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CONTRACTIBLE FIBERS OF POLYNOMIAL FUNCTIONS

ZAHID RAZA*

Abstract. In this short note, we investigate the topology of complex polynomials $f(x, y)$ in two variables. The description of the topology of the corresponding level curves $\mathcal{C}_t : f(x, y) = t$ is directly related to the vanishing of the leading coefficients $c_j(t)$ of the discriminant of the polynomial $f(x, y) - t$, regarded as polynomials in t.

In particular, we look for condition such that $\mathcal{C}_t$ is smooth and contractible, since this implies that the corresponding polynomial is a coordinate on $\mathbb{C}^2$. A detailed description including the homotopy types of the curves $\mathcal{C}_t$ and the associated geometry is so far known only up to the degree 2, and we will try to extend it to the degree 3.

Key words : plane curves, polar curves, discriminant.
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1. Introduction

Let $f : \mathbb{C}^n \longmapsto \mathbb{C}$ be a polynomial map, $n \geq 2$. It is known that a complex polynomial function might not be a locally trivial topological fibration over the complement in $\mathbb{C}$ of its critical values. The following general result by R. Thom, A. Varchenko and J.-L. Verdier is well-known, see [13], [15], [16], [14].

Theorem 1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a complex polynomial map, with $n \geq 2$. Then there is a minimal finite set $B(f)$ in $\mathbb{C}$ such that*

$$f : \mathbb{C}^n \setminus f^{-1}(B(f)) \to \mathbb{C} \setminus B(f)$$

is a locally trivial fibration.

This bifurcation set $B(f)$ contains two type of points : the critical values $C(f)$ of f and the critical values at infinity $I(f)$ of f.

*Abdus Salam School of Mathematical Sciences GC University, 68-B New Muslim Town, Lahore, Pakistan. Email: zraza@sms.edu.pk.

In this note, we concentrate on the simpler case $n = 2$. First we recall the usual definition of $I(f)$ in terms of jumps of the Milnor number of the singularities of f at infinity and a result by Krasiǹski.

In fact, we make this result more precise in Proposition 5. It is surprising that this explicit and constructive description of the set of critical values at infinity $I(f)$, which apparently goes back to Krasiǹski and Hà is present neither in Durfee's excellent survey [7], nor in the recent monograph by Tibăr [14].

In Theorem 6, we relate the degree of $\Delta(x)$ to the topology of the affine curve $\mathcal{C}' : f = 0$ supposed to be smooth and irreducible. We also investigate the relation between the possible values of $\deg \Delta(x)$ and the geometry of the curve $\mathcal{C}'$ for $d = 3$.

2. Teissier's result and Krasiǹski's formula

First we recall what is a critical value at infinity in the case $n = 2$.

Definition 1. *Let $f(x, y) \in \mathbb{C}[x, y]$ be a reduced polynomial and let $\mathcal{C}$ be the projective closure of the affine curve $\mathcal{C}' : f(x, y) = 0$. Let $\mathcal{C}_\infty = \mathcal{C} \cap \mathcal{L}_\infty$ be the set of points at infinity of $\mathcal{C}$. We consider the projective curves $\mathcal{C}_t$ (possibly with multiple factors) given by the equations*

$$\mathcal{C}_t : F(x, y, z) - tz^d = 0$$

where $d = \deg f$ and $F(x, y, z) = z^d f(x/z, y/z)$. Clearly $\mathcal{C}_\infty = \mathcal{C}_t \cap \mathcal{L}_\infty$ for any $t \in \mathbb{C}$. Let $\mu_p^t = \mu_p(\mathcal{C}_t)$ be the Milnor number of $\mathcal{C}_t$ at $p \in \mathcal{C}_\infty$. If a multiple component of $\mathcal{C}_t$ passes through p, then we set $\mu_p^t = +\infty$. Let

$$\mu_p^{min} = \min\{\mu_p^t : t \in \mathbb{C}\}. \tag{1}$$

With this notation one has

$$I(f) = \{t \in \mathbb{C} : \exists\ p \in \mathcal{C}_\infty \ s.t.\ \mu_p^t > \mu_p^{min}\}\,, \tag{2}$$

see, for instance, [8] *or* [5]*, pp. 20-22. The elements of $I(f)$ are called critical values at infinity of the polynomial function $f : \mathbb{C}^2 \longmapsto \mathbb{C}$. Different equivalent definitions of critical values at infinity are discussed in* [7]*. For every $t \in \mathbb{C}$ we put*

$$\lambda^t(f) = \sum_{p \in \mathcal{C}_\infty} (\mu_p^t - \mu_p^{min}).$$

Lemma 2. *Let $\mathcal{C}'$ be a curve defined by the reduced polynomial $f(x, y) = 0$ such that the line $x = a$ intersects the curve $\mathcal{C}'$ in a finite number of points. Then for any point $p = (a, b) \in \mathcal{C}'$ one has*

$$(f, \frac{\partial f}{\partial y})_p = \mu_p(f) + (f, x - a)_p - 1\,. \tag{3}$$

This is a two dimensional case of a formula due to Teissier, see [12], Chap.2, Proposition 1.2 or (see in [11], Chap.2, Theorem 5).

Definition 2. *Let $\mathcal{C} \subset \mathbb{P}^2$ be a reduced projective curve given by $F = 0$ and $q = [q_0 : q_1 : q_2] \in \mathbb{P}^2$ such that $q_0 \frac{\partial F}{\partial x} + q_1 \frac{\partial F}{\partial Y} + q_2 \frac{\partial F}{\partial z} \neq 0$ in $\mathbb{C}[x, y, z]$. The polar curve $\Gamma_q(\mathcal{C})$ of $\mathcal{C}$ with respect to q is the projective plane curve defined by the equation*

$$q_0 \frac{\partial F}{\partial x} + q_1 \frac{\partial F}{\partial Y} + q_2 \frac{\partial F}{\partial z} = 0.$$

Lemma 3. *For any point $p \neq q$ on the projective curve $\mathcal{C}$, one has*

$$(\mathcal{C}, \Gamma_q(\mathcal{C}))_p = \mu_p(\mathcal{C}) + (\mathcal{C}, \bar{pq})_p - 1$$

where $\bar{pq}$ is the line passing through the points p and q.

Proof. We choose coordinates on $\mathbb{P}^2$ so that $p = [0 : 0 : 1]$, $q = [0 : 1 : 0]$. Then the above formula reduces to Teissier's local formula at the origin

$$(f, \tfrac{\partial f}{\partial y})_0 = \mu_0(f) + (f, X)_0 - 1.$$

□

Let $f(x, y) \in \mathbb{C}[x, y]$ be a reduced polynomial such that $deg_y f = deg f = d > 1$. Let $\Delta(x, T) = disc_y(f(x, y) - T) = \mathcal{R}_y(f(x, y) - T, \frac{\partial f}{\partial y})$ be the y discriminant regard as a polynomial in $\mathbb{C}[x, T]$. Let us write

$$\Delta(x, T) = \Delta_0(T)x^N + \cdots + \Delta_N(T), \text{where } \Delta_0(T) \neq 0 \in \mathbb{C}[T].$$

Let $\Delta(x) = disc_y f(x, y)$ be the y-discriminant of the polynomial f.

Proposition 4. *Let f be a polynomial of two variables x, y which is is x-regular, i.e. the equality $\deg_y f = \deg f$ holds. Then, for any value of $t \in \mathbb{C}$, we have*

$$\textstyle\sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}_t) = c - deg_x(\Delta(x, t))$$

where c is a constant independent of t.

This formula is due to Krasiǹski, for details see [10].

3. Main Results

In this section, we shall establish our main results. First we improve the result in Proposition 4 by calculating the value of the constant c in the following Proposition.

Proposition 5. *Let $f(x, y) \in \mathbb{C}[x, y]$ be a reduced polynomial such that $deg_y f = deg f = d > 1$ and let $\mathcal{C}$ be the projective closure of the affine curve $f(x, y) = 0$. Let $\mathcal{C}_\infty$ be the set of points at infinity of $\mathcal{C}$ and let $k = \#\mathcal{C}_\infty$. Then*

$$deg \Delta(x) = d(d - 2) + k - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}). \tag{4}$$

Proof. If $F(x, y, z)$ be the homogenous polynomial corresponding to $f(x, y)$, so $\mathcal{C}$ is defined by $F(x, y, z) = 0$. Then $\frac{\partial F}{\partial y}$ corresponds to $\frac{\partial f}{\partial y}$ because we have assumed that $deg_y f = deg f$. Let $q = (0 : 1 : 0)$, note that $q \notin \mathcal{C}$ and consider the corresponding polar curve $\Gamma_q(\mathcal{C}) : \frac{\partial F}{\partial y} = 0$. Then by well known properties of the discriminant, we have

$$\deg \Delta(x) = \deg \mathcal{R}_y(f, \frac{\partial f}{\partial y}) = \sum_{p \in \mathbb{C}^2} (f, \frac{\partial f}{\partial y})_p$$

$$= \sum_{p \in \mathcal{C} \setminus \mathcal{L}_\infty} (\mathcal{C}, \Gamma_q(\mathcal{C}))_p = \sum_{p \in \mathcal{C}} (\mathcal{C}, \Gamma_q(\mathcal{C}))_p - \sum_{p \in \mathcal{L}_\infty} (\mathcal{C}, \Gamma_q(\mathcal{C}))_p.$$

By Bézout s' Theorem and Lemma 3 for $\mathcal{C}, \Gamma_q(\mathcal{C})$, we get

$$\deg \Delta(x) = d(d-1) - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}) - \sum_{p \in \mathcal{C}_\infty} ((\mathcal{C}, \bar{pq})_p - 1)$$

$$= d(d-1) - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}) - d + \sum_{p \in \mathcal{C}_\infty} 1 = d(d-2) + k - \sum_{p \in \mathcal{C}_\infty} \mu_p(\mathcal{C}).$$

□

Remark 1. *In the above Propositions we need the condition $deg f_y = deg f = d > 1$. Indeed, we can not apply the affine from of the Bézout 's Theorem to get the equality*

$$\deg \mathcal{R}_y(f, \frac{\partial f}{\partial y}) = \sum_{p \in \mathbb{C}^2} (f, \frac{\partial f}{\partial y})_p$$

in the proof of Proposition 5, because the point $[0 : 1 : 0]$ *at infinity lies on both curves defined by the equations F and $\frac{\partial F}{\partial y}$. As an explicit example, consider $f(x, y) = x^2 y - x$. Then $disc_y(x, t) = 1$, so we can derive no information about the critical values at infinity. But we know that f has a critical value at infinity, namely $t = 0$.*

Theorem 6. *Let $f(x, y) = t \in \mathbb{C}[x, y]$ be a reduced (without multiple factors) polynomial such that $deg_y f = deg f = d > 1$. Assume that the fiber $\mathcal{C}'_t = f^{-1}(t)$ is smooth and irreducible, then*

$$deg_x \Delta(x, t) = d - 1 + b_1(\mathcal{C}'_t).$$

Proof. Let $f(x, y) = y^d + a_1(x) y^{d-1} + \cdots + a_d(x) - t$ be a reduced (without multiple factors) polynomial such that $deg_y f = deg f = d > 1$ and let $\mathcal{C}$ be the projective closure of the affine curve $f(x, y) = 0$. Let k be the cardinality of the set of points at infinity of $\mathcal{C}$. Let $\Delta(x) = disc_y f(x, y)$ be the y-discriminant of the polynomial f, then

$$\chi(\mathcal{C}'_t) = \chi(\mathcal{C}_t) - k.$$

Using the Corollary (5.4.4) in [5], p. 162, we get

$$\chi(\mathcal{C}'_t) = \chi(\mathcal{C}_{d,smooth}) + \sum_{i=1}^{k} \mu_{p_i}(\mathcal{C}, p) - k$$

By Proposition 5, we get

$$\chi(\mathcal{C}'_t) = \chi(\mathcal{C}_{d,smooth}) + d(d-2) + k - deg_x\Delta(x,t) - k$$

but

$$\chi(\mathcal{C}_{d,smooth}) = 2 - 2g = 3d - d^2$$

so

$$\chi(\mathcal{C}'_t) = d - deg_x\Delta(x,t).$$

On the other hand $\mathcal{C}'_t$ is a affine variety of dimension one, so has the homotopy type of a CW-complex of dimension one i.e. a bouquet of $b_1(\mathcal{C}'_t)$ circles S^1. It follows the result. □

Corollary 7. *If the fiber $\mathcal{C}'_t$ is smooth and irreducible. Then $\mathcal{C}'_t$ is contractible iff*

$$deg_x\Delta(x,t) = d - 1.$$

Example 1. *We give detail explanation when the fibers are contractible for $d = 3$ i.e. when the conditions of the above corollary hold. The general form of a polynomial of degree three is $f(x,y) = y^3 + (ax^2 + c)y + Ax^3 + Bx^2 + Cx - t$. Then the y-discriminant of $f(x,y)$ is:*
$\Delta(x,t) = (-4a^3 - 27A^2)x^6 - 54ABx^5 + (-54AC - 12a^2c - 27B^2)x^4 + (54At - 54BC)x^3 + (54Bt - 27C^2 - 12ac^2)x^2 + 54Ctx - 4c^3 - 27t^2.$
Now the fiber $\mathcal{C}'_t$ is contractible $\Longleftrightarrow deg_x\Delta(x,t) = 3-1 = 2$. So the coefficients of x^6, x^5, x^4, x^3 must be zero i.e.
$x^6 : 4a^3 + 27A^2 = 0$
$x^5 : AB = 0$
$x^4 : 18Ac + 4a^2c + 9B^2 = 0$
$x^3 : At - BC = 0$
$x^2 : 18Bt - 9C^2 - 4ac^2 \neq 0$
Case(i) *If $A = 0$. Then $a = 0$ and $B = 0$, the coefficient of x^2 is non-zero iff $C \neq 0$. So the polynomial becomes $f(x,y) = y^3 + cy + Cx - t$, which is equivalent to a linear form i.e. f is a global coordinate change on $\mathbb{C}^2$. Hence $\mathcal{C}'_t$ is contractible.*
Case(ii) *If $A \neq 0$.*
Then $a \neq 0,\ B = 0\ \ t = 0$, and polynomial becomes

$$f(x,y) = \underbrace{y^3 + ax^2y + Ax^3}_{} + Cx + cy.$$

$$f_3 = y^3 + ax^2y + Ax^3.$$

$\Longrightarrow f_3$ *has a double root i.e.* $f_3 = (y-ux)^2(y+2ux)$, *so* $A = 2u^3,\ a = -3u^2$. *Then the coefficient of* x^4 *becomes*

$$18u^3C + 18u^4c = 0 \Longrightarrow u = \tfrac{-C}{c}.$$

$$\text{If} \quad c = 0 \Longrightarrow C = 0.$$

So the coefficient of x^2 *becomes zero, which is impossible. If* $c \neq 0$. *The polynomial becomes reducible* $f(x,y) = (y-ux)((y-ux)(y+2ux)+c)$, *which is impossible. So* $deg_x\Delta(x,t) \neq 2$ *in this case. Hence fiber is not contractible.*

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