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LIMIT SETS OF WEAKLY CONTRACTING RELATIONS WITH EVENTUAL CONDENSATION

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Abstract. Barnsley's formula for the attractor of a hyperbolic IFS with condensation is generalized for the omega-limit set of a weakly contracting set-valued map with eventual condensation. The latter need not be a contraction, as well as its omega-limit set need not be an attractor.

Key words : Set-valued map, weak contraction, eventual condensation, attractor, limit relation, limit set, fractal.
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1. Introduction

The well known fractal called *Pythagoras tree* represents the attractor of an Iterated Function System, shortly IFS, with condensation [2] (see Figure 1). This IFS consists of a constant set-valued mapping with the "hypotenuse's square" as value (the "condensation") together with two similitudes, which map this square onto the other two squares related to the given right triangle.

As a multi-function, or a relation, this IFS with condensation is contracting with respect to the Hausdorff-Pompeiu metrics.

Weakly contracting relations have been considered in [3, 4], where existence of the attractor, as well as some characteristics of the set-valued dynamics, such as Shadowing Property, Asymptotic Phase Property, denseness of periodic points on the attractor, have been proved.

Let (X, d) denote a complete metric space and let $\mathcal{P}(X)$ denote the set of all nonempty subsets of X.

A *relation on* X is a subset $f \subset X \times X$. Any relation can be regarded as a *multi-function (set-valued map)* $f : X \to \mathcal{P}(X)$, associating to each

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$x \in X$ a subset $f(x)$ of X. These two aspects of relations (set theoretical and functional) allow one to apply set operations, such as union, intersection and closure, on the one hand, and the functional operations, such as composition, inverse and identity, on the other hand. We will make use of both these meanings of relations.

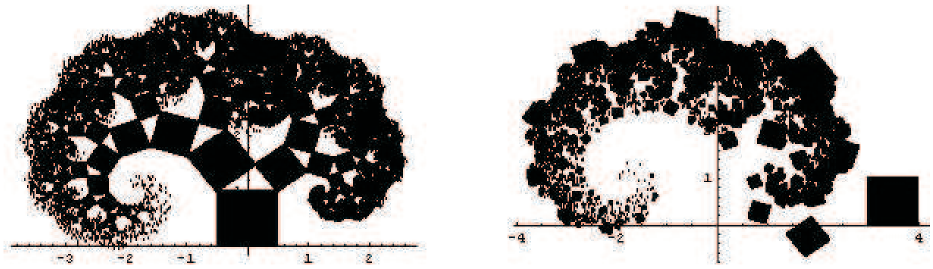

FIGURE 1. The Pythagoras tree (left) and the attractor of the same IFS with another condensation set (right)

We say that a compact-valued multi-function (relation) $f : X \to \mathcal{P}(X)$ is an *eventual condensation* if there exists a non-empty compact $K \subset X$ and a natural n_0 such that for all $n \geq n_0$ the multi-function f^n is constant with value K.

M. Barnsley [2] has studied the structure of the attractor for a hyperbolic IFS with condensation.

Here we give a two-fold generalization of Barnsley's formula. Firstly, we relax the hyperbolicity (contractivity) condition of the IFS up to a weak contractivity assertion for a multi-function, and, secondly, we replace the condensing component by eventually condensing one. It is worth noting that such a multi-function need not be even eventually (weakly) contracting, and its limit set need not be an attractor.

2. Weak contractions

In the sequel $\mathcal{P}_{b,cl}(X)$ and $\mathcal{P}_{cp}(X)$ will denote the spaces of nonempty bounded and closed and, respectively, compact subsets of X, endowed with the Hausdorff-Pompeiu metrics (see, e.g. [8]), defined for any $A, B \in \mathcal{P}_{b,cl}(X)$ by

$$H(A,B) = \max\{\varrho(A,B), \varrho(B,A)\},$$

where $\varrho(A,B) = \sup_{a \in A} \inf_{b \in B} d(a,b)$.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* [5, 8] if:

- φ is monotonically increasing, i.e. $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;
- $\varphi^n(t) \to 0$, as $n \to +\infty$, for all $t \geq 0$.

Following [5, 8], we call the multi-function $f : X \to \mathcal{P}_{b,cl}(X)$ a *weak contraction*, if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(f(x), f(y)) \le \varphi(d(x, y)), \qquad \forall\, x, y \in X.$$

In this case we will say also that f is a *contraction with respect to* φ, or that f is a φ-*contraction* (see [8]). Notice that for $\varphi(t) = \lambda t$ with $0 \le \lambda < 1$ the multi-function f is contracting.

Given a multi-function $f : X \to \mathcal{P}(X)$ one can construct the *Nadler-Hutchinson mapping* $f_* : \mathcal{P}(X) \to \mathcal{P}(X)$, defined for any $A \in \mathcal{P}(X)$ by $f_*(A) := \overline{f[A]}$, where $f[A] = \bigcup\limits_{a\in A} f(a)$ and bar denotes the closure.

Theorem 1. [4] *Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weak contraction with respect to a right continuous comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then the Nadler-Hutchinson mapping $f_* : \mathcal{P}_{b,cl}(X) \to \mathcal{P}_{b,cl}(X)$, $f_*(A) = \overline{f[A]}$, is also a φ-contraction, i.e. for any $A, B \in \mathcal{P}_{b,cl}(X)$ the following inequality holds*

$$H(f_*(A), f_*(B)) \le \varphi(H(A, B)).$$

Corollary 2. *Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weak contraction with respect to a right continuous comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then for any $A, B \in \mathcal{P}_{b,cl}(X)$ one has:*

$$H(f_*^n(A), f_*^n(B)) \le \varphi^n(H(A, B)) \to 0, \qquad \text{as} \qquad n \to +\infty.$$

For compact-valued weakly contracting multi-functions the condition of right continuity of the comparison function can be dropped.

Theorem 3. [4] *Let $f : X \to \mathcal{P}_{cp}(X)$ be a compact-valued weakly contracting multi-function with respect to a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then the Nadler-Hutchinson mapping $f_* : \mathcal{P}_{cp}(X) \to \mathcal{P}_{cp}(X)$, $f_*(A) = \overline{f[A]}$, is also a φ-contraction.*

3. Attractors in weak contractions

There are various definitions of the attractor in dynamical systems. In ordinary dynamics (e.g. iterations of mappings) one usually means by an attractor an invariant set, which is dynamically indivisible and whose basin – the set of attracted points – is a large set. The dynamical indivisibility sometimes is understood as the existence of a dense orbit. As for the basin, it must contain a neighborhood of the attractor, or at least the nonvoid interior, sometimes positive Lebesgue measure is required.

In the case of multi-functions on compact spaces in [1] (see also [7]) the following definition has been proposed: the closed subset A is an attractor, if it is invariant and there exists a closed neighborhood V of A such that $\bigcap\limits_{n\ge 0} f^n[V]$ is contained in A. For another definition of attractor see [6].

Let X be a metric space and let $f : X \to \mathcal{P}(X)$ be a closed relation. A nonempty closed subset $A \subset X$ is called an *attractor* for f, if:

- $f[A] \supset A$;
- there is a closed neighborhood $\bar{V}_\delta = \overline{\{x \in X \mid \varrho(x, A) < \delta\}}$ of A such that $\bigcap\limits_{n \geq 0} f^n[\bar{V}_\delta] \subset A$.

Remark 1. Both of inclusions are, in fact, equalities [3].

Theorem 4. [4] *For every weak contraction $f : X \to \mathcal{P}_{b,cl}(X)$ with respect to a right continuous comparison function there exists a unique bounded and closed set A such that $\overline{f[A]} = A$.*

Theorem 5. [4] *A compact nonempty subset $A \subset X$ is an attractor for a compact-valued weakly contracting multi-function $f : X \to \mathcal{P}_{cp}(X)$ if and only if A is invariant with respect to f.*

Remark 2. The condition on the multi-function to take compact values is necessary. As a counter-example one can choose a constant multi-function on an infinite-dimensional Banach space with the unit closed ball as value.

Corollary 6. *Every compact-valued weakly contracting multi-function has a nonempty compact attractor and this attractor is unique.*

A sequence $\{x_n\}_{n \geq 0}$ is called a *chain* of the multi-function $f : X \to \mathcal{P}(X)$, if $x_{n+1} \in f(x_n)$ for all $n \geq 0$.

The following results describe the dynamics of weakly contracting multi-functions near the attractor.

Theorem 7. [4] *Let $f : X \to \mathcal{P}_{cp}(X)$ be a compact-valued weakly contracting multi-function with respect to a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then for every chain $\{x_n\}_{n \in \mathbb{N}}$ in X and every $y_0 \in X$ there exists a chain $\{y_n\}_{n \in \mathbb{N}}$ in X, starting at y_0, such that*

$$d(x_n, y_n) \leq \varphi^n(d(x_0, y_0)) \qquad (n \geq 0).$$

As a consequence we obtain the following result.

Theorem 8. [4] *[Asymptotic phase theorem for weakly contracting multi-functions] Let $f : X \to \mathcal{P}_{cp}(X)$ be a compact-valued weakly contracting multi-function and let A stand for its attractor. Then for every chain $\{x_n\}_{n \in \mathbb{N}}$ in X and every $a_0 \in A$ there exists a chain $\{a_n\}_{n \in \mathbb{N}}$ in A, starting at a_0, such that $d(x_n, a_n) \to 0$, as $n \to +\infty$.*

In what follows we will call the *big positive orbit* of a point the subset of all chains, which start at this point. It is worth noting that the big positive orbit of a point differs from the image of this point under the orbit relation (see below), just this point making the difference.

Theorem 9. [4] *The big positive orbit of every point from the attractor of a weak contracting compact-valued multi-function represents a dense subset on the attractor.*

4. Limit sets and limit relations for weak contractions

In this section we will assume that X is a complete metric space with the property that each bounded and closed subset is compact. Also, let $f \subset X \times X$ be a closed relation which, treated as a multi-function, takes bounded and closed (therefore, compact) values. Associated with f there are other relations containing f. The first one is the orbit relation $\mathcal{O}f = \bigcup_{n\geq 1} f^n$. Generally, $\mathcal{O}f(x)$ is not a closed set, nor $\mathcal{O}f$ is a closed relation. To close them we need the following definition of the *limit* of a sequence of closed subsets $\{C_n\}_{n\geq 1}$:

$$C := \limsup \{C_n\} = \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} C_k}.$$

If X is compact, then *limsup* exists and is unique. In this case for a subset C to be $\limsup \{C_n\}$ means that for every $\varepsilon > 0$ there exists a natural $N(\varepsilon)$ such that for every $n \geq N(\varepsilon)$ the set C_n is contained in the ε-neighborhood of C, and C is the smallest closed subset with this property (see, e.g. [1]). In the noncompact case *limsup* may be empty.

Obviously, convergence in the Hausdorff-Pompeiu metrics implies convergence in the above mentioned sense.

Given $x \in X$ we define ω-limit relation ωf by $\omega f(x) = \limsup \{f^n(x)\}$.

Thus, the relation $\omega f \subset X \times X$ is defined by its values as a set-valued mapping. Generally, ωf is not a closed relation. Taking the closure of each value, one obtains the recurrent relation

$$\mathcal{R}f(x) := \overline{\mathcal{O}f(x)} = \mathcal{O}f(x) \cup \omega f(x).$$

The relation $\mathcal{R}f$ also need not be closed. We get a closed relation by defining $\Omega f = \limsup \{f^n\}$.

Using Ωf instead of ωf, we obtain the closure of the orbit relation $\mathcal{O}f$:

$$\mathcal{N}f = \mathcal{O}f \cup \Omega f = \overline{\mathcal{O}f}.$$

Thus, for every closed relation f we have a tower of limit relations:

$$f \subset \mathcal{O}f \subset \mathcal{R}f \subset \mathcal{N}f.$$

In general, all these inclusions are strict. They are strict even for an eventually condensing relation. At the same time, for an actual condensation all inclusions become equalities.

The following result shows that for a weakly contracting relation some of these inclusions are equalities.

Theorem 10. *Let X be a complete metric space with the property that every bounded and closed subset is compact. Let $f : X \to \mathcal{P}_{b,cl}(X)$ be a weakly contracting multi-function and let A stand for its attractor. Then the associated limit relations satisfy the following equalities:*

(1) $\omega f = \Omega f = X \times A$;
(2) $\overline{\mathcal{O}f} = \mathcal{R}f = \mathcal{N}f$.

Proof. 1) It is known that convergence with respect to Hausdorff-Pompeiu metrics implies convergence with respect to *limsup*. Using

$$H(f^n(x), f_*^n(A)) \leq \varphi^n(H(x, A)) \to 0, \quad \text{as } n \to +\infty,$$

one has $\limsup \{f^n(x)\} = A$, so $\omega f(x) = A$ for every $x \in X$.

The second equality $\Omega f = X \times A$ follows from the previous one and from the relations (see [1])

$$\bigcap_{\varepsilon>0} \Omega f(\bar{V}_\varepsilon(x)) = \Omega f(x) = \bigcap_{\varepsilon>0} \omega f(\bar{V}_\varepsilon(x)),$$

where $\bar{V}_\varepsilon(x)$ denotes the closed ε-neighborhood of x.

2) To prove the second sentence one uses the result 1) and the definitions of the limit relations:

$$\overline{\mathcal{O}f} = \mathcal{R}f = \mathcal{O}f \cup \omega f = \mathcal{O}f \cup (X \times A) = \mathcal{O}f \cup \Omega f = \mathcal{N}f.$$

□

5. Eventually condensing multi-functions

We say that a compact-valued multi-function (relation) $f : X \to \mathcal{P}_{cp}(X)$ is an *eventual condensation* if there exists a non-empty compact $K \subset X$ (called the *condensation set*) and a natural n_0 such that for all $n \geq n_0$ the multi-function f^n is constant with value K. The smallest natural n_0 with this property is called the *time of condensation*. For $n_0 = 1$ the multi-function f is a *condensation*. In contrast with a condensation, an eventual condensation need not be contracting.

Remark 3. For an eventual condensation $f : X \to \mathcal{P}_{cp}(X)$ with condensation set K and condensation time n_0 one has

$$\mathcal{O}f = f \cup f^2 \cdots \cup f^{n_0-1} \cup (X \times K).$$

Theorem 11. *Let X be a complete metric space with the property that every bounded and closed subset is compact and let f be an eventual condensation with the condensation set K. Then*

(1) $\omega f = \Omega f = X \times K$;
(2) $\mathcal{O}f = \overline{\mathcal{O}f} = \mathcal{R}f = \mathcal{N}f$.

Proof. 1) Let n_0 be the condensation time for f. One has

$$\Omega f = \limsup\{f^n\} = \limsup\{f^n(f^{n_0})\} = X \times K.$$

Similarly, for every $x \in X$:

$$\omega f(x) = \limsup\{f^n(x)\} = \limsup\{f^n(f^{n_0}(x))\} = K.$$

So, $\omega f = \Omega f = X \times K$.

2) Since f is an eventual condensation, the orbit relation takes the form

$$\mathcal{O}f = f \cup f^2 \cdots \cup f^{n_0-1} \cup (X \times K).$$

Since $f, \ldots, f^{n_0-1}$ are closed, we have $\overline{\mathcal{O}f} = \mathcal{O}f$.

Other equalities follow from the respective definitions and the result 1). □

6. Limit sets and limit relations for weak contractions with eventual condensation

M. Barnsley [2] has obtained a nice formula for the attractor of a hyperbolic IFS with condensation. Namely, if $\{X; f_1, \ldots, f_m\}$ is a hyperbolic IFS with the attractor A, and if $f_0(x) \equiv K$ for some compact set $K \subset X$ and for all $x \in X$, then the attractor A_1 of the IFS with condensation $\{X; f_0, f_1, \ldots, f_m\}$ has the form

$$A_1 = A \cup \Big(\bigcup_{n\geq 0} f_*^n(K)\Big), \tag{1}$$

where f_* is the respective Nadler-Hutchinson mapping for the hyperbolic IFS.

Both relations, the hyperbolic IFS and the condensation, are contractions, as well as their union $F := f \cup f_0$ (relation generated by a contraction and a condensation), or in other words, $F(x) = f(x) \cup f_0(x)$.

An eventual condensation is an eventual contraction as well, i.e. some its power is a contraction (moreover, a constant). At the same time the union of a contraction and of an eventual condensation need not be even an eventual contraction.

Example 1. Consider the functions $f_0, f_1 : \mathbb{R} \to \mathbb{R}$,

$$f_0(x) = \begin{cases} 0, & x \leq 1, \\ -2x+2, & x > 1; \end{cases} \qquad f_1(x) = \begin{cases} -x/2+1, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

The function f_1 is a contraction. Moreover, it is an eventual condensation, since $f_1^2(x) \equiv 1$. The function f_0 is also an eventual condensation, since $f_0^2(x) \equiv 0$, but it is not contracting. Their union $F = f_1 \cup f_0$ is not an eventual contracting, since for every $x \in \mathbb{R}$ there exists a periodic or a preperiodic chain, starting at x. For example, for $x \in (-\infty, 0] \cup [1, +\infty)$ one has

$$x \stackrel{f_0}{\longmapsto} (-2x+2) \stackrel{f_1}{\longmapsto} x,$$

while $x \in (0,1)$ yields

$$x \overset{f_0}{\longmapsto} 0 \overset{f_1}{\longmapsto} 1 \overset{f_0}{\longmapsto} 0.$$

In what follows we state the limit sets and the limit relations as asymptotic characteristics of set-valued dynamics, generated by iterations of the relation $F = f \cup f_0$, where f is a weak contraction and f_0 is an eventual condensation. In contrast with the previous results the limit relations ωF and ΩF need not be constant. The relation F itself, generally, may admit many local attractors.

Lemma 12. *Let f be a hyperbolic IFS with the attractor A. Then for every nonempty compact $B \subset X$ and for every $m \geq 0$ one has*

$$\overline{\bigcup_{n\geq m} f^n(B)} = A \cup \big(\bigcup_{n\geq m} f^n(B)\big).$$

Proof. The sentence follows from the fact that for every nonempty compact $B \subset X$ the sequence $\{f^n(B)\}_{n\geq 0}$ converges to A, as n tends to $+\infty$. □

Theorem 13. *Let X be a complete metric space with the property that every bounded and closed subset is compact. Let f be a weakly contracting relation with the attractor A and let f_0 denote an eventual condensation with the condensation time n_0. Then for every $x \in X$ the ω-limit set with respect to the relation $F := f \cup f_0$ has the form*

$$\omega F(x) = \bigcap_{n\geq 1} \overline{\bigcup_{m\geq n} B_m(x)}, \tag{2}$$

where the sequence of subsets $\{B_m(x)\}_{m\geq 0}$ is defined as follows:

$$B_0(x) = \bigcup_{n\geq 0} f^n(x),$$

$$B_m(x) = \bigcup_{n\geq 0} f^n\big(f_0(F^{m-1}(x))\big) \quad (m \geq 1). \tag{3}$$

Proof. By definition

$$\omega F(x) = \limsup \{F^n(x)\} = \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} F^k(x)},$$

where F^k means the union of all possible compositions of k relations, consisting of f and f_0.

For convenience we will use the notation f_1 instead of f. Notice that

$$F^k(x) = \bigcup_{i_j\in\{0,1\}} (f_{i_k} \circ f_{i_{k-1}} \circ \cdots \circ f_{i_1})(x) = f_{i_k}(f_{i_{k-1}}(\ldots(f_{i_1}(x))\ldots)).$$

Denote the composition $f_{i_k} \circ f_{i_{k-1}} \circ \cdots \circ f_{i_1}$ by $f_{i_1 i_2 \ldots i_k}$.

Let W be the set of all finite words of two letters $\{0,1\}$. Define an one-to-one correspondence between W and the set of all compositions $\bigcup\limits_{k\geq 1} F^k$ as follows:

$$w = i_1 i_2 \dots i_k \longmapsto f_{i_1 i_2 \dots i_k}.$$

Denote by W_m $(m \geq 1)$ the set of all words of length at least m, and such that each of them contains at least one "0" and the last "0" in this word appears at the position m. The set W_0 consists of all words without "0". These subsets make a partition of W, i.e. $W = \bigcup\limits_{m\geq 0} W_m$ and $W_i \cap W_j = \emptyset$ for $i \neq j$.

Let prove (2), rewritten as follows:

$$\bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} F^k(x)} = \bigcap_{n\geq 1} \overline{\bigcup_{m\geq n} B_m(x)}. \tag{4}$$

Since $B_m(x) \subset \bigcup\limits_{k\geq m} F^k(x)$, one has $\bigcup\limits_{m\geq n} B_m(x) \subset \bigcup\limits_{k\geq n} F^k(x)$ for every $n \geq 1$. Therefore,

$$\bigcap_{n\geq 1} \overline{\bigcup_{m\geq n} B_m(x)} \subset \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} F^k(x)}.$$

Conversely, assume that $y \in \bigcap\limits_{n\geq 1} \overline{\bigcup\limits_{k\geq n} F^k(x)}$. So, given $\varepsilon > 0$ and $n \geq 1$, there exists $k > n$ such that $\varrho(y, F^k(x)) < \varepsilon/2$, i.e. there exists a word $w_k = i_1^{(k)} \dots i_k^{(k)} \in W$ such that $\varrho(y, f_{i_1 \dots i_k}(x)) < \varepsilon/2$ (for convenience we omit here and below the superscript, which denotes the dependence on the word w_k).

There are two possibilities:

1) There exists a natural m such that for every $k \geq m$ one has $w_k \in \bigcup\limits_{j=1}^{m} W_j$, i.e. none of these words has the letter "0" to the right of the position m. Denote by C the compact $C = F^m(x)$. Then for every $k > m$ for the mentioned words $w_k = i_1 \dots i_k$ one has $i_{m+1} = \dots = i_k = 1$ and

$$f_{i_1 \dots i_m i_{m+1} \dots i_k}(x) = f_{i_{m+1} \dots i_k}(f_{i_1 \dots i_m}(x)) \subset f_{i_{m+1} \dots i_k}(F^m(x)) = f_1^{k-m}(C).$$

Using the properties of the distance ϱ from one set to another (see, e.g. [1]) and the fact that A is the attractor for weakly contracting multi-function f_1, one has for k big enough:

$$\begin{aligned} &\varrho(y, A) \leq \varrho(y, f_{i_1 \dots i_k}(x)) + \varrho(f_{i_1 \dots i_k}(x), A) \leq \\ &\varrho(y, f_{i_1 \dots i_k}(x)) + \varrho(f_1^{k-m}(C), A) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned} \tag{5}$$

Since ε is arbitrary, (5) implies that $\varrho(y, A) = 0$, which, in turn, implies $y \in A \subset \overline{\bigcup\limits_{n\geq m} f_1^n(x)}$ for every $m \geq 0$.

2) For every $n \geq 1$ there exist naturals $k > m > n$ such that the word $w_k = i_1 \dots i_k$ belongs to W_m. In this case $f_{i_1 \dots i_k}(x) \in B_m(x)$, which, in turn, implies $y \in \bigcap_{n\geq 1} \overline{\bigcup_{m\geq n} B_m(x)}$.

Hence,

$$\bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} F^k(x)} \subset \bigcap_{n\geq 1} \overline{\bigcup_{m\geq n} B_m(x)}.$$

This accomplishes the proof of (4). □

Theorem 14. *If the sequence* $\{B_m\}_{m\geq 0}$ *from Theorem 13 satisfies the equality* $B_{m_0+1}(x) = B_{m_0}(x)$ *for some* $m_0 \geq 0$, *then it stabilizes, i.e.*

$$B_{m+1}(x) = B_m(x) \tag{6}$$

for all $m \geq m_0$, *and*

$$\omega F(x) = \overline{B_{m_0}(x)}. \tag{7}$$

Proof. Let $B_{m_0+1}(x) = B_{m_0}(x)$. We will show by induction that (6) holds for all $m \geq m_0$.

Firstly, notice that $\bigcup_{n\geq 0} f^n\big(B_m(x)\big) = B_m(x)$ for every $m \geq 0$.

Assume that (6) occurs for some $m \geq m_0$, i.e.

$$\bigcup_{n\geq 0} f^n\big(f_0(F^m(x))\big) = \bigcup_{k\geq 0} f^k\big(f_0(F^{m-1}(x))\big). \tag{8}$$

The latter implies the following inclusion:

$$f_0(F^m(x)) \subset \bigcup_{k\geq 0} f^k\big(f_0(F^{m-1}(x))\big),$$

which, in turn, yields

$$f_0(F^{m+1}(x)) = f_0(F^m(F(x))) \subset \bigcup_{k\geq 0} f^k\big(f_0(F^{m-1}(F(x)))\big) =$$
$$\bigcup_{k\geq 0} f^k\big(f_0(F^m(x))\big) = B_{m+1}(x).$$

As a consequence,

$$B_{m+2}(x) = \bigcup_{n\geq 0} f^n\big(f_0(F^{m+1}(x))\big) \subset \bigcup_{n\geq 0} f^n(B_{m+1}(x)) = B_{m+1}(x).$$

Conversely, (8) implies

$$f_0(F^{m-1}(x)) \subset \bigcup_{n\geq 0} f^n\big(f_0(F^m(x))\big),$$

which, in turn, yields

$$f_0(F^m(x)) = f_0(F^{m-1}(F(x))) \subset \bigcup_{n\geq 0} f^n\big(f_0(F^m(F(x)))\big) = \bigcup_{n\geq 0} f^n\big(f_0(F^{m+1}(x))\big) = B_{m+2}(x),$$

and

$$B_{m+1}(x) = \bigcup_{k\geq 0} f^k\big(f_0(F^m(x))\big) \subset \bigcup_{k\geq 0} f^k(B_{m+2}(x)) = B_{m+2}(x).$$

So, $B_{m+2}(x) = B_{m+1}(x)$.

Therefore, (6) holds for all $m \geq m_0$.

In these conditions the equality (7) follows directly from (2). □

Remark 4. If $f_0 : X \to \mathcal{P}_{cp}(X)$ is a condensing multi-function with the condensation set K, then (3) implies

$$B_m(x) = B_1(x) = \bigcup_{n\geq 0} f^n(K),$$

and, by Lemma 12,

$$\omega F(x) = \overline{B_1(x)} = \overline{\bigcup_{n\geq 0} f^n(K)} = A \cup \big(\bigcup_{n\geq 0} f^n(K)\big) = A_1,$$

where A_1 is the attractor of $F = f \cup f_0$. As a result, the formula (2) becomes Barnsley's formula (1).

Remark 5. Example 1 represents an IFS, consisting of two functions, each one with a unique fixed point as (global) attractor. Every hyperbolic IFS, consisting of two functions on $\mathbb{R}$, possesses a unique attractor, located between the fixed points of IFS and containing these fixed points. Thus, in our example, there are two natural candidates for the attractor: the whole segment $[0, 1]$, or a subset containing the boundary $\{0, 1\}$. None of them satisfies the definition of attractor, since every their neighborhood contains periodic chains beyond the supposed attractor. This argument is valid for every segment $[-2a + 2, a]$ with $|a| \geq 1$, which is invariant with respect to IFS.

Remark 6. A relation, consisting of a weak contraction and an eventual condensation may admit more than one local attractor, as the following example shows.

Example 2. Consider the IFS with an eventual condensation $\mathcal{F} = \{\mathbb{R}^2; f_0, f_1, f_2\}$, consisting of two contractions

$$f_1(x, y) = (0.64x - 0.48y - 0.18,\ 0.48x + 0.64y + 1.24),$$
$$f_2(x, y) = (0.36x + 0.48y + 0.32,\ -0.48x + 0.36y + 1.24),$$

and of an eventually condensing multi-function f_0 with condensation set K and condensation time $n_0 = 2$,

$$f_0(x,y) = \begin{cases} K, & \text{if} \quad y < 4.3, \\ K \cup K_1, & \text{if} \quad y = 4.3, \\ K_1, & \text{if} \quad y > 4.3, \end{cases}$$

where $K = [-0.5, 0.5] \times [0,1]$ and $K_1 = [3,4] \times [0,1]$.

Figure 2 represents two local attractors of this IFS: the first attractor (left) is the Pythagoras tree and the second one (right) is the union of the attractors from the Figure 1.

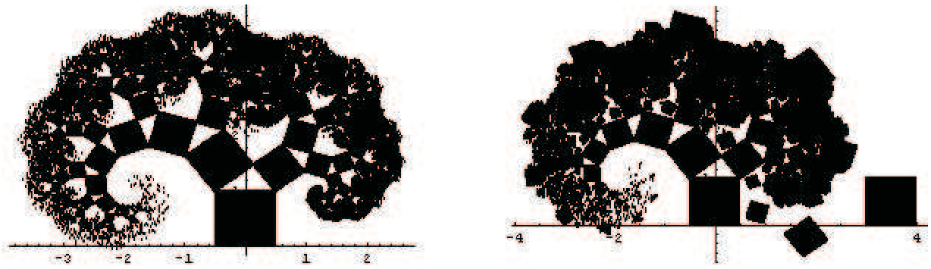

FIGURE 2. Two local attractors of an IFS with eventual condensation: the first (left) and the second, including the first one (right)

Modifying the eventual condensation f_0 (but keeping the same condensation time $n_0 = 2$), one can construct IFS's with any finite or even infinite number of distinct local attractors.

All numerical calculations and graphic objects have been done using the Computer Algebra System *Mathematica*.

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