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Gp-FINITENESS OF TENSOR PRODUCT

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Abstract. In this paper we introduce G_P finiteness of a Von-Neumann algebra and we define a G-dimension function. Then we prove a result on tensor product of fixed point algebra under group of automorphisms and finally verify a result under which the tensor product is G_P finite.

Key words : Von Neumann algebra M, the fixed point algebra under groups of automorphisms M^G, G- central support, ultra weak continuous.
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1. Introduction

A Von Neumann algebra is finite if and only if there exist a centre-valued trace. Replacing the role played by the centre by the fixed point algebra of the Von Neumann algebra under group of automorphisms, G-dimension function is introduced. The study of the classification problem of tensor product of Von Neumann algebras is a part and parcel of the study of the classification problems of Von Neumann algebra.

2. Preliminaries

Definition 1. *Let M be a Von neumann algebra and G be the group of automorphisms on M. Define the fixed point algebra M^G as the set of all a in M such that $t(a) = a$, for t in G. Denote P as set of all projections in M.*

Theorem 1. [5] *Let M_1 and M_2 be two Von Neumann algebras and $M_1 \otimes M_2$ be their tensor product. Then $M_1 \otimes M_2$ is finite if and only if M_1 and M_2 are finite.*

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Definition 2. [2] *Let M_1 and M_2 be two Von Neumann algebras on the Hilbert space H_1 and H_2 respectively, $M_1 \otimes M_2$ be their tensor product. G_1 and G_2 be groups of automorphism of M_1 and M_2 respectively. For $a \otimes b$ in $M_1 \otimes M_2$ and t_1 in G_1 and t_2 in G_2 define a map,*

$$t : M_1 \otimes M_2 \longrightarrow M_1 \otimes M_2$$

as

$$t(a \otimes b) = t_1(a) \otimes t_2(b).$$

Let $G_1 \otimes G_2$ denote the set of all such t obtained from all possible pairs t_1 in G_1 and t_2 in G_2.

For t and t' in $G_1 \otimes G_2$ define

$$(t \cdot t')(a \otimes b) = t(t'(a \otimes b)).$$

Then $G_1 \otimes G_2$ is a group of automorphism on $M_1 \otimes M_2$ under the above multiplication.

3. Main Results

Definition 3. *Let M be a Von Neumann algebra, G, a group of automorphisms on M. We define a relation between projections in M as follows. Projections e and f in M are said to be equivalent relative to G or G equivalent denoted by $e \sim_G f$ if and only if there exists $t \in G$ such that $e = t(f)$.*

Remark 1. *$\sim_G$ is an equivalence relation in P.*

Definition 4. *M, G, M^G and P as above. Then M is said to be G_P-finite if $e \sim_G f \leq e$ implies $e = f$ for every pair of projections e and f in P.*

Definition 5. *A G-dimension function on M is a mapping*

$$d : P \to M^G$$

which satisfies

(1) $d(e) > 0$ *if* $e \neq 0$,
(2) $d(e + f) = d(e) + d(f)$,
(3) $d(e) = d(f)$ *if* $e \sim_G f$
(4) $d(q) = q$ *if q is a projection in M^G for every $e, f \in P$.*

Remark 2. *If a G-dimension function $d : P \to M^G$ exists, then M is G_P-finite.*

Remark 3. *If G is a group of inner automorphisms on M ,then M^G reduecd to the centre, Z of M then d can be cosidered as a function from M to Z. Let H be a Hilbert space with dimension n. Let $M = B(H)$, e a projection in M and dim E denote dimension of the subspace $e(H)$ of H then $d(E) = \frac{\dim E}{n} I$, where I is the identity operator in H.*

Definition 6. *G- Central carrier or G-Central support of a projection e in M is defined as the smallest projection in M^G majorising e and is denoted by C_e^G.*

Remark 4. *If $e \sim_G f$ then $C_e^G = C_f^G$.*

Proposition 2. *Let M_1, M_2 be Von Neumann algebras, G_1, and G_2 groups of automorphisms on M_1 and M_2 respectively. Let $M = M_1 \otimes M_2$, $G = G_1 \otimes G_2$, then $M^G = M_1^{G_1} \otimes M_2^{G_2}$.*

Proof. Let $a \otimes b$ belongs to M^G. Then $(t_1 \otimes t_2)(a \otimes b) = a \otimes b$ for every $t_1 \otimes t_2 \in G$. Hence

$$t_1(a) \otimes t_2(b) = a \otimes b \ \forall t_1 \otimes t_2 \in G.$$

i.e.,

$$t_1(a) \otimes t_2(b) = a \otimes b \ \forall t_1 \in G_1 \text{ and } t_2 \in G_2.$$

Taking $t_2 = e_2$, the identity of G_2, $t_1(a) \otimes e_2(b) = a \otimes b)$
implies

$$t_1\,(a \otimes b) = a \otimes b.$$

implies

$$(t_1(a) - a) \otimes b = 0.$$

b is nonzero operator in M_2 implies $t_1(a) - a = 0$. Therefore

$$t_1(a) = a \ \forall t_1 \in G_1.$$

Hence a is in $M_1^{G_1}$. Similarly we can prove that b is in $M_2^{G_2}$. Therefore $a \otimes b$ is in $M_1^{G_1} \otimes M_2^{G_2}$. M^G is subset of $M_1^{G_1} \otimes M_2^{G_2}$.
Conversely, let $a \otimes b$ in $M_1^{G_1} \otimes M_2^{G_2}$ such that a in $M_1^{G_1}$ and b in $M_2^{G_2}$. Then $t_1(a) = a \ \forall t_1 \in G_1$ and $t_2(b) = b \ \forall t_2 \in G_2$. Hence $a \otimes b = t_1(a) \otimes t_2(b) \ \forall t_1 \in G_1$ and $\forall t_2 \in G_2$. Hence $a \otimes b = (t_1 \otimes t_2)(a \otimes b)$ for every $t_1 \otimes t_2 \in G_1 \otimes G_2$. This shows that on the generators of $M_1^{G_1} \otimes M_2^{G_2}$ we have

$$(t_1 \otimes t_2)(a \otimes b) = a \otimes b.$$

But if $a \otimes b$ in $M_1^{G_1} \otimes M_2^{G_2}$ is arbitrary, then $a \otimes b$ is the weak limit of a net $a^m \otimes b^m$ where a^m in $M_1^{G_1}$ and b^m in $M_2^{G_2}$. But for $t_1 \in G_1$ and $t_2 \in G_2$ the automorphism $t_1 \otimes t_2$ defined on $M_1 \otimes M_2$ is ultraweakly continuous and hence weakly continuous. Therefore $(t_1 \otimes t_2)(a^m \otimes b^m)$ converges weakly to $(t_1 \otimes t_2)(a \otimes b)$. Consider $| < (t_1 \otimes t_2)(a \otimes b) - (a \otimes b)x, y > | = | < (t_1 \otimes t_2)(a \otimes b) - (t_1 t_2)(a^m \otimes b^m) + (t_1 \otimes t_2)(a^m b^m) - (a^m - b^m) + (a^m \otimes b^m) - (a \otimes b)x, y > |$. This converges to zero. Hence $(t_1 \otimes t_2)(a \otimes b)$ is equal to $a \otimes b$ in weak topology. Therefore $a \otimes b$ lies in M^G and also M^G is the weak closure of $M_1^{G_1} \otimes M_2^{G_2}$. Hence the result. ∎

Proposition 3. *Let M_1 and M_2 be two Von Neumann algebras. G_1 and G_2 be groups of automorphisms on M_1 and M_2 respectively. Let G_1- central support of e_1 be $C_{e_1}^{G_1}$ and G_2-central support of e_2 be $C_{e_2}^{G_2}$ where e_1 and e_2 are projection in M_1 and M_2 respectively. Let $e = e_1 \otimes e_2$ and $G = G_1 \otimes G_2$. Then G-central support of e is $C_e^G = C_{e_1}^{G_1} \otimes C_{e_2}^{G_2}$.*

Proof. C_e^G is G-central support and hence it is the smallest projection in M^G majorizing e. Therefore

$$C_e^G(e_1 \otimes e_2) = (e_1 \otimes e_2).$$

Now

$$\begin{aligned} C_{e_1}^{G_1} \otimes C_{e_2}^{G_2}(e_1 \otimes e_2) &= C_{e_1}^{G_1} e_1 \otimes C_{e_2}^{G_2} e_2 \\ &= e_1 \otimes e_2, \end{aligned}$$

by the definition of $C_{e_1}^{G_1}$ and $C_{e_2}^{G_2}$. Hence by the definition of C_e^G,

$$C_{e_1}^{G_1} \otimes C_{e_2}^{G_2} \geq C_e^G. \tag{1}$$

Now let $f_1 \otimes f_2$ is any projection in M^G with let f_1 in M_1^G and f_2 in M_2^G such that

$$(f_1 \otimes f_2)(e_1 \otimes e_2) = (e_1 \otimes e_2). \tag{2}$$

Therefore

$$f_1 e_1 \otimes f_2 e_2 = e_1 \otimes e_2$$

implies

$$f_1 e_1 = m e_1 \text{ and } f_2 e_2 = m^{-1} e_2,$$

where m is a scalar. Therefore $(m^{-1} f_1)e_1 = e_1$ and $(m f_2)e_2 = e_2 m^{-1} f_1$ belongs to M_1^G implies $C_{e_1}^{G_1} \leq m^{-1} f_1$;$m f_2$ belongs to M_2^G implies $C_{e_2}^{G_2} \geq m f_2$. Therefore

$$\begin{aligned} C_{e_1}^{G_1} \otimes C_{e_2}^{G_2} &\leq m^{-1} f_1 m f_2 \\ &= (m^{-1} m)(f_1 f_2). \end{aligned}$$

Therefore

$$C_{e_1}^{G_1} \otimes C_{e_2}^{G_2} \leq f_1 \otimes f_2$$

for any projection $f_1 \otimes f_2$ in M^G. C_e^G is the smallest projection in M^G majorizing e. Therefore in particular

$$C_{e_1}^{G_1} \otimes C_{e_2}^{G_2} \leq C_e^G \tag{3}$$

From (1) and (3), result follows. □

Theorem 4. *Let M_1 and M_2 be two Von Neumann algebras G_1 and G_2 be groups of automorphism on M_1 and M_2 respectively. Let $G = G_1 \otimes G_2$ and P_1 set of all projections in M_1. Suppose $P_1 \otimes P_2$ is ordered by the relation , $e_1 \otimes e_2 \leq f_1 \otimes f_2$ if and only if $e_1 \leq f_1$ and $e_2 \leq f_2$. If M_1 is G^1_P- finite and M_2 is G^P_2- finite then $M_1 \otimes M_2$ is G_P-finite.*

Proof. Let $e_1 \otimes e_2 \sim_G f_1 \otimes f_2 \leq e_1 \otimes e_2$ in $M_1 \otimes M_2$. $e_1 \otimes e_2 \sim_G f_1 \otimes f_2$ implies there exists $t \in G$ such that $e_1 \otimes e_2 = t(f_1 \otimes f_2)$. $t \in G$ implies $t = t_1 \otimes t_2$, where $t_1 \in G_1$ and $t_2 \in G_2$. Therefore

$$e_1 \otimes e_2 = t_1(f_1) \otimes t_2(f_2)$$

implies

$$e_1 = mt_1(f_1) and e_2 = m^{-1}t_2(f_2),$$

where m is a scalar. Therefore $e_1 \sim_G f_1$and $e_2 \sim_G f_2$. Also

$$f_1 \otimes f_2 \leq e_1 \otimes e_2$$

implies

$$f_1 \leq e_1 \text{ and } f_2 \leq e_2$$

Hence

$$e_1 \sim_G f_1 \leq e_1.$$

Since M_1 is G^1_P- finite, $e_1 = f_1$. Also M_2 is G^2_P- finite and $e_2 \sim_G f_2 \leq e_2$implies $e_2 = f_2$. Therefore $e_1 \otimes e_2 = f_1 \otimes f_2$ and hence $M_1 \otimes M_2$ is G^P-finite. □

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