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ALGEBRAIC PROPERTIES OF INTEGRAL FUNCTIONS

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Abstract. For K a valued subfield of $\mathbb{C}_p$ with respect to the restriction of the p-adic absolute value $|\ |$ of $\mathbb{C}_p$ we consider the K-algebra $IK[[X]]$ of integral (entire) functions with coefficients in K. If K is a closed subfield of $\mathbb{C}_p$ we extend some results which are known for subfields of $\mathbb{C}$ (see [3] and [4]). We prove that $IK[[X]]$ is a Bézout domain and we describe some properties of maximal ideals of $IK[[X]]$.

Key words : integral functions, Bézout domain.
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1. Introduction

Consider K a valued subfield of $\mathbb{C}_p$ with respect to the restriction of the $p-adic$ absolute value of $\mathbb{C}_p$. A formal series

$$f = \sum_{i=0}^{\infty} a_i X^i \in K[[X]] \tag{1}$$

is called an *integral* (*entire*) *function* if for all $x \in K$, the sequence $S_n(x) = \sum_{i=0}^{n} a_i x^i$ is a Cauchy sequence. We denote by

$$IK[[X]] = \{f \in K[[X]],\ f \text{ is an integral function}\}.$$

It is easy to prove that $IK[[X]]$ is K-subalgebra of $K[[X]]$ with ordinary addition and multiplication. We denote by $\tilde{K}$ the completion of K with respect to $|\ |$. If $f \in IK[[X]]$, then for every $x \in K$, $S_n(x)$ is a convergent sequence in $\tilde{K}$ which tends to an element denoted by $f(x)$. We consider $A(f)$ the set of zeros of f in $\mathbb{C}_p$ counted with multiplicities.

Let K be a closed subfield of $\mathbb{C}_p$ with respect to the topology defined by $p-adic$ absolute value and $G_K = Gal(\mathbb{C}_p/\ K)$ the corresponding Galois group. If A is a multisubset of $\mathbb{C}_p$ i.e. counting some of its elements several times,

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then A is called a G_K- *invariant subset* if for every $\sigma \in G_K$, $\sigma(A) \subset A$. By definition we consider the empty set G_K-invariant. The subset A is called *discrete* if it has no finite accumulation points i.e. it is discrete as a subset of $\mathbb{C}_p$ with respect to the topology defined by the absolute value.

If R is an integral domain and $a, b \in R$ we denote by (a, b) the greatest common divisor of a, b, if there exists this element. Using Kaplansky's term (see [5], p.32) an integral domain is a GCD-*domain* if any two elements in R have a greatest common divisor. R is called *Bézout domain* if all finitely generated ideals are principal.

2. Arithmetic properties of the ring of integral functions

We consider K a subfield of $\mathbb{C}_p$. Then by Theorem of Section 6.2.3 of [6] it follows that $f \in IK[[X]]$ is a unit if and only if it is a nonzero constant of K. Moreover $A(f)$ is finite if and only if it is a polynomial.

The following two results give useful representations of G_K-invariant discrete infinite multisubsets of $\mathbb{C}_p$ by means of zeros of integral functions of $IK[[X]]$ which are not polynomials.

Proposition 1. *Let K be a closed subfield of $\mathbb{C}_p$. Then a infinite multisubset A of $\mathbb{C}_p$ is a discrete G_K-invariant subset if and only if there exists $f \in IK[[X]]$ such that $A(f) = A$.*

Proof. If $f = \sum_{i=0}^{\infty} a_i X^i \in IK[[X]]$ is not a polynomial and $\sigma \in G_K$, it follows that $\sigma(a_i) = a_i$. Let $A(f) = \{\xi_1, \xi_2, \xi_3, ...\}$ be the zero set of f. Then $\sum_{i=0}^{\infty} a_i \xi_j^i = 0$ and applying σ we obtain $\sum_{i=0}^{\infty} a_i (\sigma(\xi_j))^i = 0$. Hence $\sigma(\xi_j)$ is a root of f having the same multiplicity and $A(f)$ is G_K-invariant. Moreover $A(f)$ is a discrete set because f is an integral function.

Conversely, by Theorem of Sec. 6.2.3 of [6], for an infinite discrete multisubset $A \subset \mathbb{C}_p$ we can construct a function $f \in I\mathbb{C}_p[[X]]$ given by

$$f(x) = x^m \prod_{i=1}^{\infty} \left(1 - \frac{x}{\xi_i}\right)$$

such that $A(f) = A$, where the product is on the nonzero roots counting multiplicities. We can write this function as $f = \sum_{i=m}^{\infty} a_i X^i$, with $a_i \in \mathbb{C}_p$. Now for each $\sigma \in G_K$ consider $f^\sigma = \sum_{i=m}^{\infty} \sigma(a_i) X^i \in \mathbb{C}_p[[X]]$. Since $|\sigma(a_i)| = |a_i|$ it follows that $\lim\limits_{n\to\infty} |\sigma(a_n)|^{\frac{1}{n}} = 0$ and f^σ is also an integral function. We want to show that it has the same zero set. For this we remark that the values of the partial sums $s_k(\xi_j) = \sum_{i=m}^{k} a_i \xi_j^i$ tend to zero for every j. Then for the partial sum s_k^σ of f^σ we obtain

$$|s_k^\sigma(\xi_j)| = |\sum_{i=m}^{k} \sigma(a_i)\xi_j| = |\sum_{i=m}^{k} \sigma(a_i)\sigma(\sigma^{-1}(\xi_j))|$$

$$|\sigma(\sum_{i=m}^{k} a_n\sigma^{-1}(\xi_j))| = |s_k(\sigma^{-1}(\xi_i)| \to 0$$

because $\sigma^{-1}(\xi_j) \in A$. Hence f and f^σ have the same roots and this implies $f = f^\sigma$. Thus $\sigma(a_i) = a_i$ and $f(X) \in IK[[X]]$. □

Proposition 2. *Let K be a closed subfield of $\mathbb{C}_p$, $f \in IK[[X]]$ and a G_K-invariant subset $A_1 \subset A(f)$. Then there exists a divisor $g \in IK[[X]]$ of f such that $A(g) = A_1$.*

Proof. The statement is easy to prove when f is a polynomial. Thus we consider $f \in IK[[X]]$ which is not a polynomial. Then by Proposition 1, $A(f)$ is G_K-invariant discrete subset of $\mathbb{C}_p$. Since A_1 is G_K-invariant, we can construct a function $g \in IK[[X]]$ such that $A(g) = A_1$. Because $A_2 = A(f) \setminus A(g)$ is also G_K-invariant we can find a function $h \in IK[[X]]$ such that $A_2 = A(h)$. These $A(g)$ and $A(h)$ are disjoint subsets of $A(f)$ so the multiplication of these two functions have the zero set $A(f)$. Hence g and h are the divisors of f. □

Now we prove that $IK[[X]]$ is a GCD and a Bézout domain.

Theorem 3. *If K is a closed subfield of $\mathbb{C}_p$, then any finite or infinite set of functions from $IK[[X]]$ has a greatest common divisor in $IK[[X]]$.*

Proof. Consider a set of functions $\{f_i\}_{i\in I}$ from $IK[[X]]$ and let $\{A(f_i)\}_{i\in I}$ be their zero sets respectively. By Proposition 1, these zero sets are G_K-invariant. Consider their intersection $A = \bigcap_{i\in I} A_i$, which obviously is a discrete G_K-invariant set. Then we can find a function $d \in IK[[X]]$ such that $A = A(d)$ and it is obviously their greatest common divisor. □

Corollary 4. *If K is a closed subfield of $\mathbb{C}_p$, then $IK[[X]]$ is an integrally closed domain.*

Proof. Since every GCD-domain is integrally closed (see [5], Theorem 50, p.33) it is enough to use Theorem 3. □

If K is a subfield of $\mathbb{C}$, it is known (see [3], Theorem 9) that $IK[[X]]$ is a Bézout domain. The proof uses Mittag-Leffler Theorem for an unbounded domain. Since in the case of $\mathbb{C}_p$ Mittag-Leffler Theorem is proved only for particular bounded domains (see [6], Sec.6.4.5), so we'll use an infinite interpolation theorem to extend Helmer's result to a closed subfield of $\mathbb{C}_p$.

Theorem 5. *Let K be a closed subfield of $\mathbb{C}_p$. Then $IK[[X]]$ is a Bézout domain.*

Proof. Since $IK[[X]]$ is a GCD-domain it is enough to show that the greatest common divisor of a finite number of integral functions from $IK[[X]]$ can be written as linear combination of the functions.

If d is the greatest common divisor of the integral functions $f_1, \cdots, f_n$ we must find $h_i \in IK[[X]]$, $i = 1, 2, ..., n$ such that $h_1 f_1 + \cdots + h_n f_n = d$. It is easy to see that it is sufficient to prove the statement for $n = 2$. Without loss of generality we can assume that $d = 1$ and we'll prove that there exist $h_1, h_2 \in IK[[X]]$, such that $f_1 h_1 + f_2 h_2 = 1$. By [6], Sec. 6.2.3 we can write

$$f_1(x) = x^m \prod_{i=1}^{\infty} \left(1 - \frac{x}{\alpha_i}\right) \quad \text{and} \quad f_2(x) = \prod_{i=1}^{\infty} \left(1 - \frac{x}{\beta_i}\right)$$

By [2], Theorem 2.2 there exists $g \in IK[[X]]$ such that, for every i, $g(\beta_i) = \frac{1}{f_1(\beta_i)}$. Hence all β_i are the roots of $gf_1 - 1$ and by [6], Theorem of Sec. 6.2.3 it follows that

$$g(x)f_1(x) - 1 = C \prod_{i=1}^{\infty} \left(1 - \frac{x}{\gamma_i}\right) = f_2(x) C \prod_{\gamma_i \notin A(f_2)} \left(1 - \frac{x}{\gamma_i}\right)$$

Now by taking $h_1 = g,\ h_2 = -C \prod_{\gamma_i \notin A(f_2)} \left(1 - \frac{X}{\gamma_i}\right)$, it follows the theorem. □

Definition 1. Let K be a closed subfield of $\mathbb{C}_p$. An *ideal* I of $IK[[X]]$ is called *fixed* if $\bigcap_{f \in I} A(f)$ is non empty, otherwise it is called *free*. Thus I is a fixed ideal if all integral functions in the ideal have common zeros, otherwise it is a free ideal.

Now we can prove two corollaries of Theorem 5.

Corollary 6. *Let K be a closed subfield of $\mathbb{C}_p$. Every free ideal $IK[[X]]$ is not finitely generated.*

Proof. Suppose contrary. If the ideal is finitely generated then it is a principal ideal. Then it is a fixed ideal, a contradiction which implies the corollary. □

Corollary 7. *Let K be a closed subfield of $\mathbb{C}_p$ and I a free ideal of $IK[[X]]$. Then I does not contain any nonconstant polynomial.*

Proof. If I contains a polynomial $P \in K[X]$, then we can consider that it has the smallest degree. By Division Theorem for integral functions it follows that P divides each function of I. Hence it follows the statement. □

Theorem 8. *Let K be a closed subfield of $\mathbb{C}_p$ and let $\{A_\alpha\}_{\alpha \in J}$ be a family of G_K-invariant subsets such that*

i) The family $\{A_\alpha\}_{\alpha \in J}$ is closed under finite set intersection.

ii) $\bigcap_{\alpha \in J} A_\alpha$ *is empty.*
If $F_\alpha = \{f \in IK[[X]] \ : \ f(z_\alpha) = 0 \ \forall \ z_\alpha \in A_\alpha\}$*, then* $I = \{F_\alpha\}_{\alpha \in J}$ *is a free ideal. Conversely if* I *is a free ideal of* $IK[[X]]$ *generated by the family* $\{f_\alpha\}$ *and* $Z_\alpha = \{z \in K \ : \ f_\alpha(z) = 0\}$*, then the family* $\{Z_\alpha\}$ *will satisfy the conditions i) and ii).*

Proof. Suppose $I = \{F_\alpha\}_{\alpha \in J}$. Consider $f, g \in I$ such that $f \in F_\alpha$ and $g \in F_\beta$. By *i*) there exists $F_\gamma = F_\alpha \bigcap F_\beta$ such that f and g both will vanish on F_γ. This implies that $f - g \in F_\gamma \subset I$. Now take $f \in IK[[X]]$ and $g \in I$. Then fg belongs to the same family F_α. Hence I is an ideal and from *ii*) it follows that I is a free ideal.

Conversely suppose that $I = < \{f_\alpha\}_{\alpha \in J} >$ is a free ideal. Then by Theorem 5, *i*) holds and *ii*) follows from the definition of a free ideal. □

3. Maximal ideals of $IK[[X]]$

In this section we describe some properties of the maximal ideals of $IK[[X]]$.

Theorem 9. *Let* $K = \mathbb{C}_p$*. Then every maximal fixed ideal of* $IK[[X]]$ *is of the form* $I(z_0) = \{f \in IK[[X]] \mid f(z_0) = 0\}$ *for some* $z_0 \in K$*. Moreover the field* $IK[[X]]/I(z_0)$ *is isomorphic to* K*.*

Proof. Consider $I(z_0) = \{f \in IK[[X]] \mid f(z_0) = 0\}$ and the mapping $\Psi : IK[[X]] \longrightarrow K$ defined as $\Psi(f(z)) = f(z_0)$ which is a homomorphism. The kernel of this homomorphism is $I(z_0)$. It implies that $I(z_0)$ is a maximal fixed ideal. Now suppose that I is a maximal fixed ideal but not of the above form i.e. it has two fixed points $z_1, z_2 \in \bigcap_{f \in I} A(f)$. Then I is contained properly in $I(z_1)$ and $I(z_2)$, a contradiction which implies that I has above form. Finally, by using the first isomorphism theorem we have $IK[[X]]/I(z_0) \simeq K$. □

The free ideals are characterized in Theorem 8. Now we are interested in extra conditions to characterize the maximal free ideals.

Theorem 10. *A free ideal* M *of* $IK[[X]]$ *is maximal if and only if* $A(M)$ *satisfies the following condition in addition to the conditions of Theorem* 3

iii) If $D = \{z_n\}_{n=1}^{\infty}$ *is any infinite discrete* G_K*-invariant subset of* K *such that* $D \bigcap A(f)$ *is non-empty for every* $f \in M$*, then there exists* $f \in M$ *such that* $D = A(f)$*.*

Proof. Suppose M is free ideal and *iii*) holds. If M is not maximal, then there is an ideal N properly containing M. Suppose $g \in N$ and apply *i*) of Theorem 8 to $A(N)$. Then $A(g) \bigcap A(f)$ is non-empty for every $f \in N$, and hence for every $f \in M$. By *iii*), $g \in M$ then it implies that M is maximal free ideal.

Conversely, suppose M is maximal free ideal. If there was an infinite discrete

G_K-invariant subset D violating *iii*), then any $g \in IK[[X]]$ such that $A(g) = D$ would generate together with M an ideal N properly containing M, which is contradiction of maximality. □

Theorem 11. *If M is a maximal free ideal, then $IK[[X]]/M$ contains a subfield isomorphic to the field $K(X)$ of all rational functions.*

Proof. Since M is a free ideal, no polynomial belongs to it. If p_1, p_2 are two distinct polynomials, then $p_1 \not\equiv p_2 (\text{mod } M)$. Hence $IK[[X]]/M$ contains as a subring all polynomials. So $IK[[X]]/M$ contains $K(X)$ as a subfield. □

If $K = \mathbb{C}_p$ and we don't wish to fix the elements of $\mathbb{C}_p$, then we will prove that $I\mathbb{C}_p[[X]]/M$ is isomorphic to $\mathbb{C}_p$. For this we need two lemmas.

Lemma 12. *The field $I\mathbb{C}_p[[X]]/M$ is algebraically closed.*

Proof. If $f \in M$, then M contains all functions h vanishing on the distinct points of $A(f)$, because f divides h. Since M is a maximal free ideal, by using Theorem 10, M contains all functions with the simple zeros at the distinct points of $A(f)$. Now consider a nonconstant polynomial

$$\Phi(X, Y) = f_0(X) + f_1(X)Y + \ldots + f_n(X)Y^n$$

with coefficients $f_0, f_1, \ldots, f_n \in I\mathbb{C}_p[[X]]$, where f_n is not in M. Choose any sequence $\{x_k\}$ from $A(M) = \bigcup_{f \in M} A(f)$. Now for any fixed k, the $\Phi(x_k, Y)$ is a polynomial with coefficients in $\mathbb{C}_p$ and has n roots in $\mathbb{C}_p$. If y_k is one of these roots, we can construct functions $g \in I\mathbb{C}_p[[X]]$ such that $g(x_k) = y_k$ for $k = 1, 2, \ldots$. Hence $\Phi(X, g(X)) \equiv 0 (mod\ M)$ and this implies the lemma. □

Lemma 13. *The field $I\mathbb{C}_p[[X]]/M$ has the power c of the continuum.*

Proof. Since $I\mathbb{C}_p[[X]]$ contains a countable dense subset, its power is at most c. Hence the power of $I\mathbb{C}_p[[X]]/M$ is at most c. But all the elements from $\mathbb{C}_p$ are incongruent (mod M), so the power of $I\mathbb{C}_p[[X]]/M$ is equal to c. □

Theorem 14. *If M is a maximal free ideal, then $I\mathbb{C}_p[[X]]/M$ is isomorphic to $\mathbb{C}_p$.*

Proof. By Lemmas 1 and 2, we obtain that $I\mathbb{C}_p[[X]]/M$ is algebraically closed and of transcendence degree c over $\mathbb{Q}$. Using a theorem of Steinitz it follows that it is isomorphic to $\mathbb{C}$. Since $\mathbb{C} \simeq \mathbb{C}_p$ (see [6], p.145), the theorem holds. □

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