Journal of Prime Research in Mathematics Vol. **3***(2007), 154-161*

ON THE CONVERGENCE OF STRICTLY PSEUDO-CONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we establish the weak and strong convergence theorems for strictly pseudo-contractive mappings in the framework of q-uniformly smooth Banach spaces. Our results improve and extend the corresponding ones announced by Reich, Acedo, Marino, Xu and some others.

Key words : Nonexpansive mapping; Strictly pseudo-contractive mapping; q-uniformly smooth; Fixed point
AMS SUBJECT: 47H09; 47H10.

1. Introduction and Preliminaries

Throughout this paper, we assume that E is an arbitrary real Banach space and $J_q(q > 1)$ denotes the generalized duality mapping from E into 2^{E^*} given by

$$J_qx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. Let T be a mapping. We denote the fixed points set of T by $F(T)$. It is well known (see, for example, [8]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$, and that if E^* is strictly convex, then J_q is single-valued. We shall denote the single-valued duality mapping by j_q.

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The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x \leq 1,\ \|y\| \leq \tau\|\}.$$

E is uniformly smooth if and only if $\lim_{\tau\to\infty} \frac{\rho_E(\tau)}{\tau} = 0$. Let $q > 1$. E is said to be q-uniformly smooth (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. It is known that if E is q-uniformly smooth [8], then there is a constant $c_q > 0$ such that

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. \tag{1.1}$$

Let C be a subset of E. A mapping T is said to be a k-strict pseudo-contractive [2] with a sequence $\{k_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$ if for any $x,\ y \in C$, there exist $j_q(x-y) \in J_q(x-y)$ and a constant $k \in [0,1)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^q - k\|(I-T)x - (I-T)y\|^q, \tag{1.2}$$

for all $n \geq 1$. If I denotes the identity operator, then (1.2) can be written in the form

$$\langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq k\|(I-T)x - (I-T)y)\|^q. \tag{1.3}$$

In Hilbert spaces, (1.2) (and hence (1.3)) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + (1-2k)\|(I-T)x - (I-T)y\|^2. \tag{1.4}$$

These mappings are extensions of nonexpansive mappings which satisfy the inequality (1.4) with $k = \frac{1}{2}$. That is, $T : C \to C$ is nonexpansive if there exists a sequence $\{k_n\}$ such that

$$\|Tx - Ty\| \leq k_n\|x-y\|$$

for all $n \geq 1$ and $x, y \in C$.

The normal Mann's iterative process [4] generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \geq 1, \tag{1.5}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval (0,1).

If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Normal Mann's iterative process (1.5) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [6]).

Recently, Marino and Xu [5] extended the results of Reich [6] from nonexpansive mappings to strict pseudo-contractions and obtained a weak convergence theorem in Hilbert spaces. More precisely, they gave the following results.

Theorem 1.1 (Marino and Xu [5]). *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a k-strict pseudo-contraction for some $0 \le k < 1$ and assume that T admits a fixed point in C. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Normal Mann's algorithm (1.5). Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $k < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Very recently, Acedo and Xu [1] still in the framework of Hilbert spaces introduced the following cyclic algorithm.

Let C be a closed convex subset of a Hilbert space H and let $\{T_i\}_{i=0}^{N-1}$ be N k-strict pseudo-contractions on C such that $F = \bigcap_{i=0}^{N-1} F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}$ be a sequence in (0,1). The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$
\begin{aligned}
&x_1 = \alpha_0 x_0 + (1-\alpha_0)T_0 x_0,\\
&x_2 = \alpha_1 x_1 + (1-\alpha_1)T_1 x_1,\\
&\vdots\\
&x_N = \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1} x_{N-1},\\
&x_{N+1} = \alpha_N x_N + (1-\alpha_N)T_0 x_N,\\
&\vdots
\end{aligned}
$$

In general, x_{n+1} is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]} x_n, \tag{1.6}$$

where $T_{[n]} = T_i$, with $i = n \pmod N$, $0 \le i \le N-1$. They also proved a weak convergence theorem for k-strict pseudo-contractions in Hilbert spaces by cyclic algorithm (1.6). More precisely, they obtained the following theorem:

Theorem 1.2 (Acedo and Xu [1]). *Let C be a closed convex subset of a Hilbert space H. Let $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $T_i : C \to C$ be a k_i-strict pseudo-contraction for some $0 \le k_i < 1$. Let $k = \max\{k_i : 1 \le i \le N\}$. Assume the common fixed point the set $\bigcap_{i=0}^{N-1} F(T_i)$ of $\{T_i\}_{i=0}^{N-1}$ is nonempty. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the cyclic algorithm (1.6). Assume that the control sequence $\{\alpha_n\}$ is chosen so that $k + \epsilon \le \alpha_n \le 1 - \epsilon$ for all n and some $\epsilon \in (0,1)$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.*

Motivated by Acedo and Xu [1] and Marino and Xu [5], this paper establishes the weak and strong convergence theorems of iteration process (1.6) for k-strict pseudo-contractions in q-uniformly Banach spaces. Our results extend the Acedo and Xu [1]'s from Hilbert spaces to q-uniformly smooth Banach spaces.

In order to prove our main results, we need the following Lemmas.

Lemma 1.1 (Krüppel [3]). *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space E, and let $T : C \to E$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C such that $\{x_n\}$ converges weakly to some point x. Then there exists an increasing continuous function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ depending on the diameter of C such that $h(\|x - Tx\|) \leq \liminf_{n\to\infty} \|x_n - Tx_n\|$.*

Lemma 1.2. *Let E be a real Banach space, C a nonempty subset of E and $T : C \to C$ a k-strictly pseudo-contractive mapping. Then T is uniformly L-Lipschitzian.*

Proof. It follows from the definition of T that

$$\langle x - Tx - (y - Ty), j(x - y)\rangle \geq k\|x - T - (y - Ty)\|^q.$$

Observing that $\|j(x - y)\| = \|x - y\|^{q-1}$, we have

$$k\|x - Tx - (y - Ty)\|^q \leq \|x - Tx - (y - Ty)\|\|x - y\|^{q-1}.$$

That is, $k^{\frac{1}{q-1}}\|x - Tx - (y - Ty)\| \leq \|x - y\|$, which yields that

$$\|Tx - Ty\| \leq (\frac{1}{k^{\frac{1}{q-1}}} + 1)\|x - y\|.$$

This completes the proof. □

Lemma 1.3 (Tan and Xu [7]). *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \leq r_n + t_n, \text{ for all } n \in N.$$

If (i) $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists. (ii) If $\sum_{n=1}^{\infty} t_n < \infty$ and $\{r_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} r_n = 0$.

Lemma 1.4. *Let H be a real q-uniformly smooth Banach space. Let C be a nonempty closed convex subset of E and $T : C \to C$ a k-strictly pseudo-contractive mapping for some $0 \leq k < 1$ and the fixed point set of T is nonempty. Then $(I - T)$ is demi-closed at zero.*

Proof. Let $\{x_n\}$ be a sequence in C which is generated by the Mann's algorithm $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T^n x_n$ and converges weakly to x^* and $\{x_n - Tx_n\}$ converges strongly to 0. Our aim is to prove $x^* = Tx^*$. Let $p \in F(T)$. Next we consider the mapping $\alpha_n I + (1 - \alpha_n)T^n$, where the sequence $\{\alpha_n\}$ is chosen such that $1 - (\frac{qk}{c_q})^{\frac{1}{q-1}} < a \leq \alpha_n$, for some constant $a \in (0, 1)$. It follows from

(1.1), (1.3) and (1.6) that

$$\begin{aligned}
&\|(\alpha_n I + (1-\alpha_n)T)x_n - (\alpha_n I + (1-\alpha_n)T)y_n\|^q \\
&= \|x_n - y_n - (1-\alpha_n)((I-T)x_n - (I-T)y_n)\|^q \\
&\leq \|x_n - y_n\|^q - q(1-\alpha_n)\langle (I-T)x_n - (I-T)y_n), j_q(x_n - y_n)\rangle \\
&\quad + c_q(1-\alpha_n)^q\|(I-T)x_n - (I-T)y_n)\|^q \\
&\leq \|x_n - y_n\|^q - q(1-\alpha_n)(k\|(I-T)x_n - (I-T)y_n)\|^q \\
&\quad + c_q(1-\alpha_n)^q\|(I-T)x_n - (I-T)y_n)\|^q \\
&= \|x_n - y_n\|^q - (1-\alpha_n)[qk - c_q(1-\alpha_n)^{q-1}]\|(I-T)x_n - (I-T)y_n)\|^q.
\end{aligned} \tag{1.7}$$

Noting that the assumptions $1 - (\frac{qk}{c_q})^{\frac{1}{q-1}} < a \leq \alpha_n$, we have

$$\|(aI + (1-a)T)x_n - (aI + (1-a)T)y_n\|^q \leq \|x_n - y_n\|^q.$$

That is,

$$\|(aI + (1-a)T)x_n - (aI + (1-a)T)y_n\| \leq \|x_n - y_n\|. \tag{1.8}$$

It follows that the mapping $(aI + (1-a)T)$ is a nonexpansive. Observing (1.8) and taking $y_n = p$, we have that

$$\|x_{n+1} - p\| \leq \|x_n - p\|.$$

Since Lemma 1.3, we know the limit $\lim_{n\to\infty}\|x_n - p\|^q$ exists. Then there exists $R > 0$ such that $\|x_n - p\| \leq R$, $\forall n \geq 1$. Let $B_R = \{x \in E : \|x_n - p\| \leq R\}$ and let $K = C \cap B_R$. Then K is nonempty closed convex and bounded, and $\{x_n\} \subset K$. It follows from Lemma 1.1 that there exists an increasing continuous function $h : [0,\infty) \to [0,\infty)$ with $h(0) = 0$ depending on the diameter of C such that

$$h(\|x^* - (aI + (1-a)T)x^*\|) \leq \liminf_{n\to\infty}\|x_n - (aI + (1-a)T)x_n\| \tag{1.9}$$

On the other hand, we have

$$\|x_n - (aI + (1-a)T)x_n\| \leq (1-a)\|x_n - Tx_n\| \to 0, \tag{1.10}$$

as $n \to \infty$. That is,

$$\lim_{n\to\infty}\|x_n - (aI + (1-a)T)x_n\| = 0.$$

Observe that (1.10) yields that

$$h(\|x^* - (aI + (1-a)T)x^*\|) \leq 0. \tag{1.11}$$

That is,

$$\|x^* - (aI + (1-a)T)x^*\| \leq (1-a)\|x^* - Tx^*\| = 0 \tag{1.12}$$

One can easily see $x^* = Tx^*$. This completes the proof. □

2. Main Results.

Theorem 2.1. *Let C be a closed convex subset of a real q-uniformly smooth Banach E which satisfies the opial's condition. Let $N \geq 1$ be an integer. Let, for each $0 \leq i \leq N-1$, $T_i : C \to C$ be a k_i-strictly pseudo-contractive mapping for some $0 \leq k_i < 1$. Let $k = \max\{k_i : 1 \leq i \leq N-1\}$. Assume the common fixed point the set $\bigcap_{i=0}^{N-1} F(T_i)$ of $\{T_i\}_{i=0}^{N-1}$ is nonempty. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the cyclic algorithm (1.6). Assume that the control sequence $\{\alpha_n\}$ is chosen so that*

$$1 - (\frac{qk}{c_q})^{\frac{1}{q-1}} < a \leq \alpha_n \leq b < 1,$$

for all $n \geq 1$. Then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i\}_{i=0}^{N-1}$.

Proof. Let $p \in F = \bigcap_{i=0}^{n-1} F(T_i)$. It follows from (1.1), (1.3) and (1.6) that

$$\begin{aligned} &\|x_{n+1} - p\|^q \\ &= \|x_n - p - (1-\alpha_n)[(I - T_{[n]})x_n - (I - T_{[n]})p]\|^q \\ &\leq \|x_n - p\|^q - q(1-\alpha_n)\langle (I - T_{[n]})x_n - (I - T_{[n]})p, j_q(x_n - p)\rangle \\ &\quad + c_q(1-\alpha_n)^q \|(I - T_{[n]})x_n - (I - T_{[n]})p\|^q \\ &\leq \|x_n - p\|^q - q(1-\alpha_n)k\|(I - T^n)x - (I - T^n)y\|^q \\ &\quad + c_q(1-\alpha_n)^q \|(I - T_{[n]})x_n - (I - T_{[n]})p\|^q \\ &\leq \|x_n - p\|^q - (q(1-\alpha_n)k - c_q(1-\alpha_n)^q)\|(I - T_{[n]})x_n - (I - T_{[n]})p\|^q. \end{aligned} \tag{2.1}$$

It follows from the assumptions that

$$\|x_{n+1} - p\| \leq \|x_n - p\|, \tag{2.2}$$

which implies $\{x_n\}$ is bounded. It follows from Lemma 1.3 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Consider (2.1) again yields that

$$\begin{aligned} &(q(1-\alpha_n)k - c_q(1-\alpha_n)^q)\|(I - T_{[n]})x_n - (I - T_{[n]})p\|^q \\ &\leq \|x_n - p\|^q - \|x_{n+1} - p\|^q. \end{aligned} \tag{2.3}$$

Observing that the assumptions $1 - (\frac{qk}{c_q})^{\frac{1}{q-1}} < a \leq \alpha_n \leq b < 1$, we have

$$\begin{aligned} &(1-b)[qk - c_q(1-a)^{q-1}]\|(I - T_{[n]})x_n - (I - T_{[n]})p\|^q \\ &\leq \|x_n - p\|^q - \|x_{n+1} - p\|^q. \end{aligned} \tag{2.4}$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we obtain

$$\lim_{n\to\infty} \|x_n - T_{[n]}x_n\| = 0. \tag{2.5}$$

It follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \alpha_n)\|x_n - T_{[n]}x_n\| = 0, \tag{2.6}$$

Since $\{x_n\}$ is bounded, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that x_{n_i} converges weakly to $\bar{x}$. We may further assume $n_i = k \pmod N$ for all i. From (2.6), we also have x_{n_i+j} converges weakly to $\bar{x}$ for all $j \geq 0$, which yields that

$$\|x_{n_i+j} - T_{[k+j]}x_{n_i+j}\| = \|x_{n_i+j} - T_{[n_i+j]}x_{n_i+j}\| \to 0.$$

An application of Lemma 1.4 yields that $\bar{x} \in F(T_{[k+j]})$ for all j which implies that $\bar{x} \in F$.

Next we show $\{x_n\}$ converges weakly to $\bar{x}$. Suppose the contrary, if $\{x_n\}$ has another subsequence $\{n_j\}$ which converges weakly to $\widehat{x}$ such that $\widehat{x} \neq \bar{x}$, then we must have $\widehat{x} \in F$ and since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$ and E satisfies *Opial* condition, it follows from a standard argument that $\widehat{x} = \bar{x}$. Thus $\{x_n\}$ converges weakly to $\bar{x} \in F$. This completes the proof. □

Remark 2.2. Our result improves Acedo and Xu [1]'s results from Hilbert spaces to more general Banach spaces. From Theorem 2.1, we can easy to get Acedo and Xu [1]'s results (Theorem 1.2). That is, Acedo and Xu [1]'s a result is a special case of our Theorem 2.1 in Hilbert spaces.

Theorem 2.3. *Let C be a closed convex compact subset of a q-uniformly smooth Banach E. Let $N \geq 1$ be an integer. Let, for each $0 \leq i \leq N-1$, $T_i : C \to C$ be a k_i-strict pseudo-contraction for some $0 \leq k_i < 1$. Let $k = \max\{k_i : 1 \leq i \leq N-1\}$. Assume the common fixed point the set $\bigcap_{i=0}^{N-1} F(T_i)$ of $\{T_i\}_{i=0}^{N-1}$ is nonempty. Given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the cyclic algorithm (1.6). Assume that the control sequence $\{\alpha_n\}$ is chosen so that*

$$1 - (\frac{qk}{c_q})^{\frac{1}{q-1}} < a \leq \alpha_n \leq b < 1,$$

for all $n \geq 1$. Then $\{x_n\}$ converges strongly to a common fixed point p of the family $\{T_i\}_{i=0}^{N-1}$.

Proof. We only conclude the difference. By compactness of C this immediately implies that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a fixed point of T, say p. Combining (2.2) with Lemma 1.3, we have $\lim_{n\to\infty} \|x_n - p\| = 0$. This completes the proof. □

Acknowledgments

The authors are extremely grateful to the referees for useful suggestions that improved the content of the paper.

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