Journal of Prime Research in Mathematics Vol. **4***(2008), 118-126*

ON THE GRACEFULNESS OF THE DIGRAPHS $n - \vec{C}_m$ FOR m ODD

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Abstract. A digraph $D(V,E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f^{'} : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u, v) = [f(v) - f(u)] \pmod{|E| + 1}$ for every directed edge (u, v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of $D(V,E)$, while $f^{'}$ is called the induced edge's graceful labeling of D. In this paper we discuss the gracefulness of the digraph $n - \vec{C}_m$ and prove that $n - \vec{C}_m$ is a graceful digraph for $m = 5, 7, 9, 11, 13$ and even n.

Key words:Digraph, directed cycles, graceful graph, graceful labeling.
AMS SUBJECT:05C65.

1. Introduction

A graph $G(V,E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f^{'} : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u, v) = |f(u) - f(v)|$ for every edge (u, v) is a bijection. Here, f is called a graceful labeling(graceful numbering) of G, while $f^{'}$ is called the induced edge's graceful labeling of G. A digraph $D(V,E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f^{'} : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f^{'}(u, v) = [f(v) - f(u)] \pmod{|E| + 1}$ for every directed edge (u, v) is a bijection, where $[v] \pmod{u}$ denotes the least positive residue of v modulo n. In this case, f is called a graceful labeling (graceful numbering) of D and $f^{'}$ is called the induced edge's graceful labeling of D (see[3]).

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Let C_m and $\vec{C}_m$ denote the cycle and directed cycle on m vertices, respectively, $n \cdot C_n$ and $n - C_m$ denote the graphs obtained from any n copies of C_m which have just one common vertex and have just one common edge, respectively. At the same time, let $n \cdot \vec{C}_m$ and $n - \vec{C}_m$ denote the digraphs obtained from any n copies of the directed cycle $\vec{C}_m$ which have just one common vertex and have just one common edge, respectively.

As to the gracefulness of $n \cdot \vec{C}_m$ we know the following results: Ma proved in [3] that the gracefulness of $n \cdot \vec{C}_3$ implies that n is even, at same times he conjectured that the condition that n is even was also sufficient for $n \cdot \vec{C}_3$ to be graceful. In [5], Jirimutu et al has showed this conjecture. It was showed that $n \cdot \vec{C}_{2k}$ is graceful for every integer $n \geq 1$ and $k \geq 1$ in [6], and $n \cdot \vec{C}_{2k+1}$ is graceful for n even and $k = 2, 3$ in [7].

About the gracefulness of $n - \vec{C}_m$, to our knowledge, there are no so much result: It was showed in [3] that $n - \vec{C}_3$ is graceful when n is even, and it was proved in [6] that the necessary condition for $n - \vec{C}_m$ to be graceful is $mn \equiv 0 \pmod 2$. In [8], we have showed that $n - \vec{C}_m$ is graceful if $m = 4, 6, 8, 10$ and n is even.

In this paper, we will further discuss the gracefulness of the digraph $n - \vec{C}_m$ and prove the digraph $n - \vec{C}_m$ is graceful if $m = 5, 7, 9, 11, 13$ and n is even.

2. Main Results

Let $\vec{C}_m^1, \vec{C}_m^2, \cdots, \vec{C}_m^n$ denote the n directed cycles in $n - \vec{C}_m$. The two vertices of the common edge of $\vec{C}_m^i$'s are denoted by v_0 and v_{m-1}, and other $m-2$ vertices of the $\vec{C}_m^i$ are denoted by v_j^i $(j = 1, \ldots, m-2; i = 1, 2, \cdots, n)$, respectively. For convenience, we put $v_0^1 = v_0^2 = \cdots = v_0^n = v_0, v_m^1 = v_m^2 = \cdots = v_m^n = v_m$, and take subscripts j's modulo m. Obviously, $|E(n - \vec{C}_m)| = (m-1)n + 1$.

Suppose that $n - \vec{C}_m$ is graceful and f and f' are its graceful labeling and the induced edge's graceful labeling, respectively.

For every i, it is easy to see that

$$\sum_{j=0}^{m-1} \left[f(v_j^i) - f(v_{j-1}^i) \right] \equiv \sum_{j=0}^{m-1} f(v_j^i) - \sum_{j=0}^{m-1} f(v_{j-1}^i) = 0 \pmod{((m-1)n+2)},$$

which means that there is an integer k_i such that

$$\sum_{j=0}^{m-1} \left[f(v_j^i) - f(v_{j-1}^i) \right] = k_i((m-1)n+2), \ (1 \leq i \leq n). \tag{1}$$

This implies that there is an integer k such that

$$\sum_{i=1}^{n}\sum_{j=0}^{m-1}\left[f(v_j^i)-f(v_{j-1}^i)\right]=k((m-1)n+2). \tag{2}$$

On the other hand, setting $q=|E(n-\vec{C}_m)|=(m-1)n+1$ and $d=[f(v_0)-f(v_{m-1})]$, by definition we have

$$\sum_{i=1}^{n}\sum_{j=0}^{m-1}\left[f(v_j^i)-f(v_{j-1}^i)\right]=(n-1)d+\frac{1}{2}q(q+1)=k(q+1). \tag{3}$$

From the above discussion we obtain the necessary condition as follows.

$$(n-1)d\equiv 0 \pmod{\frac{q+1}{2}}. \tag{4}$$

In the argument below we always take $f(v_0)=0$ and $f(v_{m-1})=\frac{q+1}{2}$. Thus, $d=[f(v_0)-f(v_{m-1})]=[-\frac{q+1}{2}]\equiv\frac{q+1}{2} \pmod{q+1}$, which satisfies the condition given in (4).

Theorem 1. *For every even integer n, the digraph $n-\vec{C}_5$ is graceful.*

Proof. We have had $f(v_0)=0$ and $f(v_4)=2n+1$. For other vertices, define:

$$f(v_1^i)=\begin{cases} i, & 1\le i\le \frac{n}{2},\\ n+i, & \frac{n}{2}+1\le i\le n,\end{cases}$$

$$f(v_2^i)=n+\frac{n}{2}+1-i, 1\le i\le n,$$

$$f(v_3^i)=\begin{cases} 2n+1+i, & 1\le i\le \frac{n}{2},\\ 3n+1+i, & \frac{n}{2}+1\le i\le n.\end{cases}$$

Firstly, we show that f is an injective mapping from $V(n-\vec{C}_5)$ into $\{0,1,\cdots,4n+1\}$.

Put $S_j=\{f(v_j^i)|1\le i\le n\},\ 0\le j\le 4$. Then

$$\begin{aligned}
S_0 &= \{f(v_0)\}=\{0\},\\
S_1 &= \{f(v_1^i)|1\le i\le n\}=\{i|1\le i\le \tfrac{n}{2}\}\cup\{n+i|\tfrac{n}{2}+1\le i\le n\},\\
&= \{1,2,\cdots,\tfrac{n}{2}\}\cup\{n+\tfrac{n}{2}+1,n+\tfrac{n}{2}+2,\cdots,2n\}\\
S_2 &= \{f(v_2^i)|1\le i\le n\}=\{n+\tfrac{n}{2}+1-i|1\le i\le n\}\\
&= \{\tfrac{n}{2}+1,\tfrac{n}{2}+2,\cdots,n+\tfrac{n}{2}\}\\
S_3 &= \{f(v_3^i)|1\le i\le n\}=\{2n+1+i|1\le i\le \tfrac{n}{2}\}\cup\{3n+1+i|\tfrac{n}{2}+1\le i\le n\},\\
&= \{2n+2,2n+3,\cdots,2n\tfrac{n}{2}+1\}\cup\{3n+\tfrac{n}{2}+2,3n+\tfrac{n}{2}+3,\cdots,4n+1\}\\
S_4 &= \{f(v_4)\}=\{2n+1\}.
\end{aligned}$$

Hence, $S_i\cap S_j=\varnothing$ for $i\ne j, i,j\in\{0,1,2,3,4\}$, which yields that f is an injection from $V(n-\vec{C}_5)$ into $\{0,1,\cdots,4n+1\}$.

Secondly, we show the induced edges labeling f' is a bijective mapping from $E(n-\vec{C}_5)$ onto $\{1,2,\cdots,4n+1\}$.

Set $B_j=\{[f(v_{j+1}^i)-f(v_j^i)] \pmod{4n+2} | 1\le i\le n\},\ 0\le j\le 4$, and $B=B_0\cup B_1\cup B_2\cup B_3\cup B_4$. Then

$$
\begin{aligned}
B_0 &= \{[\,f(v_1^i)-f(v_0)] \pmod{(\,4n+2\,)} | 1\le i\le n\}\\
&= B_{01}\cup B_{02} \text{ where}\\
&\quad B_{01}=\{i|1\le i\le \tfrac{n}{2}\}=\{1,2,\cdots,\tfrac{n}{2}\},\\
&\quad B_{02}=\{n+i|\ fracn2+1\le i\le n\}=\{n+\tfrac{n}{2}+1,n+\tfrac{n}{2}+2,\cdots,2n\}\\
B_1 &= \{[\,f(v_2^i)-f(v_1^i)] \pmod{(\,4n+2\,)} | 1\le i\le n\}\\
&= B_{11}\cup B_{12} \text{ where}\\
&\quad B_{11}=\{n+\tfrac{n}{2}+1-2i|1\le i\le \tfrac{n}{2}\}=\{\,\tfrac{n}{2}+1,\tfrac{n}{2}+3,\cdots,\ n+\tfrac{n}{2}-1\}\\
&\quad B_{12}=\{\tfrac{n}{2}+1-2i|\tfrac{n}{2}+1\le i\le n\}=\{2n+\tfrac{n}{2}+3,2n+\tfrac{n}{2}+5,\cdots,3n+\tfrac{n}{2}+1\}\\
B_2 &= \{[\,f(v_3^i)-f(v_2^i)] \pmod{(\,4n+2\,)} | 1\le i\le n\}\\
&= B_{21}\cup B_{22} \text{ where}\\
&\quad B_{21}=\{\tfrac{n}{2}+2i|1\le i\le \tfrac{n}{2}\}=\{\,\tfrac{n}{2}+2,\tfrac{n}{2}+4,\cdots,n+\tfrac{n}{2}\}\\
&\quad B_{22}=\{n+\tfrac{n}{2}+2i|\tfrac{n}{2}+1\le i\le n\}=\{2n+\tfrac{n}{2}+2,2n+\tfrac{n}{2}+4,\cdots,3n+\tfrac{n}{2}\}\\
B_3 &= \{[\,f(x_4)-f(x_3^i)] \pmod{(\,4n+2\,)} | 1\le i\le n\}\\
&= B_{31}\cup B_{32} \text{ where}\\
&\quad B_{31}=\{4n+2-i|1\le i\le \tfrac{n}{2}\}=\{\,3n+\tfrac{n}{2}+2,3n+\tfrac{n}{2}+3,\cdots,4n-1\}\\
&\quad B_{32}=\{3n+2-i|\tfrac{n}{2}+1\le i\le n\}=\{2n+2,2n+3,\ \cdots,2n+\tfrac{n}{2}+1\,\}\\
B_4 &= \{[\,f(x_0)-f(x_4^i)] \pmod{(\,4n+2\,)} | 1\le i\le n\}=\{2n+1\}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
B &= B_0\cup B_1\cup B_2\cup B_3\cup B_4\\
&= B_{01}\cup B_{11}\cup B_{21}\cup B_{02}\cup B_4\cup B_{32}\cup B_{22}\cup B_{12}\cup B_{31}\\
&= \{1,2,\cdots,\tfrac{n}{2}\}\cup\{\,\tfrac{n}{2}+1,\tfrac{n}{2}+3,\cdots,\ n+\tfrac{n}{2}-1\}\\
&\quad \cup\{\,\tfrac{n}{2}+2,\tfrac{n}{2}+4,\cdots,n+\tfrac{n}{2}\}\cup\{n+\tfrac{n}{2}+1,n+\tfrac{n}{2}+1,\cdots,2n\}\\
&\quad \cup\{2n+1\}\cup\{2n+2,2n+3,\ \cdots,2n+\tfrac{n}{2}+1\}\\
&\quad \cup\{2n+\tfrac{n}{2}+2,2n+\tfrac{n}{2}+4\ \cdots,3n+\tfrac{n}{2}\}\\
&\quad \cup\{2n+\tfrac{n}{2}+3,2n+\tfrac{n}{2}+5,\cdots,3n+\tfrac{n}{2}+1\}\\
&\quad \cup\{\,3n+\tfrac{n}{2}+2,3n+\tfrac{n}{2}+3,\cdots,4n-1\}\\
&= \{\,1,\ 2,\cdots,4n+1\,\}.
\end{aligned}
$$

which implies that f' is surjective, hence, bijective. So we prove that $n-\vec{C}_5$ is a graceful digraph for even n. □

Theorem 2. *For every even integer n, the digraph $n-\vec{C}_7$ is graceful.*

Proof. Define

$$f(v_0)=0, f(v_6)=3n+1$$

and

$$f(v_j^i)=\begin{cases}\frac{j-1}{4}(5n+1)+i, & j=1,5,\ 1\le i\le n;\\ 4n+1+i, & j=3,\ 1\le i\le \frac{n}{2};\\ 2n+i, & j=3,\ \frac{n}{2}+1\le i\le n;\\ jn+2-i, & j=2,4,\ 1\le i\le \frac{n}{2};\\ \frac{8}{j}+2-i, & j=2,4,\ \frac{n}{2}+1\le i\le n.\end{cases}$$

Firstly, we show that f is an injective mapping from $V(n-\vec{C}_7)$ into $\{0,1,\cdots,6n+1\}$.

Set $S_j=\{f(v_j^i)|1\le i\le n\},\ 0\le j\le 6$ and $S=S_0\cup S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_6$. Then

$$\begin{aligned}
S_0 &= \{f(v_0)\}=\{0\}\\
S_1 &= \{f(v_1^i)|1\le i\le n\}=\{i|1\le i\le n\}=\{1,2,\cdots,n\}\\
S_2 &= \{f(v_2^i)|1\le i\le n\}\}\\
&= S_{21}\cup S_{22},\ \text{where}\\
&\quad S_{21}=\{f(v_2^i)|1\le i\le \tfrac{n}{2}\}=\{n+\tfrac{n}{2}+2,n+\tfrac{n}{2}+3,\cdots,2n+1\},\\
&\quad S_{22}=\{f(v_1^i)|\tfrac{n}{2}+1\le i\le n\}=\{3n+2,3n+3,\cdots,3n+\tfrac{n}{2}+1\}\\
S_3 &= \{f(v_3^i)|1\le i\le n\}\\
&= S_{31}\cup S_{32},\ \text{where}\\
&\quad S_{31}=\{f(v_3^i)|1\le i\le \tfrac{n}{2}\}=\{4n+2,4n+3,\cdots,4n+\tfrac{n}{2}+1\},\\
&\quad S_{32}=\{f(v_3^i)|\tfrac{n}{2}+1\le i\le n\}=\{2n+\tfrac{n}{2}+1,2n+\tfrac{n}{2}+2,\cdots,3n\},\\
S_4 &= \{f(v_4^i)|1\le i\le n\}\\
&= S_{41}\bigcup S_{42},\ \text{where}\\
&\quad S_{41}=\{f(v_4^i)|1\le i\le \tfrac{n}{2}\}=\{3n+\tfrac{n}{2}+2,3n+\tfrac{n}{2}+3,\cdots,4n+1\},\\
&\quad S_{42}=\{f(v_4^i)|\tfrac{n}{2}+1\le i\le n\}=\{n+1,n+2,\cdots,n+\tfrac{n}{2}\},\\
S_5 &= \{f(v_5^i)|1\le i\le n\}=\{5n+2,5n+3,\cdots,6n+1\}\\
S_6 &= \{f(v_6)\}=\{3n+1\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
S &= S_0\cup S_1\cup S_2\cup S_3\cup S_4\cup S_5\cup S_6\\
&= S_0\cup S_1\cup S_{42}\cup S_{21}\cup S_{32}\cup S_6\cup S_{22}\cup S_{41}\cup S_{31}\cup S_5\\
&= \{0\}\cup\{1,2,\cdots,n\}\cup\{n+1,n+2,\cdots,n+\tfrac{n}{2}\}\\
&\quad \cup\{n+\tfrac{n}{2}+2,n+\tfrac{n}{2}+3,\cdots,2n+1\}\cup\{2n+\tfrac{n}{2}+1,2n+\tfrac{n}{2}+2,\cdots,3n\}\\
&\quad \cup\{3n+1\}\cup\{3n+2,3n+3,\cdots,3n+\tfrac{n}{2}+1\}\\
&\quad \cup\{4n+2,4n+3,\cdots,4n+\tfrac{n}{2}+1\}\cup\{5n+2,5n+3,\cdots,6n+1\}\\
&\quad \subseteq\{1,2,\cdots,6n+1\}.
\end{aligned}$$

It is clear that the labels of each vertices are different. So, f is a injection from $V(n-\vec{C}_7)$ into $\{0,1,\cdots,6n+1\}$.

Secondly, we show the induced edges labeling f' is a bijective mapping from $E(n-\vec{C}_7)$ onto $\{1,2,\cdots,6n+1\}$.

Set $B_j = \{[f(v^i_{j+1}) - f(v^i_j)] \pmod{6n+2} | 1 \leq i \leq n\}$ $(0 \leq j \leq 6)$ and let $B = B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$. Then

$$\begin{aligned}
B_0 &= \{[f(v^i_1) - f(v_0)] \pmod{6n+2} | 1 \leq i \leq n\} = \{1, 2, \cdots, n\}\\
B_1 &= \{[f(v^i_2) - f(v^i_1)] \pmod{6n+2} | 1 \leq i \leq n\}\\
&= B_{11} \cup B_{12},\ \text{where}\\
&\quad B_{11} = \{[2n+2-2i] \pmod{6n+2} | 1 \leq i \leq \tfrac{n}{2}\} = \{n+2, n+4 \cdots, 2n\},\\
&\quad B_{12} = \{[4n+2-2i] \pmod{6n+2} | \tfrac{n}{2}+1 \leq i \leq n\} = \{2n+2, 2n+4, \cdots, 3n\},\\
B_2 &= \{[f(v^i_3) - f(v^i_2)] \pmod{6n+2} | 1 \leq i \leq n\}\\
&= B_{21} \bigcup B_{22},\ \text{where}\\
&\quad B_{21} = \{[2n-1+2i] \pmod{6n+2} | 1 \leq i \leq \tfrac{n}{2}\} = \{2n+1, 2n+3, \cdots, 3n-1\},\\
&\quad B_{22} = \{[4n+2i] \pmod{6n+2} | \tfrac{n}{2}+1 \leq i \leq n\} = \{5n+2, 5n+4, \cdots, 6n\},\\
B_3 &= \{[f(v^i_4) - f(v^i_3)] \pmod{6n+2} | 1 \leq i \leq n\}\\
&= B_{31} \bigcup B_{32},\ \text{where}\\
&\quad B_{31} = \{[6n+3-2i)] \pmod{6n+2} | 1 \leq i \leq \tfrac{n}{2}\} = \{5n+3, 5n+5, \cdots, 6n+1\},\\
&\quad B_{32} = \{[6n+3-2i] \pmod{6n+2} | \tfrac{n}{2}+1 \leq i \leq n\} = \{4n+3, 4n+5, , \cdots, 5n+1\},\\
B_4 &= \{[f(v^i_5) - f(v^i_4)] \pmod{6n+2} | 1 \leq i \leq n\}\\
&= B_{41} \bigcup B_{42},\ \text{where}\\
&\quad B_{41} = \{[n-1+2i] \pmod{6n+2} | 1 \leq i \leq \tfrac{n}{2}\} = \{n+1, n+3, \cdots, 2n-1\},\\
&\quad B_{42} = \{[3n+2i] \pmod{6n+2} | \tfrac{n}{2}+1 \leq i \leq n\} = \{4n+2, 4n+4, \cdots, 5n\},\\
B_5 &= \{[f(v^i_6) - f(v^i_5)] \pmod{6n+2} | 1 \leq i \leq n\} = \{4n+2-i | 1 \leq i \leq n\}\\
&= \{3n+2, 3n+3, \cdots, 4n+1\}\\
B_6 &= \{[f(v^i_0) - f(v^i_6)] \pmod{6n+2} | 1 \leq i \leq n\} = \{3n+1\}.
\end{aligned}$$

Hence, $B = B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6$ is the set of labels of all edges, and

$$\begin{aligned}
B &= B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6\\
&= B_0 \cup B_{41} \cup B_{11} \cup B_{21} \cup B_{12} \cup B_6 \cup B_5 \cup B_{42} \cup B_{32} \cup B_{22} \cup B_{31}\\
&= \{1, 2, \cdots, n\} \cup \{n+1, n+3, \cdots, 2n-1\} \cup \{n+2, n+4, \cdots, 2n\}\\
&\quad \cup \{2n+1, 2n+3, \cdots, 3n-1\} \cup \{2n+2, 2n+4, \cdots, 3n\} \cup \{3n+1\}\\
&\quad \cup \{3n+2, 3n+5, \cdots, 4n+1\} \cup \{4n+2, 3n+4, \cdots, 5n\}\\
&\quad \cup \{4n+3, 4n+5, \cdots, 5n+1\} \cup \{5n+2, 5n+4, \cdots, 6n\}\\
&\quad \cup \{5n+3, 5n+5, \cdots, 6n+1\}\\
&= \{1, 2, \ldots, 6n+1\}.
\end{aligned}$$

It shows that f' is a bijection from $E(n - \vec{C}_7)$ onto $\{1, 2, \cdots, |E(n - \vec{C}_7)|\}$. So we conclude that $n - \vec{C}_7$ is graceful for even n. □

Theorem 3. *For every even integer n, the digraph $n - \vec{C}_9$ is graceful.*

Proof. Define

$$f(v_0) = 0,\ f(v_8) = 4n+1$$

and

$$f(v_j^i) = \frac{j-1}{2}n + i,\ j = 1,5 \ \text{ and } \ 1 \leq i \leq \ n,$$

$$f(v_2^i) = \begin{cases} 2n+1-i, & 1 \leq i \leq \frac{n}{2}, \\ 7n+3-i, & \frac{n}{2}+1 \leq i \leq n, \end{cases}$$

$$f(v_j^i) = \frac{j+7}{2} + 1 + i,\ j = 3,7 \ \text{ and } \ 1 \leq i \leq \ n,$$

$$f(v_j^i) = \frac{j}{2}(n+1) + 2n - 1 - i,\ j = 4,6 \ \text{ and } \ 1 \leq i \leq \ n,$$

Similar to the proof of Theorem 1 and Theorem 2, it can be shown that this assignment provides a graceful labeling of $n - \vec{C}_9$ for even n. Hence $n - \vec{C}_9$ is graceful for even n. □

Theorem 4. *For every even integer n, the digraph $n - \vec{C}_{11}$ is graceful.*

Proof. Define

$$f(v_0) = 0,\ f(v_{10}) = 5n + 1$$

and

$$f(v_j^i) = \frac{3(j-1)}{8}n + i, j = 1,9 \text{ and } 1 \leq i \leq n,$$

$$f(v_j^i) = 2(j-1)n + i, j = 3,5 \text{ and } 1 \leq i \leq n,$$

$$f(v_j^i) = (\frac{j+4}{6} + 8)n + 2 - i,\ j = 2,8 \text{ and } 1 \leq i \leq n,$$

$$f(v_j^i) = \frac{36-5j}{2}n + 1 - i,\ j = 4,6 \text{ and } 1 \leq i \leq n,$$

$$f(v_7^i) = \begin{cases} 5n+1+i, & 1 \leq i \leq \frac{n}{2} \\ n-1+i, & \frac{n}{2}+1 \leq i \leq n \ . \end{cases}$$

Similar to the proof of Theorem 1 and Theorem 2, it can be shown that this assignment provides a graceful labeling of $n - \vec{C}_{11}$ for even n. Hence $n - \vec{C}_{11}$ is graceful for even n. □

Theorem 5. *For every even integer n, the digraph $n - \vec{C}_{13}$ is graceful.*

Proof. Define

$$f(v_0) = 0,\ f(v_{12}) = 6n + 1$$

and

$$f(v_j^i) = \frac{j-1}{4}n + i,\ j = 1,5 \ \text{ and } \ 1 \leq i \leq \ n\ ,$$

$$f(v_j^i) = \frac{7j-2}{4} + \frac{j}{2} - i,\ j = 2,6 \ \text{ and } \ 1 \leq i \leq \ n,$$

$$f(v_j^i) = \frac{7+3j}{4}n + 1 + i,\ j = 3,7 \ \text{ and } \ 1 \leq i \leq \ n,$$

$$f(v_9^i) = \begin{cases} 10n + 2 + i, & 1 \le i \le \frac{n}{2}, \\ 5n + i, & \frac{n}{2} + 1 \le i \le n, \end{cases}$$

$$f(v_{10}^i) = 4n + 1 - i, \quad 1 \le i \le n,$$

$$f(v_{11}^i) = 11n + 1 + i, \quad 1 \le i \le n,$$

Similar to the proof of Theorem 1 and Theorem 2, it can be shown that this assignment provides a graceful labeling of $n - \vec{C}_{13}$ for even n. Hence $n - \vec{C}_{13}$ is graceful for even n. □

In Figure 1, we give graceful labelings of $8 - \vec{C}_5$,$8 - \vec{C}_7$,$8 - \vec{C}_9$,$8 - \vec{C}_{11}$ and $8 - \vec{C}_{13}$.

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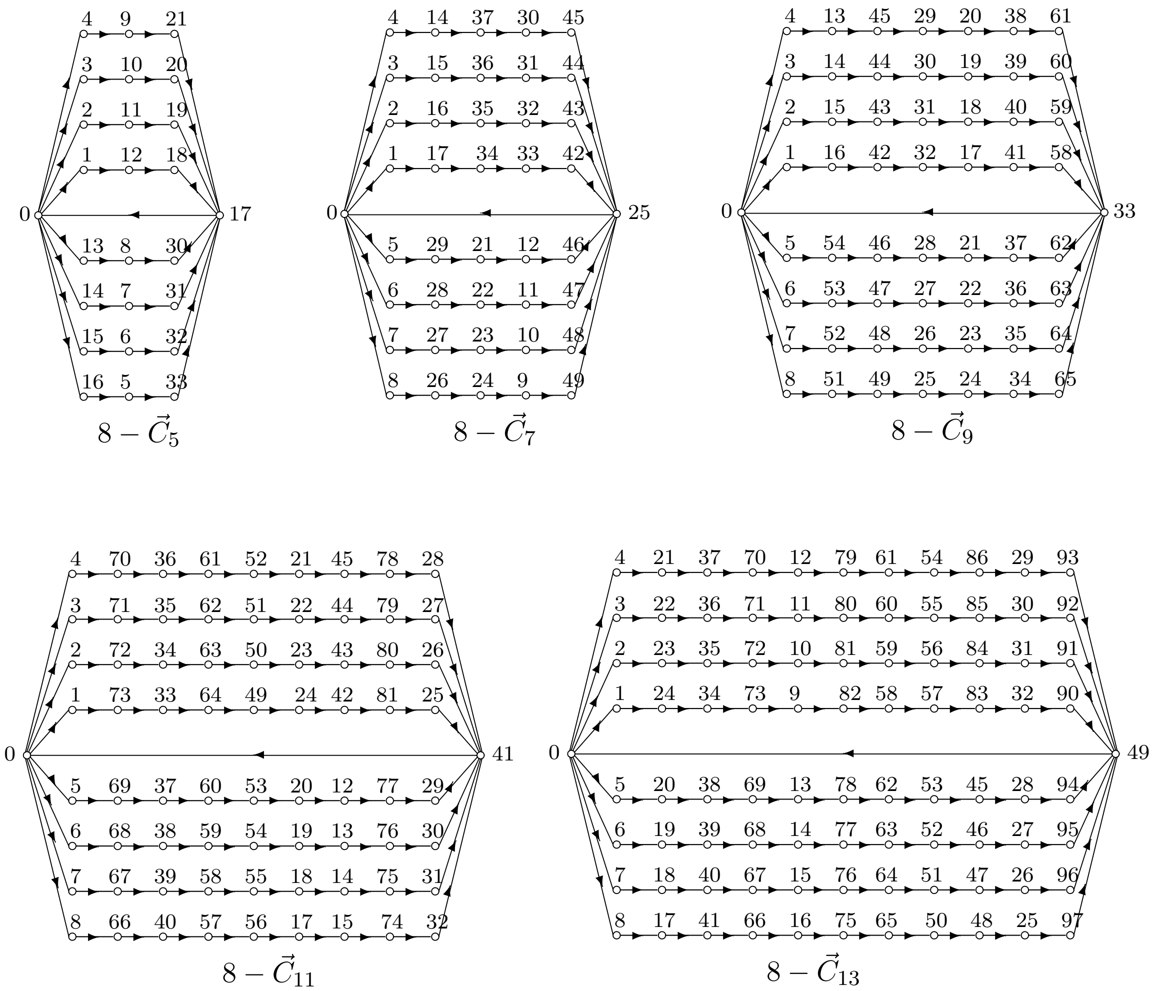

Figure 1: Graceful labelings of $8-\vec{C}_5$,$8-\vec{C}_7$,$8-\vec{C}_9$,$8-\vec{C}_{11}$ and $8-\vec{C}_{13}$.