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ALGEBRAIC PROPERTIES OF SPECIAL RINGS OF FORMAL SERIES

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Abstract. The K-algebra $K_S[[X]]$ of Newton interpolating series is constructed by means of Newton interpolating polynomials with coefficients in an arbitrary field K (see Section 1) and a sequence S of elements K. In this paper we prove that this algebra is an integral domain if and only if S is a constant sequence. If K is a non-archimedean valued field we obtain that a K-subalgebra of convergent series of $K_S[[X]]$ is isomorphic to Tate algebra (see Theorem 3) in one variable and by using this representation we obtain a general proof of a theorem of Strassman (see Corollary 1). In the case of many variables other results can be found in [2].

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1. Formal Newton interpolating series

Let K be a field and $S = \{\alpha_n\}_{n\geq 1}$ a fixed sequence of elements of K. We consider the polynomials

$$u_0 = 1,\ u_i = \prod_{j=1}^{i} (X - \alpha_j)\ ,\ i \geq 1 \tag{1}$$

and the set of formal series

$$K_S[[X]] = \{f = \sum_{i=0}^{\infty} a_i u_i \mid a_i \in K\}, \tag{2}$$

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two such expressions being regarded as equal if and only if they have the same coefficients. We call an element f from $K_S[[X]]$ a (formal) *Newton interpolating series* with coefficients in K defined by the sequence S. It is easy to see that every polynomial $P \in K[X]$ can be written uniquely in the form $P = \sum_{i=0}^{p} a_i u_i$ with $a_i \in K$, where p is the degree of P and every u_i is defined by (1). Thus we can consider $K[X]$ a $K-$subalgebra of $K_S[[X]]$. If $u_i,\ u_j$ are given by (1), we obtain that for every k, $\max\{i,j\} \le k \le i+j$ there exist the elements $d_k(i,j)$ in K uniquely defined such that

$$u_i u_j = \sum_{k=\max\{i,j\}}^{i+j} d_k(i,j) u_k, \tag{3}$$

where the coefficients satisfy the following properties

$$d_k(i,j) = d_k(j,i),\ d_{i+j}(i,j) = 1. \tag{4}$$

$K_S[[X]]$ becomes a commutative K-algebra which contains $K[X]$ under the addition and multiplication of two elements $f = \sum_{i=0}^{\infty} a_i u_i, g = \sum_{i=0}^{\infty} b_i u_i \in K_S[[X]]$ defined by

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i) u_i \tag{5}$$

and

$$fg = \sum_{k=0}^{\infty} c_k u_k \tag{6}$$

with

$$c_k = \sum_{(\alpha,\beta)\in I(k)} d_k(\alpha,\beta) a_\alpha b_\beta, \tag{7}$$

where

$$I(k) = \{(\alpha,\beta) \in \mathbb{N}\times\mathbb{N} \mid \max\{\alpha,\beta\} \le k,\ \alpha+\beta \ge k\} \tag{8}$$

and $d_k(\alpha,\beta)$ are given in (3).

If $f = \sum_{i=0}^{\infty} a_i u_i \in K_S[[X]]$, the smallest index i for which the coefficient a_i is different from zero will be called the *order* of f and will be denoted by $o(f)$. We agree to attach the order $+\infty$ to the element 0 from $K_S[[X]]$. If we fix a positive real number $\delta < 1$ and define the norm $\|f\|_o$ of an element f of $K_S[[X]]$ by the formula

$$\|f\|_o = \delta^{o(f)}, \tag{9}$$

$K_S[[X]]$ becomes a $K-$ultrametric normed vector space, where the norm on K is trivial. Moreover for $f, g \in K_S[[X]]$

$$\|fg\|_o \le \min\{\|f\|_o, \|g\|_o\} \tag{10}$$

holds and $K_S[[X]]$ becomes a topological K-algebra. It follows easily that $K_S[[X]]$ is a complete $K-$ultrametric normed vector space and

$$f = \lim_{n\to\infty} \sum_{i=0}^{n} a_i u_i. \tag{11}$$

This implies as in the classical case (see for example [4], Ch.VII, §1) that distributive law holds also for infinite sums and in particular we obtain that

$$u_j \sum_{i=0}^{\infty} a_i u_i = \sum_{i=0}^{\infty} a_i u_i u_j. \tag{12}$$

For a fix $f \in K_S[[X]]$ we denote $L_f : K_S[[X]] \to K_S[[X]]$ the K-linear application defined by

$$L_f(g) = fg. \tag{13}$$

Then, by (3) and (12)

$$L_f(u_i) = fu_i = \sum_{j=i}^{\infty} f_{j,i} u_j, \tag{14}$$

where $f_{j,i} \in K$ are uniquely determined by f. Since

$$L_f(u_{i+1}) = \sum_{j=i+1}^{\infty} f_{j,i+1} u_j = fu_i(X - \alpha_{i+1}) = \sum_{j=i}^{\infty} f_{j,i} u_j (X - \alpha_{i+1})$$

$$= \sum_{j=i}^{\infty} f_{j,i}(u_{j+1} + (\alpha_{j+1} - \alpha_{i+1})u_j) = \sum_{j=i+1}^{\infty} (f_{j-1,i} + (\alpha_{j+1} - \alpha_{i+1}) f_{j,i}) u_j,$$

we get

$$f_{j,i+1} = f_{j-1,i} + (\alpha_{j+1} - \alpha_{i+1}) f_{j,i}, \text{ for } j = i+1, i+2, \dots . \tag{15}$$

Now we consider $g = \sum_{j=0}^{\infty} b_j u_j$. Then by (12) and (14)

$$fg = \sum_{j=0}^{\infty} b_j (fu_j) = \sum_{j=0}^{\infty} b_j (\sum_{i=j}^{\infty} f_{i,j} u_i) = \sum_{j=0}^{\infty} (\sum_{i=0}^{j} b_i f_{j,i}) u_j. \tag{16}$$

If $f = \sum_{j=0}^{\infty} a_j u_j$ and we take $i = 0$ in (14) we get for every j

$$f_{j,0} = a_j. \tag{17}$$

Now for a fix j, by (15) we obtain that

$$f_{j,j} = f_{j-1,j-1} + (\alpha_{j+1} - \alpha_j) f_{j,j-1}$$

and by recurrence for all k, $1 \le k \le j$ we will get

$$f_{j,j} = \sum_{r=0}^{k} f_{j-r,j-k} \prod_{s=1}^{k-r} (\alpha_{j+1} - \alpha_{j-k+s}), \tag{18}$$

where $\prod_{s=1}^{0} (\alpha_{j+1} - \alpha_{j+1-s}) = 1$.

Similarly for i, j such that $1 \le k \le i < j$,

$$f_{j,i} = f_{j-1,i-1} + (\alpha_{j+1} - \alpha_i) f_{j,i-1}$$

and generally

$$f_{j,i} = \sum_{r=0}^{k} f_{j-r,i-k} P_{j-r,i-k}(\alpha_{j+2-k}, ..., \alpha_{j+1}, \alpha_{i-k+1}, ..., \alpha_i), \tag{19}$$

where $P_{j-r,j-k}$ are polynomials with integer coefficients and

$$P_{j,i-k} = (\alpha_{j+1} - \alpha_i)(\alpha_{j+1} - \alpha_{i-1})...(\alpha_{j+1} - \alpha_{i-k+1}). \tag{20}$$

Now by putting $j = n = k$ in (18) we obtain

$$f_{n,n} = \sum_{r=0}^{n} f_{n-r,0} \prod_{s=1}^{n-r} (\alpha_{n+1} - \alpha_s) = \sum_{r=0}^{n} a_{n-r} \prod_{s=1}^{n-r} (\alpha_{n+1} - \alpha_s). \tag{21}$$

In order to study the $K-$ algebra $K_S[[X]]$ we need the following lemma which is an easy consequence of (21).

Lemma 1. *If $\alpha_{n+1} = \alpha_i$ for $i < n$ then $f_{n,n} = f_{i-1,i-1}$.*

Now we describe when $K_S[[X]]$ is an integral domain.

Theorem 1. *$K_S[[X]]$ is an integral domain if and only if S is a constant sequence.*

Proof. If S is a constant sequence then $K_S[[X]]$ is an integral domain because it is isomorphic to the $K-$algebra $K[[X]]$ of formal power series over K.

Conversely, we suppose that $K_S[[X]]$ is an integral domain and now we have to show that S is a constant sequence. Suppose contrary that S is not a constant sequence. By (16) it follows that a nonzero element $f = \sum_{i=0}^{\infty} a_i u_i$ is a zero divisor in $K_S[[X]]$ if and only if for every nonnegative integer i there exists b_i in K, not all equal to zero, such that

$$\sum_{i=0}^{j} b_i f_{j,i} = 0. \tag{22}$$

Assume first $\alpha_2 \neq \alpha_1$ and we have to construct the elements a_i and b_i such that they satisfy (22).

We take $f_{0,0} = a_0 = 1$, $b_0 = 0$ then (22) holds for $j = 0$. Now by (21) we obtain

$$f_{1,1} = a_0 + (\alpha_2 - \alpha_1)a_1. \tag{23}$$

Hence, if we take $a_1 = \frac{-a_0}{\alpha_2 - \alpha_1}$ and $b_1 = 1$ we get $f_{1,1} = 0$ and (22) also holds for $j = 1$. If $j = 2$, (22) becomes

$$b_0 a_2 + b_1 f_{2,1} + b_2 f_{2,2} = 0. \tag{24}$$

Since by (15) and (17) $f_{2,1} = f_{1,0} + (\alpha_3 - \alpha_1)f_{2,0} = a_1 + (\alpha_3 - \alpha_1)a_2$ and $f_{2,2} = (\alpha_3 - \alpha_2)a_1 + (\alpha_3 - \alpha_2)(\alpha_3 - \alpha_1)a_2$, we can choose either $a_2 = -\frac{a_1}{\alpha_2 - \alpha_1}$ and $b_2 = 1$, if $\alpha_3 = \alpha_2$ or $b_2 = -f_{2,1}$, $a_2 = 1$ which implies $f_{2,2} = 1$, if $\alpha_3 = \alpha_1$. If α_3 is different from α_1 and α_2 we can choose a_2 such that $f_{2,2} \neq 0$ and b_2 follows from (24).

Now, we suppose that for $n \geq 3$ we found a_j and b_j not all equal to zero such that (22) holds for $j < n$, $j > 1$ and if $\alpha_{j+1} \neq \alpha_2$, then $f_{j,j} \neq 0$. We construct a_n and b_n such that (22) is verified for $j = n$ and if $\alpha_{n+1} \neq \alpha_2$, $f_{n,n} \neq 0$. There are four possibilities for α_{n+1}. First if $\alpha_{n+1} \neq \alpha_i$, for all $i < n+1$, then by (21) the coefficient of a_n in $f_{n,n}$ is different from zero. Thus we can choose a_n such that $f_{n,n}$ is different from zero and hence we can take b_n such that (22) holds for $j = n$ in this case. Second if $\alpha_{n+1} = \alpha_1$, then by Lemma 1 $f_{n,n} = a_0 \neq 0$ and by taking $a_n = 1$ we can find b_n such that (22) holds. Third if $\alpha_{n+1} = \alpha_2$, then by Lemma 1 $f_{n,n} = f_{1,1} = 0$. Hence by (19) and (20) with $j = n$ and $k = i$ we find that $f_{n,i}, i = 2, ..., n-1$ does not contain a_n and $f_{n,1} = a_{n-1} + (\alpha_2 - \alpha_1)a_n$. Since $b_1 = 1$ we can choose $b_n = 1$ and by (22) with $j = n$ we can find a_n such that (22) holds when $j = n$. Last if α_{n+1} is different from α_1 and α_2 but $\alpha_{n+1} = \alpha_i$ for some $2 < i < n+1$ we suppose that u is the least such i. Thus $\alpha_{n+1} = \alpha_u$ for some $3 \leq u < n+1$ and by Lemma 1, $f_{n,n} = f_{u-1,u-1} \neq 0$, because $\alpha_u \neq \alpha_2$. Then we take $a_n = 1$, $b_n = -\frac{b_1 f_{n,1} + ... + b_{n-1} f_{n,n-1}}{f_{n,n}}$ and (22) holds for $j = n$. Hence $K_S[[X]]$ is not an integral domain, a contradiction and the theorem is true if $\alpha_2 \neq \alpha_1$.

Now suppose $\alpha_1 = \alpha_2 = ... = \alpha_{i_0-1} \neq \alpha_{i_0}$ and we can take $a_0 = a_1 = ... = a_{i_0-3} = 0$ and $a_{i_0-2} = 1$ such that

$$\begin{aligned} f = (X - \alpha_1)^{i_0-2} + a_{i_0-1}(X - \alpha_1)^{i_0-1} + a_{i_0}(X - \alpha_1)^{i_0-1}(X - \alpha_{i_0}) + ... \\ = (X - \alpha_1)^{i_0-2} h, \end{aligned} \tag{25}$$

where $h = 1 + a_{i_0-1}(X - \alpha_1) + a_{i_0}(X - \alpha_1)(X - \alpha_{i_0}) + ... \in K_{S'_f}[[X]]$ and the sequence $S'_f = \{\alpha'_1 = \alpha_1, \alpha'_2 = \alpha_{i_0}, \alpha'_3 = \alpha_{i_0+1}, ...\}$. Thus in S'_f $\alpha'_1 \neq \alpha'_2$ and by the previous case we can find a nonzero element $g \in K_{S'_f}[[X]]$ such that $hg = 0$. We define the K-linear embedding $T_f : K_{S'_f}[[X]] \to K_S[[X]]$ such that $T_f(g) = (X - \alpha_1)^{i_0-2} g$. Hence $(X - \alpha_1)^{i_0-2} T_f(gh) = T_f(g)T_f(h)$ and $g_1 = (X - \alpha_1)^{i_0-2} g = T_f(g)$ is a nonzero element of $K_S[[X]]$ such that

$g_1 f = 0$, a contradiction which implies the theorem. □

2. Convergent Newton interpolating series

Let K be a field. We call $(K, |\ |)$ a *valued field*, if $|\ |$ is a non-trivial non-archimedean absolute valuation on K (see [3], Ch. 2). We consider $S = \{\alpha_n\}_{n\geq 1}$ a fixed sequence of elements of the valuation ring $A(K)$ of K and we denote

$$\mathcal{H}K_S[[X]] = \left\{ f = \sum_{i=0}^{\infty} a_i u_i \in K_S[[X]] : \lim_{i\to\infty} |a_i| = 0 \right\}. \tag{26}$$

If $f = \sum\limits_{i=0}^{\infty} a_i u_i \in \mathcal{H}K_S[[X]]$, then we can define

$$|f| = \sup_i |a_i|. \tag{27}$$

Theorem 2. *If K is a valued field and $S = \{\alpha_n\}_{n\geq 1}$ is a fixed sequence of elements of $A(K)$, then $\mathcal{H}K_S[[X]]$ is a $K-$subalgebra of $K_S[[X]]$ and $|\ |$ defined by (27) is a non-archimedean absolute value on $\mathcal{H}K_S[[X]]$. Moreover if K is a complete valued field, then $\mathcal{H}K_S[[X]]$ becomes a $K-$Banach algebra.*

Proof. Let $f, g = \sum\limits_{i=0}^{\infty} b_i u_i$ be elements of $\mathcal{H}K_S[[X]]$. Then, by (5) and (27) we obtain $|f \pm g| = \sup\limits_i \{|a_i \pm b_i|\} \leq \max\{|f|, |g|\}$. Similarly, since $u_i \in A(K)[X]$, by (3), (6) and (7), it follows that $d_k(i,j) \in A(K)$ and $|fg| = \sup\limits_k |c_k| \leq |f||g|$. Moreover if we choose $i(f)$ the greatest index i such that $|a_i| = |f|$, then by (4) and (7) $\left|c_{i(f)+i(g)}\right| = \left|a_{i(f)}\right| \left|b_{i(g)}\right| = |f||g|$ and $|fg| = |f||g|$. Hence $\mathcal{H}K_S[[X]]$ is a $K-$subalgebra of $K_S[[X]]$ and $|\ |$ defined by (27) is a non-archimedean absolute value on $\mathcal{H}K_S[[X]]$.

Now we prove that $\mathcal{H}K_S[[X]]$ is complete, when K is complete. We take $f^{[t]} = \sum\limits_{i=0}^{\infty} a_{i,t} u_i$, $t \geq 1$, a Cauchy sequence of elements from $\mathcal{H}K_S[[X]]$. Since

$$|a_{i,t+1} - a_{i,t}| \leq \left| f^{[t+1]} - f^{[t]} \right|, \tag{28}$$

for a fixed i, each sequence $a_{i,t}$, $t = 1, 2, ...$ is a Cauchy sequence in K. For $i \in \mathbf{N}$, let $a_i \in K$ be the limit of this sequence. Set $f = \sum\limits_{i=0}^{\infty} a_i u_i \in K_S[[X]]$ We have to prove that f is an element of $\mathcal{H}K_S[[X]]$ and $\lim\limits_{t\to\infty} \left| f - f^{[t]} \right| = 0$. We may assume $\left| f^{[s]} - f^{[t]} \right| \leq \frac{1}{t}$ for all $s \geq t$, $t = 1, 2, ...$. By (28) we obtain $|a_{i,s} - a_{i,t}| \leq \frac{1}{t}$, $s = t, t+1, ...$. Now the continuity of $|\ |$ implies that $|a_i - a_{i,t}| \leq \frac{1}{t}$, for all $i \in \mathbf{N}, t \in \mathbf{N}^*$. Since, for each t, $f^{[t]} \in \mathcal{H}K_S[[X]]$, then for

i big enough we have $|a_{i,t}| \leq \frac{1}{t}$. Hence, if $\epsilon > 0$, we can choose t and i_0 such that $\frac{1}{t} < \epsilon$ and for every $i \geq i_0$ $|a_i| < \epsilon$, it follows that $f \in \mathcal{H}K_S[[X]]$. Furthermore, we have $\left|f - f^{[t]}\right| = \sup_i |a_i - a_{i,t}| \leq \frac{1}{t}$ and this implies $\lim_{t\to\infty} \left|f - f^{[t]}\right| = 0$. □

Theorem 3. *If K is a complete valued field and $S = \{\alpha_n\}_{n\geq 1}$ is a fixed sequence of elements of $A(K)$, then the $K-$Banach algebra $\mathcal{H}K_S[[X]]$ is isometric isomorphic to Tate algebra $K < X >$.*

Proof. Consider $P = \sum_{i=0}^{n} b_i X^i \in K[X]$ written also in the form $P = \sum_{i=0}^{n} a_i u_i \in K_S[[X]]$. Then

$$b_i = a_i + \sum_{j=i+1}^{n} a_j Q_{i,j}(\alpha_1, ..., \alpha_j), \tag{29}$$

where $Q_{i,j}$ are homogeneous polynomials with integral coefficients. Suppose $|P|_{\mathcal{H}K_S[[X]]} = |a_{i_0}|$, where i_0 is the greatest index with this property. Since $|Q_{i,j}(\alpha_1, ..., \alpha_j)| \leq 1$, it follows that $|b_{i_0}| = |a_{i_0}|$ and $|b_i| \leq \max_{j\geq i}\{|a_j|\}$. Hence $|P|_{K<X>} = |P|_{\mathcal{H}K_S[[X]]}$.

Now, by means of (29) we define $\phi : \mathcal{H}K_S[[X]] \to K < X >$ such that

$$\phi\left(\sum_{i=0}^{\infty} a_i u_i\right) = \sum_{i=0}^{\infty} b_i X^i, \tag{30}$$

where

$$b_i = a_i + \sum_{j=i+1}^{\infty} a_j Q_{i,j}(\alpha_1, ..., \alpha_j). \tag{31}$$

Similarly we can define (see [3], p. 354) $\psi : K < X > \to \mathcal{H}K_S[[X]]$ such that

$$\psi\left(\sum_{i=0}^{\infty} b_i X^i\right) = \sum_{i=0}^{\infty} a_i u_i, \tag{32}$$

where

$$a_i = b_i + \sum_{j=i+1}^{\infty} b_j S_{i,j}(\alpha_1, ..., \alpha_j). \tag{33}$$

Then the maps ϕ and ψ are well defined and continuous with respect to the corresponding norms. The relations (30) and (32) imply that the restricted mappings ϕ and ψ are inverse to each other on $K[X]$. Since $K[X]$ is dense in $\mathcal{H}K_S[[X]]$ and $K < X >$ we obtain that ϕ and ψ are inverse to each other and hence ϕ is bijective map. In fact ϕ is the identity map on $K[X]$ so ϕ is also a $K-$algebra morphism. So we obtain that $\mathcal{H}K_S[[X]]$ and $K < X >$ are isomorphic $K-$algebras. □

By using Theorem 3 we obtain a simple proof of Strassman's Theorem in general case (see [3], Sec. 6.2.1).

Corollary 1. *Let K be a complete valued field. Then a nonzero power series f from $K < X >$ has finitely many zeros in $A(K)$.*

Proof. Suppose contrary. If $\{\alpha_k\}_{k\geq 1}$ are infinitely many distinct zeros of f in $A(K)$, we consider $S = \{\alpha_n\}_{n\geq 1}$. Since $\mathcal{H}K_S[[X]]$ and $K < X >$ are isometric isomorphic K-algebras we can write $f = \sum_{i=0}^{\infty} a_i u_i$, where $\lim_{i\to\infty} a_i = 0$. Hence f converges for every $x \in A(K)$ and $f(\alpha_k) = 0$, for every k. This implies successively $a_0 = 0$, $a_1 = 0$ and generally $a_i = 0$ for every i, a contradiction which implies the theorem. □

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