal{K})$ such that*

$$\|\boldsymbol{m}_\varepsilon\|_{L^2(0,T;H^{1,2}(\mathcal{K}))} + \|\partial_t \boldsymbol{m}_\varepsilon\|_{L^2(0,T;H^{-1,2}(\mathcal{K}))} \le C_1(\mathcal{K}).$$

The families $\{\boldsymbol{m}_\varepsilon\}_{\varepsilon>0}$ and $\{\boldsymbol{v}_\varepsilon \cdot \boldsymbol{n}^\perp\}_{\varepsilon>0}$ are relatively compact in $L^2_{loc}(\Pi)$.

Proof is just a slight modification of [7, proof of proposition 2]. □

3. Kinetic equation

Method of kinetic equation has been created and applied recently to study a wide range of problems, for example, to study the equations of isentropic gas dynamics and p-systems, and the first and second order quasilinear conservation laws [6, 8]. In this paper we make use of the version of method, that has been constructed in [8].

In order to state a theorem on kinetic equation corresponding to problem (1), recall some facts from the measure theory. Further $\mathbb{M}(\mathbb{R}^n)$ denotes the Banach space of bounded Radon measures on $\mathbb{R}^n$. Recall that a mapping σ: $\mathbb{R}^2_x \times (0,T) \mapsto \mathbb{M}(\mathbb{R}^n)$ is said to be bounded weakly* measurable and 1-periodic if for all $F \in L^1_{loc}(\mathbb{R}^2_x \times (0,T); C_0(\mathbb{R}^n))$ the function $(\boldsymbol{x},t) \mapsto \int_{\mathbb{R}^n_p} F(\boldsymbol{x},t,p)d\sigma_{x,t}(p)$ is measurable and $\int_{\mathbb{R}^n_p} F(\boldsymbol{x},t,p)d\sigma_{x+e_i,t}(p) = \int_{\mathbb{R}^n_p} F(\boldsymbol{x}-\boldsymbol{e}_i,t,p)d\sigma_{x,t}(p)$, $i=1,2$. Here we use the standard notation $\sigma_{x,t} = \sigma(\boldsymbol{x},t)$ as if measures $\sigma_{x,t}$ were parametrized by $(\boldsymbol{x},t)$ and, in line with the notation from [3], we say that $\sigma \in L^\infty_w(\mathbb{R}^2_x \times (0,T); \mathbb{M}(\mathbb{R}^n))$.

Now let us consider in detail the notion and properties of Young measures associated with a sequence of approximate saturations u_ε: $\mathbb{R}^2_x \times (0,T) \mapsto [0,1]$. We start with the observation that, by Tartar's theorem [3, section 3.2], there exist a subsequence, still denoted by u_ε, and a family of probability Radon measures $\mu_{x,t}$ supported uniformly on $[0,1]$ such that

$$h(u_\varepsilon) \to \overline{h} \text{ weakly* in } L^\infty(Q), \quad \overline{h} = \int_{\mathbb{R}_\lambda} h(\lambda)d\mu_{x,t}(\lambda) \quad \forall\, h \in C(\mathbb{R}_\lambda). \tag{11}$$

The mapping $(\boldsymbol{x},t) \mapsto \mu_{x,t}$ is weakly* measurable and 1-periodic in $\boldsymbol{x}$. In particular, the distribution function of the Young measure $\mu_{x,t}$

$$f(\boldsymbol{x},t,s) := \int_{(-\infty,s]} d\mu_{x,t}(\lambda)$$

is 1-periodic in $\boldsymbol{x}$, monotone and right continuous in s. In terms of f, measure $\mu_{x,t}$ is the Stieltjes measure, $\mu_{x,t} = d_\lambda f(\boldsymbol{x}, t, \lambda)$.

Set $q_\varepsilon := \partial_n u_\varepsilon$. The functions q_ε: $\mathbb{R}^2_x \times (0,T) \mapsto \mathbb{R}_q$ are measurable and 1-periodic in $\boldsymbol{x}$. From (8) it follows that the sequence $(u_\varepsilon, q_\varepsilon)$ is bounded in L^2, which along with Ball's theorem [3, section 4.2] yields the following.

Lemma 4. *There exist a subsequence still denoted by $(u_\varepsilon, q_\varepsilon)$ and a nonnegative measure-valued 1-periodic in $\boldsymbol{x}$ function $\sigma \in L^\infty_w(Q, \mathbb{M}(\mathbb{R}_\lambda \times \mathbb{R}_q))$ such that $\int_{\mathbb{R}_\lambda \times \mathbb{R}_q} d\sigma_{x,t}(\lambda, q) = 1$, $h(u_\varepsilon, q_\varepsilon) \to \overline{h}$ weakly in $L^r(Q)$ $(1 < r \le 2/p)$, $\overline{h} = \int_{\mathbb{R}_\lambda \times \mathbb{R}_q} h(\lambda, q) d\sigma_{x,t}(\lambda, q)$ for a.e. $(\boldsymbol{x}, t) \in Q$ for all continuous functions h: $\mathbb{R}_\lambda \times \mathbb{R}_q \mapsto \mathbb{R}$ satisfying the growth condition $|h(\lambda, q)| \le c(1 + |\lambda| + |q|)^p$ $(0 \le p < 2)$, and the probability measure $\sigma_{x,t}$ is supported in $[0,1] \times \mathbb{R}_q$.*

Measure $\sigma_{x,t}$ has additional useful properties [8, lemmas 9, 12] as follows.

Lemma 5. *The following bound and identity hold true:*

$$\int_Q \Big(\int_{\mathbb{R}_\lambda \times \mathbb{R}_q} q^2 d\sigma_{x,t}(\lambda, q) \Big) d\boldsymbol{x} dt < \infty, \quad \partial_n f(\boldsymbol{x}, t, \lambda) = - \int_{\mathbb{R}_q} q d\sigma_{x,t}(\lambda, q).$$

The identity here is understood in the distributions sense, i.e., in the sense of the integral equality

$$\int_{Q \times \mathbb{R}_\lambda} f(\boldsymbol{x}, t, \lambda) \partial^*_n \zeta(\boldsymbol{x}, t, \lambda) d\boldsymbol{x} dt d\lambda = \int_{Q \times \mathbb{R}_\lambda \times \mathbb{R}_q} \zeta(\boldsymbol{x}, t, \lambda) q d\sigma_{x,t}(\lambda, \boldsymbol{q}) d\boldsymbol{x} dt, \tag{12}$$

for arbitrary smooth 1-periodic in $\boldsymbol{x}$ and vanishing for large $|\lambda|$ functions $\zeta(\boldsymbol{x}, t, \lambda)$.

In particular, the function

$$\chi(\boldsymbol{x}, t, s) := \int_{(-\infty, s] \times \mathbb{R}_q} q^2 d\sigma_{x,t}(\lambda, q)$$

is 1-periodic in $\boldsymbol{x}$, monotone and right continuous in s, and the Stieltjes measure $d_\lambda \chi(\boldsymbol{x}, t, \lambda)$ is supported on $[0,1]$ for a.e. $(\boldsymbol{x}, t) \in \mathbb{R}^2_x \times (0,T)$.

The following theorem introduces the notion of the kinetic equation:

Theorem 6. *There exists a nonnegative 1-periodic in $\boldsymbol{x}$ defect measure $M \in \mathbb{M}(\mathbb{R}^2_x \times (0,T) \times \mathbb{R}_\lambda)$ with $spt\, M \subset \mathbb{R}^2_x \times (0,T) \times [0,1]$ such that this measure, the Stieltjes measure $d_\lambda \chi$, and the distribution function f satisfy the integral* *equality*

$$\int_{Q\times\mathbb{R}_\lambda} \big\{\partial_t\zeta - a'(\lambda)\boldsymbol{m}\cdot\nabla_x\zeta + (n_1/|\boldsymbol{n}|^2)a'(\lambda)g(\lambda)\partial_n\zeta + \mathbb{A}\boldsymbol{c}'(\lambda)\cdot\nabla_x\zeta + b'(\lambda)\partial_n^*\partial_n\zeta\big\}\, f(\boldsymbol{x},t,\lambda)d\boldsymbol{x}dtd\lambda$$
$$+\int_{Q\times\mathbb{R}_\lambda} a'(\lambda)(n_1/|\boldsymbol{n}|^2)\Big(\int_\lambda^\infty g'(s)f(\boldsymbol{x},t,s)ds\Big)\partial_n\zeta d\boldsymbol{x}dtd\lambda$$
$$+\int_Q\int_{\mathbb{R}_\lambda} b'(\lambda)\partial_\lambda\zeta d_\lambda\chi(\boldsymbol{x},t,\lambda)d\boldsymbol{x}dt + \int_{Q\times\mathbb{R}_\lambda}\partial_\lambda\zeta dM(\boldsymbol{x},t,\lambda)$$
$$+\int_{\Omega\times\mathbb{R}_\lambda} f_0(\boldsymbol{x},\lambda)\zeta(\boldsymbol{x},0,\lambda)d\boldsymbol{x}d\lambda = 0 \quad (13)$$

for all 1-periodic in $\boldsymbol{x}$ smooth functions $\zeta(\boldsymbol{x},t,\lambda)$ vanishing in a neighborhood of the plane $\{t=T\}$ and for sufficiently large λ. Moreover,

$$f(\cdot,t,\cdot)\to f_0 \text{ weakly* in } L^\infty(\Omega\times\mathbb{R}_\lambda), \text{ as } t\searrow 0. \quad (14)$$

In (13)*,* (14)

$$f_0(\boldsymbol{x},\lambda) = \begin{cases} 1 & \text{if } \lambda \ge u_0(\boldsymbol{x}), \\ 0 & \text{if } \lambda < u_0(\boldsymbol{x}), \end{cases} \quad (15)$$

$$\boldsymbol{m} = \lim_{\varepsilon\searrow 0}\boldsymbol{m}_\varepsilon, \quad \boldsymbol{m}\in L^r_{loc}(\Pi)\ \forall r<\infty. \quad (16)$$

Remark 1. *On the strength of proposition 3, limiting relation* (16) *is strong.*

In the sense of distributions the integral equality (13) and the limiting relation (14) are equivalent to the following kinetic equation and Cauchy's data:

$$\partial_t f - a'(\lambda)\mathrm{div}_x(\boldsymbol{m}f) + a'(\lambda)g(\lambda)\partial_n^*(n_1 f/|\boldsymbol{n}|^2)$$
$$+\mathbb{A}\boldsymbol{c}'(\lambda)\cdot\nabla_x f - b'(\lambda)\partial_n^*\partial_n f + a'(\lambda)\partial_n^*\left((n_1/|\boldsymbol{n}|^2)\int_{-\infty}^\lambda g'(s)f(\boldsymbol{x},t,s)ds\right)$$
$$+\partial_\lambda\big(b'(\lambda)\partial_\lambda\chi + M\big) = 0, \text{ in } Q\times\mathbb{R}_\lambda, \quad (17)$$

$$f(\boldsymbol{x},0,\lambda) = f_0(\boldsymbol{x},\lambda), \text{ in } \Omega\times\mathbb{R}_\lambda. \quad (18)$$

Proof of theorem 6 is analogous to [8, proof of theorem 5]. □

4. Proof of theorem 1

In the case $\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{e}_1 = 0$ formula (10) reduces to $\boldsymbol{m}_\varepsilon = -\boldsymbol{v}_\varepsilon$. Hence, on the strength of Remark 1, there exist a subsequence from $\varepsilon > 0$ and a 1-periodic in $\boldsymbol{x}$ vector-field $\boldsymbol{v} \in L^2(0,T;H^{1,2})$ such that $\boldsymbol{v}_\varepsilon \to \boldsymbol{v}$ strongly in $L^2_{loc}(\Pi)$ and weakly in $L^2(0,T;H^{1,2})$, as $\varepsilon \searrow 0$. Thus equality (13) takes the form

$$\int_{Q\times\mathbb{R}_\lambda} \big\{\partial_t\zeta + a'(\lambda)\boldsymbol{v}\cdot\nabla_x\zeta + \mathbb{A}\boldsymbol{c}'(\lambda)\cdot\nabla_x\zeta + b'(\lambda)\partial_n^*\partial_n\zeta\big\} f(\boldsymbol{x},t,\lambda)d\boldsymbol{x}dtd\lambda$$
$$+ \int_Q\int_{\mathbb{R}_\lambda} b'(\lambda)\partial_\lambda\zeta d_\lambda\chi(\boldsymbol{x},t,\lambda)d\boldsymbol{x}dt + \int_{Q\times\mathbb{R}_\lambda}\partial_\lambda\zeta dM(\boldsymbol{x},t,\lambda)$$
$$+ \int_{\Omega\times\mathbb{R}_\lambda} f_0(\boldsymbol{x},\lambda)\zeta(\boldsymbol{x},0,\lambda)d\boldsymbol{x}d\lambda = 0. \quad (19)$$

Equation (19) has already been studied in detail and the following result for it has been established [8, theorem 7].

Proposition 7. *From equation* (19) *it follows that the distribution function* $f(\boldsymbol{x},t,\lambda)$ *satisfies the equality* $f(\boldsymbol{x},t,\lambda)\big(1-f(\boldsymbol{x},t,\lambda)\big)=0$ *a.e. in* $\Pi\times\mathbb{R}_\lambda$*. In other words,* f *attains the values 0 and 1 only.*

On the strength of proposition 7, we shortly complete justification of theorem 1 by following arguments of [8, remark 3] and [9, Sec. 8, 9].

5. Notion of H-measures

Now we turn to consideration of the case, when the genuine nonlinearity condition holds and $\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{e}_1 \neq 0$, in general. In this section we consider in detail the general properties of H-measures corresponding to subsequences of $\{u_\varepsilon\}$. We start with the following observation.

Remark 2. *System* (5)–(7) *admits the kinetic equation of the form* (17)*, in which* $dM := dM_\varepsilon(\boldsymbol{x},t,\lambda) = \varepsilon|\nabla_x u_\varepsilon|^2 d\gamma_{u_\varepsilon(x,t)}(\lambda)d\boldsymbol{x}dt$*,* $d_\lambda\chi := d_\lambda\chi_\varepsilon(\boldsymbol{x},t,\lambda) = |\partial_n u_\varepsilon(\boldsymbol{x},t)|^2 d\gamma_{u_\varepsilon(x,t)}(\lambda)$*,* $\gamma_{u_\varepsilon(x,t)}$ *is the parametrized Dirac measure on* $\mathbb{R}_\lambda$ *concentrated at the point* $\lambda = u_\varepsilon(\boldsymbol{x},t)$*,* $\boldsymbol{m} := \boldsymbol{m}_\varepsilon$ *is given by* (10)*, and*

$$f := f_\varepsilon(\boldsymbol{x},t,\lambda) = \begin{cases} 1 & \text{if } \lambda \geq u_\varepsilon(\boldsymbol{x},t), \\ 0 & \text{if } \lambda < u_\varepsilon(\boldsymbol{x},t). \end{cases}$$

This remark is quite clear in view of the evident representation

$$\varphi(u_\varepsilon(\boldsymbol{x},t)) = -\int_{\mathbb{R}} \varphi'(\lambda) f_\varepsilon(\boldsymbol{x},t,\lambda)d\lambda \quad \forall\,\varphi\in C_0^1(\mathbb{R}). \quad (20)$$

Moreover, due to (20) and (11), distribution functions f_ε of the Dirac measures $\gamma_{u_\varepsilon(x,t)}$ and the distribution function f of the Young measure $\mu_{x,t}$ are connected by the limiting relation

$$f_\varepsilon \to f \text{ weakly* in } L^\infty(Q\times\mathbb{R}_\lambda), \text{ as } \varepsilon\searrow 0. \quad (21)$$

Introduce the set

$$\mathcal{E} := \{\lambda_0\in\mathbb{R}\,|\, f(\cdot,\cdot,\lambda)\to f(\cdot,\cdot,\lambda_0) \text{ strongly in } L^1_{loc}(\Pi), \text{ as } \lambda\to\lambda_0\}.$$

From [4, lemma 4] and Panov's theorem on a modification of Tartar's H-measures [4, theorem 3] the following two propositions follow immediately.

Lemma 8. *The complement of $\mathcal{E}$ in $\mathbb{R}$ is at most countable and for any $\lambda \in \mathcal{E}$ the limiting relation $f_\varepsilon(\cdot,\cdot,\lambda) \underset{\varepsilon\searrow 0}{\longrightarrow} f(\cdot,\cdot,\lambda)$ weakly* in $L^\infty(\Pi)$ holds true.*

Theorem H. (Existence of H-measures). *There exist a family of locally finite Radon measures $\{\nu^{pq}\}_{p,q\in\mathcal{E}}$ on $\Pi\times\mathbb{S}^2$ and a subsequence $\{f^k(\lambda)-f(\lambda)\}_{k\in\mathbb{N}}$ from $\{f_\varepsilon(\lambda)-f(\lambda)\}_{\varepsilon>0}$, $\lambda\in\mathcal{E}$, such that for any $\Phi_1,\Phi_2\in C_0(\Pi)$ and $\psi\in C(\mathbb{S}^2)$ the equality*

$$\int_{\Pi\times\mathbb{S}^2}\Phi_1(\boldsymbol{x},t)\overline{\Phi_2(\boldsymbol{x},t)}\psi(\boldsymbol{y})d\nu^{pq}(\boldsymbol{x},t,\boldsymbol{y}) =$$

$$\lim_{k\nearrow\infty}\int_{\mathbb{R}^3}\mathcal{F}[\Phi_1(f^k(p)-f(p))](\boldsymbol{\xi})\overline{\mathcal{F}[\Phi_2(f^k(q)-f(q))](\boldsymbol{\xi})}\psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right)d\boldsymbol{\xi} \quad (22)$$

holds for all $p,q\in\mathcal{E}$.

In the formulation of theorem H and further $f^k := f_{\varepsilon_k}$, $\{\varepsilon_k\}$ is the extracted subsequence from $\{\varepsilon\searrow 0\}$, $\bar{\varphi}$ is the complex conjugate of φ. By $\mathcal{F}$ we denote the Fourier transform in $\boldsymbol{x}$ and t: $\mathcal{F}[\varphi](\boldsymbol{\xi}) = \int_{\mathbb{R}^3}\varphi(\boldsymbol{x},t)e^{2\pi i(\xi_0 t+\xi_1 x_1+\xi_2 x_2)}d\boldsymbol{x}dt$ for any integrable function φ. If φ is originally defined merely for $t\in[0,T]$ then it is supposed to be equal to zero outside $[0,T]$.

Family of measures $\{\nu^{pq}\}_{p,q\in\mathcal{E}}$ is called the *H-measure associated with the extracted subsequence* $\{f^k-f\}$. The following properties are consequences of the general theory of H-measures.

Lemma 9. ***(1)*** *For any finite set $E := \{p_1,\dots,p_n\}\subset\mathcal{E}$ the set of measures $(\nu^{p_ip_j})_{i,j=1,\dots,n}$ is hermitian non-negative, i.e., $\nu^{p_ip_j} = \bar{\nu}^{p_jp_i}$ and $\sum_{i,j=1}^n\langle\nu^{p_ip_j},\Phi_i\overline{\Phi}_j\psi\rangle\geq 0$ for all $\Phi_1,\dots,\Phi_n\in C_0(\Pi)$ and $\psi\in C(\mathbb{S}^2)$, $\psi\geq 0$* [10, corollary 1.2].
(2) *Mapping $(p,q)\mapsto\nu^{pq}$ is continuous from $\mathcal{E}\times\mathcal{E}$ into $\mathbb{M}(\Pi\times\mathbb{S}^2)$* [4, theorem 3].
(3) *For any $p,q\in\mathcal{E}$ measure ν^{pq} is absolutely continuous with respect to Lebesgue's measure on Π. As the functional defined on $C(Q\times\mathbb{S}^2)$, it admits the natural extension on $L^2(Q,C(\mathbb{S}^2))$ and therefore the decomposition $d\nu^{pq}(\boldsymbol{x},t,\boldsymbol{y}) = d\Lambda^{pq}_{x,t}(\boldsymbol{y})d\boldsymbol{x}dt$ takes place. Mapping $(\boldsymbol{x},t)\mapsto\Lambda^{pq}_{x,t}$, where $(\boldsymbol{x},t)\in\Pi$, is 1-periodic in $\boldsymbol{x}$, belongs to $L^2_{w,loc}(\Pi,\,\mathbb{M}(S^2))$, and is uniquely defined by ν^{pq}. For a.e. $(\boldsymbol{x},t)\in\Pi$ the mapping $\lambda\mapsto\Lambda^{\lambda\lambda}_{x,t}$ is right-continuous from $\mathcal{E}$ into the space* $\mathbb{M}(\mathbb{S}^2)$ [5, proposition 3, corollary 1].
(4) *Fix $(\boldsymbol{x},t)$ in the set of the full measure on Q such that the mapping $\Lambda^{\lambda\lambda}_{x,t}\in\mathbb{M}(\mathbb{S}^2)$ is well-defined, according to the previous item. For $\lambda\in\mathcal{E}$ by $L(\lambda)\subset\mathbb{R}^3$ denote the minimal linear subspace, which contains the support of the measure $\Lambda^{\lambda\lambda}_{x,t}$. Among the subspaces $L(\lambda)$ choose the subspace $L := L(\lambda_0)$*

with the maximal dimension. The following result on stabilization of linear span of supports of measures $\Lambda^{\lambda\lambda}_{x,t}$ *is valid: there exists a number* $\delta > 0$ *such that, for any* λ *in the set* $[\lambda_0, \lambda_0 + \delta] \cap \mathcal{E}$*, the subspace* $L(\lambda)$ *coincides with* L [5, lemma 3].

(5) $f^k(\cdot,\cdot,\lambda) \underset{k\nearrow\infty}{\to} f(\cdot,\cdot,\lambda)$ *strongly in* $L^2_{loc}(\Pi)$ *for* $\lambda \in \mathcal{E}$ *if and only if* $\nu^{\lambda\lambda} \equiv 0$ [10].

In item 3 of lemma 9 by $L^2_{w,loc}(\Pi, \mathbb{M}(\mathbb{S}^2))$ there is denoted the space of weakly measurable with respect to Lebesgue's measure on Π mappings $(\boldsymbol{x}, t) \mapsto \Lambda_{x,t}$ from Π into $\mathbb{M}(\mathbb{S}^2)$ such that its restriction to Q has the finite norm

$$\|\Lambda\|_{L^2_w(Q,\mathbb{M}(\mathbb{S}^2))} = \Big(\int_Q \|\Lambda_{x,t}\|^2_{\mathbb{M}(\mathbb{S}^2)} d\boldsymbol{x}dt\Big)^{1/2}, \quad \forall\, \Lambda \in L^2_w(Q, \mathbb{M}(\mathbb{S}^2)).$$

6. The localization principle for H-measures. Proof of theorem 2

The following additional property of the H-measure is crucial for justification of theorem 2:

Theorem 10. (The localization principle.) *For a.e.* $\lambda \in \mathbb{R}$*, the support of the* H*-measure* $\nu^{\lambda\lambda}$*, associated with the extracted subsequence* $\{f^k - f\}$*, lies in the intersection of the sets*

$$\{(\boldsymbol{x}, t, \boldsymbol{y}) \in \Pi \times \mathbb{S}^2 \mid n_1(\boldsymbol{x})y_1 + n_2(\boldsymbol{x})y_2 = 0\}$$

and

$$\Big\{(\boldsymbol{x}, t, \boldsymbol{y}) \in \Pi \times \mathbb{S}^2 \mid y_0 - \sum_{r=1}^{2}\big[a'(\lambda)m_r(\boldsymbol{x}, t) - \sum_{k=1}^{2} A_{rk}c'_k(\lambda) - (1/2)b'(\lambda)\partial_n n_r(\boldsymbol{x})\big]y_r = 0\Big\}.$$

Proof of theorem 10 is analogous to [9, proof of theorem 3 and corollary 1]. □

We start proof of theorem 2 with justification of the triviality of the H-measure due to the localization principle and the genuine nonlinearity condition:

Lemma 11. *If condition G holds then the* H*-measure* $\nu^{\lambda\lambda}$ *is zero measure for a.e.* $\lambda \in \mathbb{R}$.

Proof. On the strength of theorem 10, the support of the H-measure $\nu^{\lambda\lambda}$ lies in the set $E^\lambda_0 := E_1 \cap (E^\lambda_2 \cup E^\lambda_3)$ for a.e. $\lambda \in [0, 1]$, where

$$E_1 := \big\{(\boldsymbol{x}, t, \boldsymbol{y}) \in \Pi \times \mathbb{S}^2 \,|\, n_1(\boldsymbol{x})y_1 + n_2(\boldsymbol{x})y_2 = 0\big\},$$

$$E_2^\lambda := \Big\{(\boldsymbol{x},t,\boldsymbol{y}) \in \Pi \times \mathbb{S}^2 \ : \ y_0 - \sum_{r=1}^{2}\Big(a'(\lambda)m_r(\boldsymbol{x},t) - \sum_{k=1}^{2} A_{rk}c_k'(\lambda) - (1/2)b'(\lambda)\partial_n n_r(\boldsymbol{x})\Big)y_r = 0, \ n_1(\boldsymbol{x}) \neq 0\Big\},$$

and

$$E_3^\lambda := \Big\{(\boldsymbol{x},t,\boldsymbol{y}) \in \Pi \times \mathbb{S}^2 \ : \ y_0 - \sum_{r=1}^{2}\Big(a'(\lambda)m_r(\boldsymbol{x},t) - \sum_{k=1}^{2} A_{rk}c_k'(\lambda) - (1/2)b'(\lambda)\partial_n n_r(\boldsymbol{x})\Big)y_r = 0, \ n_1(\boldsymbol{x}) = 0\Big\}.$$

Observe that if $n_1(\boldsymbol{x}) \neq 0$ then the equality $n_1(\boldsymbol{x})y_1 + n_2(\boldsymbol{x})y_2 = 0$ yields that either $y_2 \neq 0$ and $y_1/y_2 = -n_2(\boldsymbol{x})/n_1(\boldsymbol{x})$ or $y_1 = y_2 = 0$. But in the latter case obviously we have $E_2 \cap \{(\boldsymbol{x},t,\boldsymbol{y}) \in \Pi \times \mathbb{S}^2 \,|\, y_1 = y_2 = 0\}$ is the empty set. Thus we conclude that

$$\begin{aligned} E_1 \cap E_2^\lambda = \Big\{(\boldsymbol{x},t,\boldsymbol{y}) \in \Pi \times \mathbb{S}^2 \ : \ & n_1(\boldsymbol{x})y_1 + n_2(\boldsymbol{x})y_2 = 0, \\ & n_1(\boldsymbol{x})y_0 - \Big[a'(\lambda)(m_1(\boldsymbol{x},t)n_2(\boldsymbol{x}) - m_2(\boldsymbol{x},t)n_1(\boldsymbol{x})) \\ & - \sum_{k=1}^{2} A_{1k}c_k'(\lambda)n_2(\boldsymbol{x}) + \sum_{k=1}^{2} A_{2k}c_k'(\lambda)n_1(\boldsymbol{x}) \\ & -(1/2)b'(\lambda)(n_2(\boldsymbol{x})\partial_n n_1(\boldsymbol{x}) - n_1(\boldsymbol{x})\partial_n n_2(\boldsymbol{x}))\Big]y_2 = 0, \quad n_1(\boldsymbol{x}) \neq 0, \quad y_2 \neq 0\Big\}. \end{aligned} \tag{23}$$

The similar simple considerations show that

$$\begin{aligned} E_1 \cap E_3^\lambda = \Big\{(\boldsymbol{x},t,\boldsymbol{y}) \in \Pi \times \mathbb{S}^2 \ : \ & \\ & n_2(\boldsymbol{x})y_0 - \Big[a'(\lambda)\big(m_1(\boldsymbol{x},t)n_2(\boldsymbol{x}) - m_2(\boldsymbol{x},t)n_1(\boldsymbol{x})\big) \\ & - \sum_{k=1}^{2} A_{1k}c_k'(\lambda)n_2(\boldsymbol{x}) + \sum_{k=1}^{2} A_{2k}c_k'(\lambda)n_1(\boldsymbol{x}) \\ & - (1/2)b'(\lambda)\big(n_2(\boldsymbol{x})\partial_n n_1(\boldsymbol{x}) - n_1(\boldsymbol{x})\partial_n n_2(\boldsymbol{x})\big)\Big]y_1 = 0, \\ & n_1(\boldsymbol{x}) = 0, \quad y_2 = 0, \quad y_1 \neq 0\Big\}, \end{aligned} \tag{24}$$

where the summand involving zero multiplier $n_1(\boldsymbol{x})$ are added in order to make further outline more lucid.

Now we prove the assertion of the lemma by the contradiction method. Let us suppose that the H-measure $\nu^{\lambda\lambda}$ is nontrivial, i.e., that there exists a nonnegative interval $I \subset [0,1]$ such that $\nu^{\lambda\lambda}$ is not identical to zero measure on $Q \times \mathbb{S}^2$ for all $\lambda \in I \cap \mathcal{E}$.

On the strength of this assumption and assertion 3 of lemma 9, for any $\lambda \in I \cap \mathcal{E}$ there exists the set $V_\lambda \subset Q$ having a positive Lebesgue measure such that for all $(\boldsymbol{x}, t) \in V_\lambda$ the measure $\Lambda^{\lambda\lambda}_{x,t}$ is not identical to zero measure on $\mathbb{S}^2$. Consider the union $V := \cup_{\lambda \in I \cap \mathcal{E}} V_\lambda$. Clearly it has a positive Lebesgue measure and for any $(\boldsymbol{x}, t) \in V$ there exists $\lambda \in I \cap \mathcal{E}$ such that the dimension of the minimal linear subspace $L(\lambda) \subset \mathbb{R}^3$ containing the support of $\Lambda^{\lambda\lambda}_{x,t}$ is greater than or equal to one. For an arbitrarily fixed $(\boldsymbol{x}, t) \in V$ among all subspaces $L(\lambda)$ select a subspace $L = L(\lambda_0)$ whose dimension is maximal. Moreover, make this selection along the whole range $\lambda \in [0,1] \cap \mathcal{E}$ so that, in general, λ_0 does not necessarily belong to $I \cap \mathcal{E}$. Clearly $\dim L \geq 1$ thanks to the above construction of V, and $\lambda_0 < 1$ since the function $\lambda \mapsto \Lambda^{\lambda\lambda}_{x,t}$ is identically equal to zero for $\lambda > 1$ and is right-continuous in λ, due to item 3 of lemma 9. On the strength of assertion 4 of lemma 9, we conclude that there exists a number $\delta > 0$ such that for all $\lambda \in [\lambda_0, \lambda_0 + \delta] \cap \mathcal{E}$ the subspace $L(\lambda)$ coincides with L and, in particular, $\dim L(\lambda) \geq 1$. The above arguments show that if the H-measure $\nu^{\lambda\lambda}$ is nontrivial then there exists a set $V \subset Q$ with $\operatorname{meas} V > 0$ such that for an arbitrarily fixed $(\boldsymbol{x}, t) \in V$ there is an interval $[\lambda_0, \lambda_0 + \delta]$ such that for any $\lambda \in [\lambda_0, \lambda_0 + \delta] \cap \mathcal{E}$ the dimension of the minimal subspace $L(\lambda)$ containing the support of $\Lambda^{\lambda\lambda}_{x,t}$ is greater than or equal to one, and, moreover, $L(\lambda) = L(\lambda_0) = L$. In particular, $L \cap \mathbb{S}^2$ is nonempty and $(\boldsymbol{x}, t, \boldsymbol{y})$ belongs to $\operatorname{supp} \nu^{\lambda\lambda}$ for the fixed $(\boldsymbol{x}, t) \in V$ and for any $\boldsymbol{y} \in L \cap \mathbb{S}^2$. Since $\operatorname{supp} \nu^{\lambda\lambda} \subset E_1 \cap (E_2^\lambda \cup E_3^\lambda)$ and representations (23) and (24) take place, the latter means that for any vector $\boldsymbol{y} \in L \cap \mathbb{S}^2$ for all $\lambda \in [\lambda_0, \lambda_0 + \delta]$ the set of relations

$$
\begin{aligned}
&n_1(\boldsymbol{x})y_0 - \Big[a'(\lambda)(m_1(\boldsymbol{x},t)n_2(\boldsymbol{x}) - m_2(\boldsymbol{x},t)n_1(\boldsymbol{x})) - \sum_{k=1}^{2} A_{1k}c_k'(\lambda)n_2(\boldsymbol{x}) \\
&+ \sum_{k=1}^{2} A_{2k}c_k'(\lambda)n_1(\boldsymbol{x}) - (1/2)b'(\lambda)(n_2(\boldsymbol{x})\partial_n n_1(\boldsymbol{x}) - n_1(\boldsymbol{x})\partial_n n_2(\boldsymbol{x}))\Big]y_2 = 0, \\
&n_1(\boldsymbol{x}) \neq 0, \quad y_2 \neq 0
\end{aligned}
\tag{25}
$$

or the set of relations

$$
\begin{aligned}
&n_2(\boldsymbol{x})y_0 - \Big[a'(\lambda)(m_1(\boldsymbol{x},t)n_2(\boldsymbol{x}) - m_2(\boldsymbol{x},t)n_1(\boldsymbol{x})) - \sum_{k=1}^{2} A_{1k}c_k'(\lambda)n_2(\boldsymbol{x}) \\
&+ \sum_{k=1}^{2} A_{2k}c_k'(\lambda)n_1(\boldsymbol{x}) - (1/2)b'(\lambda)(n_2(\boldsymbol{x})\partial_n n_1(\boldsymbol{x}) - n_1(\boldsymbol{x})\partial_n n_2(\boldsymbol{x}))\Big]y_1 = 0, \\
&n_2(\boldsymbol{x}) \neq 0, \quad y_1 \neq 0
\end{aligned}
\tag{26}
$$

is valid.

Whichever the set of relations holds, upon differentiation of either (25) or (26) with respect to λ on $[\lambda_0, \lambda_0+\delta]$ for a fixed $\boldsymbol{y} \in L \cap \mathbb{S}^2$, we conclude that on subintervals of $[\lambda_0, \lambda_0+\delta]$ either the function

$$\lambda \mapsto \sum_{k=1}^{2} \big(A_{1k} c_k''(\lambda) n_2(\boldsymbol{x}) - A_{2k} c_k''(\lambda) n_1(\boldsymbol{x})\big) \\ + (1/2) b''(\lambda) \big(n_2(\boldsymbol{x}) \partial_n n_1(\boldsymbol{x}) - n_1(\boldsymbol{x}) \partial_n n_2(\boldsymbol{x})\big)$$

is identical zero if $a''(\lambda) = 0$ or (and) $m_1(\boldsymbol{x},t) n_2(\boldsymbol{x}) - m_2(\boldsymbol{x},t) n_1(\boldsymbol{x}) = 0$, or the functions $\lambda \mapsto a''(\lambda)$ and

$$\lambda \mapsto \sum_{k=1}^{2} \big(A_{1k} c_k''(\lambda) n_2(\boldsymbol{x}) - A_{2k} c_k''(\lambda) n_1(\boldsymbol{x})\big) \\ + (1/2) b''(\lambda) \big(n_2(\boldsymbol{x}) \partial_n n_1(\boldsymbol{x}) - n_1(\boldsymbol{x}) \partial_n n_2(\boldsymbol{x})\big)$$

are linearly dependent. This contradicts with condition G. Consequently, the assumption that the H-measure $\nu^{\lambda\lambda}$ is nontrivial on some nondegenerate set in $\mathbb{R}_\lambda$ is incorrect. This conclusion finishes the proof of the lemma. □

Lemma 11 and item 5 of lemma 9 yield the limiting relation $f^k(\cdot,\cdot,\lambda) \to f(\cdot,\cdot,\lambda)$ strongly in $L^1_{loc}(\Pi)$ for a.e. $\lambda \in \mathbb{R}$ and almost everywhere in $\Pi \times \mathbb{R}_\lambda$, as $k \nearrow \infty$. Since f^k attains only two values, either 0 or 1, and f is monotonous non-decreasing and right-continuous in λ for a.e. $(\boldsymbol{x}, t)$, this limiting relation implies that f has the form

$$f(\boldsymbol{x},t,\lambda) = \begin{cases} 1 & \text{if } \lambda \geq \tilde{u}(\boldsymbol{x},t), \\ 0 & \text{if } \lambda < \tilde{u}(\boldsymbol{x},t) \end{cases}$$

with some function $\tilde{u} \in L^\infty(\Pi)$, $0 \leq \tilde{u} \leq 1$. From formula (20) and limiting relation (21) it follows that $\tilde{u}$ coincides with the weak limit $u = \text{w-}\lim\limits_{k\nearrow\infty} u_{\varepsilon_k}$ and that $\|u_{\varepsilon_k}\|_{L^2(Q)} \underset{k\nearrow\infty}{\longrightarrow} \|u\|_{L^2(Q)}$. Hence $u_{\varepsilon_k} \underset{k\nearrow\infty}{\longrightarrow} u$ strongly in $L^2(Q)$. On the strength of this limiting relation, we complete the proof of theorem 2 following the arguments of [8, remark 3] and [9, sections 8, 9]. □

Remark 3. (Suppression of fine oscillations in the genuinely nonlinear model.) *We end this paper noticing that, as a byproduct of the proof of theorem 2, we also established a property of the genuinely nonlinear model to rule out fine oscillations developing from initial data, in the following sense:*

Suppose that model (1a)–(1c) *is genuinely nonlinear and is provided with highly oscillatory initial data* $u_0^k \in L^\infty$, $k = 1, 2, \ldots$, *in the sense that* $u_0^k \to u_0$ *weakly* in* L^∞, *as* $k \nearrow \infty$. *Then there exists a subsequence of entropy solutions* (u^k, p_*^k), *corresponding to initial data* u_0^k, *which tends strongly in*

$L^\infty(0,T;L^\infty) \times L^2(0,T;H^{1,2})$, as $k \nearrow \infty$, to an entropy solution (u,p_*), corresponding to initial data $u_0 \in L^\infty$.

Acknowledgments: The work was partially supported by National Research Program for Universities of Higher Education Commission of Pakistan (The project title: "Modern Mathematical Analysis for Phenomena of Anisotropic Diffusion and Acoustic Wave Propagation in Porous Media") and by the Russian Foundation for Basic Research (the grant code: 07-01-00309).

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