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MATRIX LIE RINGS THAT CONTAIN AN ABELIAN SUBRING

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ABSTRACT. Let k be a field and $\bar{k}$ an algebraic closure of k. The paper is devoted to the description of subrings of the Lie ring $sl_2(\bar{k})$ that contain an abelian subring which is a one-dimensional subspace of the k-vector space $sl_2(\bar{k})$.

Key words : Lie rings, lie algebras, semi-simple matrices, nilpotent matrices.
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If R is an associative ring and $a, b \in R$, then $[ab]$ denotes the Lie product $ab-ba$ of a and b. Let n be an integer, $n \geq 2$. The ring of all $n\times n$ matrices over R is denoted by $M_n(R)$. If R is a field, then the set of all matrices in $M_n(R)$ the trace of which equals zero is a Lie ring with respect to the multiplication $[ab]$ $(a, b \in M_n(R))$. This Lie ring is denoted by $sl_n(R)$. In what follows, k denotes a field and $\bar{k}$ is an algebraic closure of k.

In [2], subrings of the Lie ring $sl_n(\bar{k})$ that contain $sl_n(P)$ has been described for $n \geq 2$ provided that the algebraic closed field $\bar{k}$ is a finite extension of its subfield P. The author of the present paper has generalized this result for arbitrary algebraic extensions $\bar{k}/P$ and used the generalization for studying subrings of the k-algebra $sl_2(\bar{k})$ that contain an abelian one-dimensional subalgebra consisting of semi-simple matrices ([1]). It turned out that any non-solvable Lie ring of this kind is isomorphic to the ring of elements which are skew symmetric relative to a suitable involution acting on some quaternion algebra. For the reader convenience it is worthwhile recalling a definition of these algebras.

Let F be a field of characteristic $\neq 2$. Let a, b be non-zero elements in F and A a four dimensional vector space over F with a basis $1, u, v, w$. We define an associative multiplication on these basis elements by the following

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conditions: the element 1 satisfies the identity relation, $u^2 = a, v^2 = b, uv = -vu = w$. We extend this multiplication linearly to a multiplication on A. The algebra A over F obtained by this construction is called a quaternion algebra. This algebra admits a unique symlectic type involution J, i. e., the anti-automorphism $J : A \to A$ such that every element of F is fixed under J whereas $u^J = -u, v^J = -v, w^J = -w$. The set of all elements $x \in A$ such that $x^J = -x$ constitute a Lie algebra over F under multiplication $[xy]$. We shall denote this algebra by $s(A)$. In addition, as is well known each quaternion algebra is either a division algebra or is isomorphic to the algebra of 2×2 matrices with entries in its center.

Recently the author observed that non-complicated arguments in similar fashion as in [1] permit us to study a more general case, namely, to describe subrings of $sl_2(\bar{k})$ containing an arbitrary one dimensional subalgebra. The present paper addresses the proof of the corresponding result. We shall use the following notations for special matrices in $sl_2(\bar{k})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Also we employ the following relations for H, X, Y:

$$[HX] = 2X, \quad [HY] = -2Y, \quad [XY] = H. \tag{1}$$

As usual, we denote by $GL_2(R)$ the general linear group of degree 2 over an associative ring R. Unlike the paper [1] the present article concerns certain solvable Lie rings. Therefore, we begin by giving the definition of a class of solvable Lie rings in terms of which our main result will be formulated.

So, let c be a non-zero element in $\bar{k}$, Q a subgroup of the additive group of the field $\bar{k}$ such that $Q \supseteq kc$. Denote by $k(Q)$ the field obtained by adjoining the set Q to k. We regard the field $\bar{k}$ as a vector space over the field $k(Q)$. Next let B be a subspace of the $k(Q)$-vector space $\bar{k}$. Define $\mathfrak{g}(c, k, Q, B)$ to be the set of matrices $qH + bX$ with $q \in Q, b \in B$. The equation

$$[q_1H + b_1X,\ q_2H + b_2X] = 2(q_1b_2 - b_1q_2)X$$

holds for every $q_i, b_i \in \bar{k}$. This shows that $\mathfrak{g}(c, k, Q, B)$ is a subring of $sl_2(\bar{k})$. It should be evident that $\mathfrak{g}(c, k, Q, B)$ is solvable and moreover, $\mathfrak{g}(c, k, Q, B)$ is abelian if $B = \{0\}$.

Now we are in a position to formulate the main result of the paper.

Theorem 1. *Let k be a field and $\bar{k}$ an algebraic closure of k. Suppose that $k \neq \mathbb{F}_3, \mathbb{F}_5$ if* char $k \neq 2$. *Let $\mathfrak{h}$ be a one-dimensional subalgebra of the Lie k-algebra $sl_2(\bar{k})$. If $\mathfrak{g}$ is a subring of $sl_2(\bar{k})$ containing $\mathfrak{h}$, then $\mathfrak{g}$ is either solvable, or there exists a quaternion algebra A over a subfield F of $\bar{k}$ such* *that $F \supseteq k$ and $\mathfrak{g}$ is isomorphic to the Lie F-algebra $s(A)$. If $\mathfrak{h}$ contains semi-simple elements only and $\mathfrak{g}$ is solvable, then $\mathfrak{g}$ is conjugate by an element of the group $GL_2(\bar{k})$ to a ring of the form $\mathfrak{g}(c, k, Q, B)$ with suitable c, Q, B.*

We take account of the results presented in [1] to reduce the proof of Theorem 1 to the consideration of the following two cases:

(1) $\mathfrak{h}$ consists of nilpotent matrices only.

(2) $\mathfrak{h}$ consists of semi-simple matrices and $\mathfrak{g}$ is a solvable Lie ring.

Proposition 2 just below deals with the first case. This proposition actually states that when case (1) arises, $\mathfrak{g}$ is conjugate to the ring $sl_2(L)$, where L is an appropriate field, i. e., $\mathfrak{g}$ is conjugate to the subset of the quaternion algebra $M_2(L)$ consisting of all elements which are skew-symmetric relative to the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

defined on $M_2(L)$.

Proposition 2. *Let k be a field of characteristic $\neq 2$, $\bar{k}$ an algebraic closure of k, $\mathfrak{g}$ a subring of the Lie ring $sl_2(\bar{k})$. Suppose $\mathfrak{g}$ contains an abelian subring $\mathfrak{h}$ which is a one-dimensional k-vector space consisting of nilpotent matrices. Then $\mathfrak{g}$ is either solvable or is conjugate by an appropriate matrix of the group $GL_2(\bar{k})$ to the Lie ring $sl_2(L)$, where L is a subfield of $\bar{k}$ containing k.*

Proof. We can replace $\mathfrak{g}$ by $x\mathfrak{g}x^{-1}$ with a suitable matrix x in $GL_2(\bar{k})$ to assume $\mathfrak{h} = kX$. Suppose $\mathfrak{g}$ is not solvable. Then $\mathfrak{g}$ contains a matrix $v = v_{11}H + v_{12}X + v_{21}Y \in sl_2(\bar{k})$ with $v_{21} \neq 0$. It is straightforward to check that $[X[Xv]] = -2v_{21}X \in \mathfrak{g}$. It follows that $kv_{21}X \subseteq \mathfrak{g}$ since $kX \subseteq \mathfrak{g}$. By induction, $kv_{21}^s X \subseteq \mathfrak{g}$ for any integer $s \geq 1$. Since v_{21} is algebraic over k, this implies the inclusion $k(v_{21})X \subseteq \mathfrak{g}$. Further,

$$[v[Xv]] = 2v_{11}v_{21}H + (-4v_{11}^2 - 2v_{12}v_{21})X + 2v_{21}^2 Y. \tag{2}$$

In the left hand side of (2), we replace v by $[v[Xv]]$ whereas in the right hand side, let us take $2v_{11}v_{21}, -4v_{11}^2 - 2v_{12}v_{21}, 2v_{21}^2$ instead of v_{11}, v_{12}, v_{21} respectively. With this having been made, we get

$$[[v[Xv]],\ [X[v[Xv]]]] = 8v_{21}^3 v \in \mathfrak{g}. \tag{3}$$

Since $\gamma X \in \mathfrak{g}$ for any $\gamma \in k(v_{21})$, relation (3) shows that

$$[[v[\gamma Xv]],\ [X[v[Xv]]]] = 8v_{21}^3 \gamma v \in \mathfrak{g}.$$

Hence $k(v_{21})v \subseteq \mathfrak{g}$. Moreover, for any $\gamma \in k(v_{21})$, we have $[v[\gamma Xv]] = \gamma[v[Xv]] \in \mathfrak{g}$, and so $\mathfrak{g}$ contains the matrix

$$y = \frac{1}{4}[v[Xv]] + \frac{1}{2}v_{21}v = v_{11}v_{21}H - v_{11}^2 X + v_{21}^2 Y.$$

Thus the ring $\mathfrak{g}$ contains a subring $\mathfrak{g}_0$ generated by y and by the set kX. Put

$$z = \begin{pmatrix} 1 & -v_{21}^{-1}v_{11} \\ 0 & 1 \end{pmatrix}.$$

Then $z\mathfrak{g}_0 z^{-1}$ is a subring of $sl_2(\bar{k})$ generated by kX and $v_{21}^2 Y$. Next we employ (1) to obtain $z\mathfrak{g}_0 z^{-1} = sl_2(k(v_{21}^2))$. So by Proposition 3 [1], $z\mathfrak{g}z^{-1} = sl_2(L)$, where L is a subfield of $\bar{k}$ containing $k(v_{21})$. The proposition is proved. □

Our next proposition in fact proves the part of Theorem 1 covering the case of solvable Lie rings.

Proposition 3. *Let k be a field, $\bar{k}$ an algebraic closure of k, $\mathfrak{g}$ a solvable subring of the Lie ring $sl_2(\bar{k})$, $\mathfrak{h}$ an abelian one-dimensional subalgebra of the Lie k-algebra $sl_2(\bar{k})$. Suppose that $\mathfrak{g} \supseteq \mathfrak{h}$ and $\mathfrak{h}$ consists of semi-simple matrices. Then $\mathfrak{g}$ is conjugate by an element of $GL_2(\bar{k})$ to a ring of the form $\mathfrak{g}(c, k, Q, B)$.*

Proof. Since $\mathfrak{h}$ consists of semi-simple matrices only, char $k \neq 2$. Replacing $\mathfrak{h}$ by $x\mathfrak{h}x^{-1}$with an appropriate matrix $x \in GL_2(\bar{k})$, one may assume $\mathfrak{h} = kcH$, where $c \in \bar{k} \setminus \{0\}$.

If $\mathfrak{g}$ contains a matrix $v_{11}H + v_{12}X + v_{21}Y$ with $v_{12} \neq 0, v_{21} \neq 0$, then according to the proof of Proposition 2 [1], there exists a quaternion algebra A such that $\mathfrak{g}$ is a Lie ring of all elements in A which are skew-symmetric relative to a symplectic type involution defined on A. But this is impossible because $\mathfrak{g}$ is solvable. Therefore for any $v_{11}H + v_{12}X + v_{21}Y$ in $\mathfrak{g}$, we have either $v_{12} = 0$ or $v_{21} = 0$. Assume that $\mathfrak{g}$ contains a matrix $v = v_{11}H + v_{12}X + v_{21}Y$ with $v_{12} \neq 0, v_{21} = 0$ and a matrix $v' = v'_{11}H + v'_{12}X + v'_{21}Y$ with $v'_{12} = 0, v'_{21} \neq 0$. Then since char $k \neq 2$, we can choose r in k so that all coefficients of the matrix $v + v' + crH$ are non-zero. As we have just shown, this contradicts to the solvability of $\mathfrak{g}$. Therefore, either $\mathfrak{g}$ consists of upper triangular matrices or $\mathfrak{g}$ consists of lower triangular matrices. Replacing $\mathfrak{g}$ by $w\mathfrak{g}w^{-1}$, where w is a monomial matrix in $GL_2(\bar{k})$, allows us to assume that all matrices in $\mathfrak{g}$ are upper triangular.

Set

$$Q = \{q \in \bar{k} \mid qH \in \mathfrak{g}\}, \quad B = \{b \in \bar{k} \mid bX \in \mathfrak{g}\}. \tag{4}$$

It should be obvious that Q is a subgroup of the additive group $\bar{k}$. Also $kc \subseteq Q$ since $kcH \subseteq \mathfrak{g}$. We seek to show that B is a subspace of the $k(Q)$-vector space $\bar{k}$. To make certain of this, we first prove that $kbX \subseteq B$ for any $b \in B$. Indeed, if $b \in B$, then $bX \in \mathfrak{g}$ and since $kcH \subseteq \mathfrak{g}$, we get $[kcH, bX] = kcbX \subseteq \mathfrak{g}$. It follows that $kc^m bX \subseteq \mathfrak{g}$ for any integer $m \geq 1$. So $k(c)bX \subseteq \mathfrak{g}$ since c is algebraic over k. But $k \subseteq k(c)$, and so $kbX \subseteq \mathfrak{g}$ as claimed. Now let again $q_1, q_2 \in Q$ and $b \in B$. Then $[q_1H, kbX] = kq_1bX \subseteq \mathfrak{g}$ hence $[q_2H, kq_1bX] = kq_1q_2bX \subseteq \mathfrak{g}$. Thus we have shown that if $b \in B$, then

$kq_1q_2b \in B$ for every $q_1, q_2 \in Q$. It is easy to see that this means in fact that B a vector space over the field $k(Q)$. So the algebra

$$\mathfrak{g}(c, k, Q, B) = \{qH + bX \mid q \in Q, b \in B\}$$

is defined, and by virtue of (4),

$$\mathfrak{g}(c, k, Q, B) \subseteq \mathfrak{g}. \tag{5}$$

If every element in $\mathfrak{g}$ is diagonal, then $\mathfrak{g}$ is abelian and $\mathfrak{g} = \mathfrak{g}(c, k, Q, \{0\})$. Suppose now that $\mathfrak{g}$ contains a matrix $v = v_{11}H + v_{12}X$ with $v_{12} \neq 0$. Then $\mathfrak{g}$ contains an element $[cH, v] = 2cv_{12}X$. Furthermore, since $kcH \subseteq \mathfrak{g}$, we have $kcv_{12}X \subseteq \mathfrak{g}$, and so, by induction, $kc^m v_{12}X \subseteq \mathfrak{g}$ for all integers $m \geq 1$. This implies $k(c)v_{12}X \subseteq \mathfrak{g}$. Specifically, $v_{12}X \in \mathfrak{g}$, and hence $v - v_{12}X = v_{11}H \in \mathfrak{g}$. Observe now that the relations $v_{12}X \in \mathfrak{g}$ and $v_{11}H \in \mathfrak{g}$ mean, according to the definitions of B and Q, that $v_{12} \in B$ and $v_{11} \in Q$. So $v \in \mathfrak{g}(c, k, Q, B)$. Consequently, $\mathfrak{g} \subseteq \mathfrak{g}(c, k, Q, B)$, and this together with (5) completes the proof of the proposition. □

Now we can prove our main result without any difficulties.

Proof of Theorem 1. If char $k = 2$, then $\mathfrak{g}$ is solvable. So further we shall suppose that char $k \neq 2$. Let $\mathfrak{h} = kd$ with $d \in sl_2(\bar{k})$. The trace of d is zero, hence the Jordan canonical form of d is either X or μH with $\mu \in \bar{k} \setminus \{0\}$. In the first case the theorem follows from Proposition 2. Now assume that the second case is valid. If $\mathfrak{g}$ is not solvable, then the theorem follows from the results proved in [1]. If $\mathfrak{g}$ is solvable, then the theorem follows from Proposition 3. □

References

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