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ON THE PARTITION DIMENSION OF SOME WHEEL RELATED GRAPHS

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Abstract. Let G be a connected graph. For a vertex $v \in V(G)$ and an ordered k-partition $\Pi = \{S_1, S_2, ..., S_k\}$ of $V(G)$, the representation of v with respect to Π is the k-vector $r(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$ where $d(v, S_i) = \min_{w \in S_i} d(v, w)(1 \leq i \leq k)$. The k-partition Π is said to be resolving if the k-vectors $r(v|\Pi)$, $v \in V(G)$, are distinct. The minimum k for which there is a resolving k-partition of $V(G)$ is called the partition dimension of G, denoted by $pd(G)$. In this paper, we give upper bounds for the cardinality of vertices in some wheel related graphs namely gear graph, helm, sunflower and friendship graph with given partition dimension k.

Key words : Resolving partition, partition dimension, gear graph, helm, sunflower and friendship graph.
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1. Introduction

If G is a connected graph, the *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path between them. The *diameter* of G, denoted by $diam(G)$ is the largest distance between two vertices in $V(G)$. For a vertex v of a graph G and a subset S of $V(G)$, the distance between v and S is $d(v, S) = \min\{d(v, x)|x \in S\}$. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be an ordered k-partition of vertices of G and let v be a vertex of G. The representation $r(v|\Pi)$ of v with respect to Π is the k-tuple $(d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. If distinct vertices of G have distinct representations with respect to Π, then Π is called a *resolving partition* for G. The cardinality of a minimal resolving partition is called the *partition dimension* of G, denoted by $pd(G)$.

If $d(x, S) \neq d(y, S)$ we shall say that the class S *distinguishes* vertices x and y. If a class S of Π distinguishes vertices x and y we shall also say that

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Π *distinguishes* x and y. From these definitions it can be observed that the property of a given partition Π of the vertices of a graph G to be a resolving partition of G can be verified by investigating the pairs of vertices in the same class. Indeed, every vertex $x \in S_i$ $(1 \leq i \leq k)$ is at distance 0 from S_i, but is at a distance different from zero from any other class S_j with $j \neq i$. It follows that $x \in S_i$ and $y \in S_j$ are distinguished either by S_i or by S_j for every $i \neq j$.

Another useful property in determining $pd(G)$ is the following lemma from 3.

Lemma 1. *Let* Π *be a resolving partition of* $V(G)$ *and* $u, v \in V(G)$. *If* $d(u, w) = d(v, w)$ *for all vertices* $w \in V(G) \setminus \{u, v\}$, *then* u *and* v *belong to different classes of* Π.

Partition dimension was firstly studied by Chartrand et. al in 3. and 4. perhaps as a variation of metric dimension. Metric dimension of a graph is defined in the following way. A subset of vertices $W = \{w_1, \ldots, w_k\}$ is called a *resolving set* for G if for every two distinct vertices $x, y \in V(G)$, there is a vertex $w_i \in W$ such that $d(x, w_i) \neq d(y, w_i)$. A resolving set containing a minimum number of vertices is called a *metric basis* for G and the number of vertices in a metric basis is its *metric dimension* denoted by $dim(G)$. They gave upper and lower bounds for partition dimension of a graph in terms of metric dimension of a graph in 3.

Theorem 2. *If* G *is a nontrivial connected graph, then*

$$pd(G) \leq dim(G) + 1.$$

Moreover, for every a, b *of positive integers with* $\lceil \frac{b}{2} \rceil + 1 \leq a \leq b + 1$, *there exist a connected graph* G *such that* $pd(G) = a$ *and* $dim(G) = b$.

Following this theorem there was an open problem. Is it the case that

$$pd(G) \geq \lceil \frac{dim(G)}{2} \rceil + 1$$

for every nontrivial connected graph G? This problem was answered in 2. We give a counter example to show that answer is NO.
We consider a wheel W_{19}. In 1, it was shown that $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$. So $dim(W_{19}) = 8$ and we show that $pd(W_{19}) \leq 4$. Let c be the central vertex and $v_0, v_1, v_2, \ldots, v_{18}$ be the rim vertices of W_{19}. We consider the following resolving partition: $\Pi = \{S_1, S_2, S_3, S_4\}$ where $S_1 = \{c, v_0, v_1, v_2, v_4, v_6, v_{13}, v_{17}\}$, $S_2 = \{v_9, v_{11}, v_{14}, v_{18}\}$, $S_3 = \{v_3, v_8, v_{10}, v_{16}\}$ and $S_4 = \{v_5, v_7, v_{12}, v_{15}\}$. This gives that $4 \geq 5$ which shows that $pd(G) \geq \lceil \frac{dim(G)}{2} \rceil + 1$ is not true in general.

The concepts of metric dimension of a graph was introduced by Slater in 14. and 15. to uniquely determine the location of an intruder in a network and this concept was studied independently by Harary and Melter in 10. It has since been extensively studied (see 1–6, 10–15). These concepts have some applications in chemistry for representing chemical compounds (see 6) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures (see 12) and arise in areas like coin weighing problems (see 13) robot navigation (see 11) and strategies for the Mastermind game (see 8 and 9).

Motivation for this paper is to answer the following extremal problem proposed in 2. for some wheel related graphs.

Problem: Determine, for each d and k, the maximum order of a graph having diameter d and partition dimension k.

Following theorem about the cardinality of vertices in graphs with diameter two and partition dimension $k \geq 2$ was proved in 2.

Theorem 3. *The maximum order of a graph with diameter two and partition dimension $k \geq 2$ is*

$$l[\binom{2l-1}{l} + 2^{2l-1}], \text{ if } k = 2l,$$

and

$$(2l+1)[\binom{2l-1}{l} + 2^{2l-1}], \text{ if } k = 2l+1.$$

In 16, it was shown that for given partition dimension k, $|V(W_n)| < \frac{k^3}{2}$ for every $k \geq 2$ and following that it was shown that $k \geq \lceil (2n)^{1/3} \rceil$. Following theorem was proved in 16.

Theorem 4. *For every $n \geq 4$*

$$\lceil (2n)^{1/3} \rceil \leq pd(W_n) \leq p + 1,$$

where p is the smallest prime number such that $p(p-1) \geq n$.

In this paper we give upper bound for cardinality of vertices in gear graph, helm, sunflower and friendship graph. In the section to follow, we give an upper bound for the cardinality of vertices in gear graph G_{2n} and show that $pd(G_{2n}) \geq \lceil \frac{1}{2} log_2(2n+1) \rceil - 1$ for every $n \geq 2$. In the third and fourth section we give upper bound for the number of vertices in helm H_n and sunflower SF_n. In the last section for the friendship graph we show that if $pd(f_n) = k$ then $n \leq \binom{k}{2}$ and this bound is attainable for every k.

2. Partition Dimension of Gear Graph

Gear graph G_{2n} is defined as follows: consider an even cycle $C_{2n} : v_0, v_1, \ldots, v_{2n-1}, v_0$, where $n \geq 2$ and a new vertex c adjacent to n vertices of C_{2n} : $v_0, v_2, \ldots, v_{2n-2}$. G_{2n} has order $2n+1$ and size $3n$. Equivalently, a gear graph is obtained from the wheel by adding a vertex between every pair of adjacent vertices of the cycle. This definition is taken from 7.

The vertices of C_{2n} in G_{2n} are of two kinds: vertices of degree two and three, respectively. The vertices of degree two will be referred to as minor vertices and vertices of degree three to as major vertices. It is not difficult to see that $pd(G_4) = 3$ ($S_1 = \{c, v_0\}$, $S_2 = \{v_1\}$ and $S_3 = \{v_2, v_3\}$), $pd(G_6) = 3$ ($S_1 = \{c, v_0, v_5\}$, $S_2 = \{v_1, v_2\}$ and $S_3 = \{v_3, v_4\}$) and $pd(G_8) = 3$ with minimal resolving partition consisting of $S_1 = \{c, v_0, v_1, v_2\}$, $S_2 = \{v_3, v_4, v_5\}$ and $S_3 = \{v_6, v_7\}$. Now we find bounds for the value of n for a given k such that $pd(G_{2n}) = k$. Suppose that $c \in S_1$.

Claim 1. There can be at most two 1's in the vector representation of a rim vertex other than the first position.

Claim 2. There can be at most two 2's in the vector representation of a minor rim vertex other than the first position.

Claim 3. The greatest number in the vector representation of a minor vertex is 4.

Claim 4. The greatest number in the vector representation of a major vertex is 3.

Now, we present some lemmas.

Lemma 5. *The number of distinct representations of the center c of gear graph with respect to partitions of $V(G_{2n})$ is 2^{k-1}.*

Proof. Let $pd(G_{2n}) = k$. Without loss of generality, we can assume that $c \in S_1$. Then c is at distance 0 from S_1 and it can be at distance 1 or 2 from other $k-1$ classes. So $k-1$ positions of the vector for c can be filled by 1 or 2. Hence there can be 2^{k-1} distinct representations for c. □

Lemma 6. *The number of distinct representations of the major rim vertices in S_1 containing the center c of gear graph with respect to partitions of $V(G_{2n})$ is $\sum_{i=0}^{2} \binom{k-1}{i} 2^{k-i-1}$.*

Proof. Let v be a major vertex in S_1. Then first entry in the vector representation of v with respect to Π is 0. So there are $k-1$ positions which can be filled by 1, 2 or 3. Since 1 can be at most twice by claim 1 then the number of possible distinct representations is $\sum_{i=0}^{2} \binom{k-1}{i} 2^{k-i-1}$. □

Lemma 7. *The number of distinct representations of the minor rim vertices in S_1 containing the center c of gear graph with respect to partitions of $V(G_{2n})$ is $\sum_{j=0}^{2} \sum_{i=0}^{2} \binom{k-1}{j} \binom{k-j-1}{i} 2^{k-i-j-1}$.*

Proof. Since the first entry in the representation is 0, there are $k-1$ positions which can be filled by 1, 2, 3 or 4. Since 1 can be at most on two positions and by claim 2, 2 can also be on at most two positions then the number of possible distinct representations is $\sum_{j=0}^{2} \sum_{i=0}^{2} \binom{k-1}{j} \binom{k-j-1}{i} 2^{k-i-j-1}$. □

Lemma 8. *The number of distinct representations of the major rim vertices in other classes than* S_1 *containing the center* c *of gear graph with respect to partitions of* $V(G_{2n})$ *is* $\sum_{i=0}^{2}\binom{k-2}{i}2^{k-i-2}$.

Proof. Let v be a vertex in any class other than S_1. Without loss of generality we may assume that $v \in S_2$. Then first entry in the vector representation of v with respect to Π is 1 and second entry is 0. So there are $k-2$ positions which can be filled by 1, 2 or 3. Since 1 can be at most twice by claim 1, then the number of possible distinct representations is $\sum_{i=0}^{2}\binom{k-2}{i}2^{k-i-2}$. □

Lemma 9. *The number of distinct representations of the minor rim vertices in other classes than* S_1 *containing the center* c *of gear graph with respect to partitions of* $V(G_{2n})$ *is* $2\sum_{j=0}^{2}\sum_{i=0}^{2}\binom{k-2}{j}\binom{k-j-2}{i}2^{k-i-j-2}$.

Proof. Since the first entry in the representation is 1 or 2 and the second entry is 0, so there are $k-2$ positions which can be filled by 1, 2, 3 or 4. Since 1 can be at most on two positions and by claim 2, 2 can also be at most on two positions then the number of possible distinct representations is $2\sum_{j=0}^{2}\sum_{i=0}^{2}\binom{k-2}{j}\binom{k-j-2}{i}2^{k-i-j-2}$. □

Theorem 10. *Let* $n \geq 2$ *and* k *denote the partition dimension of* G_{2n}. *Then* $2n+1 < 3k^4(k+2)2^{k-7}$.

Proof. If $c \in S_1$, then by Lemmas 5-7 we deduce that

$$\begin{aligned} |S_1| &\leq 2^{k-1} + \sum_{i=0}^{2}\binom{k-1}{i}2^{k-i-1} + \sum_{j=0}^{2}\sum_{i=0}^{2}\binom{k-1}{j}\binom{k-j-1}{i}2^{k-i-j-1} \\ &= 2^{k-7}(k^4 - 2k^3 + 27k^2 + 14k + 152) \\ &< 3k^4 2^{k-6} \end{aligned}$$

for every $k \geq 3$. If $l \neq 1$ then by Lemmas 8 and 9 we get

$$\begin{aligned} |S_l| &\leq \sum_{i=0}^{2}\binom{k-2}{i}2^{k-i-2} + 2\sum_{j=0}^{2}\sum_{i=0}^{2}\binom{k-2}{j}\binom{k-j-2}{i}2^{k-i-j-2} \\ &= 2^{k-7}(k^4 - 6k^3 + 35k^2 - 46k + 80) \\ &< 3k^4 2^{k-7} \end{aligned}$$

for every $k \geq 3$. This implies that

$$\begin{aligned} 2n+1 &= \sum_{l=1}^{k}|S_l| \\ &< 3k^4 2^{k-6} + 3(k-1)k^4 2^{k-7} \\ &< 3k^4(k+2)2^{k-7} \end{aligned}$$

for every $k \geq 3$. Then it is easy to see that $pd(G_{2n}) \geq 3$ for every $n \geq 2$, which implies that $2n+1 < 3k^4(k+2)2^{k-7}$, where $k = pd(G_{2n})$ for every $n \geq 2$. □

Note that the lower bound for $pd(G_{2n})$ given by Theorem 10 is asymptotically better than general bounds proposed in 2. We have $3k^4(k+2) < 2^{k+9}$ for every $k \geq 3$, which implies that $2n+1 < 2^{2k+2}$, hence the following result holds:

Corollary 11. *$pd(G_{2n}) \geq \lceil \frac{1}{2} log_2(2n+1) \rceil - 1$ for every $n \geq 2$.*

3. Partition Dimension of Helm

Helm H_n is a graph obtained from a wheel W_n with cycle C_n having a pendant edge attached to each vertex of the cycle. Helm H_n consists of the vertex set $V(H_n) = \{v_i | 0 \leq i \leq n-1\} \cup \{a_i | 0 \leq i \leq n-1\} \cup \{c\}$ and edge set $E(H_n) = \{v_i v_{i+1} | 0 \leq i \leq n-1\} \cup \{v_i a_i | 0 \leq i \leq n-1\} \cup \{v_i c | 0 \leq i \leq n-1\}$, where $i+1$ is taken modulo n. The definition of helm is taken from 7.

The vertices of $H_n \backslash \{c\}$ are of two kinds: vertices of degree four and one, respectively. The vertices of degree one will be referred to as minor vertices and vertices of degree four to as major vertices. It is not difficult to see that $pd(H_3) = 4$ ($S_1 = \{c\}$, $S_2 = \{v_0, a_0\}$, $S_3 = \{v_1, a_1\}$ and $S_4 = \{v_2, a_2\}$), $pd(H_4) = 3$ ($S_1 = \{c, v_0, a_0, v_1, a_1\}$, $S_2 = \{v_2, a_2\}$ and $S_3 = \{v_3, a_3\}$) and $pd(H_5) = 3$ with minimal resolving partition consisting of $S_1 = \{c, v_0, a_0, v_1, a_1\}$, $S_2 = \{v_2, a_2, v_3, a_3\}$ and $S_3 = \{v_4, a_4\}$. Now we find bounds for the value of n for a given k such that $pd(H_n) = k$. Suppose that $c \in S_1$.

Claim 1. There can be at most three 1's in the vector representation of a major vertex other than the first position.
Claim 2. There can be at most one 1 in the vector representation of a minor vertex.
Claim 3. There can be at most two 2's in the vector representation of a minor vertex other than the first position.
Claim 4. The greatest number in the vector representation of a major vertex is 3.
Claim 5. The greatest number in the vector representation of a minor vertex is 4.

Lemma 12. *The number of distinct representations of the center c of helm with respect to partitions of $V(H_n)$ is 2^{k-1}.*

Proof. Let $pd(H_n) = k$. Without loss of generality, we can assume that $c \in S_1$. Then c is at distance 0 from S_1 and it can be at distance 1 or 2 from other $k-1$ classes. So $k-1$ positions of the vector for c can be filled by 1 or 2. Hence there can be 2^{k-1} distinct representations for c. □

Lemma 13. *The number of distinct representations of the major vertices in S_1 containing the center c of helm with respect to partitions of $V(H_n)$ is* $\sum_{i=0}^{3}\binom{k-1}{i}2^{k-i-1}$.

Proof. Let v be a major vertex in S_1. Then first entry in the vector representation of v with respect to Π is 0. So there are $k-1$ positions which can be filled by 1, 2 or 3. Since 1 can be at most thrice by claim 1 then the number of possible distinct representations is $\sum_{i=0}^{3}\binom{k-1}{i}2^{k-i-1}$. □

Lemma 14. *The number of distinct representations of the minor vertices in S_1 containing the center c of helm with respect to partitions of $V(H_n)$ is* $\sum_{j=0}^{1}\sum_{i=0}^{2}\binom{k-1}{j}\binom{k-j-1}{i}2^{k-i-j-1}$.

Proof. Since the first entry in the representation is 0, there are $k-1$ positions which can be filled by 1, 2, 3 or 4. By claim 2, 1 can be at most on one position and by claim 3, 2 can also be at most on two positions then the number of possible distinct representations is $\sum_{j=0}^{1}\sum_{i=0}^{2}\binom{k-1}{j}\binom{k-j-1}{i}2^{k-i-j-1}$. □

Observation 1: There are $\sum_{j=0}^{1}\sum_{i=0}^{2}\binom{k-1}{j}\binom{k-j-1}{i}$ representations which appear for major as well as minor vertices for vertices in class S_1.

Lemma 15. *The number of distinct representations of the major vertices in other classes than S_1 containing the center c of helm with respect to partitions of $V(H_n)$ is* $\sum_{i=0}^{3}\binom{k-2}{i}2^{k-i-2}$.

Proof. Let v be a vertex in any class other than S_1. Without loss of generality we may assume that $v \in S_2$. Then first entry in the vector representation of v with respect to Π is 1 and second entry is 0. So there are $k-2$ positions which can be filled by 1, 2 or 3. Since 1 can be at most thrice by claim 1, then the number of possible distinct representations is $\sum_{i=0}^{3}\binom{k-2}{i}2^{k-i-2}$. □

Lemma 16. *The number of distinct representations of the minor vertices in other classes than S_1 containing the center c of helm with respect to partitions of $V(H_n)$ is* $\sum_{j=0}^{1}\sum_{i=0}^{2}\binom{k-2}{j}\binom{k-j-2}{i}2^{k-i-j-2}+\sum_{i=0}^{2}\binom{k-2}{i}2^{k-i-2}$.

Proof. Since the first entry in the representation is 1 or 2 and the second entry is 0, so there are $k-2$ positions which can be filled by 1, 2, 3 or 4. Since 1 can be at most on one position and by claim 3, 2 can also be at most on two positions then the number of possible distinct representations is $\sum_{j=0}^{1}\sum_{i=0}^{2}\binom{k-2}{j}\binom{k-j-2}{i}2^{k-i-j-2}+\sum_{i=0}^{2}\binom{k-2}{i}2^{k-i-2}$. □

Observation 2: There are $\sum_{j=0}^{1}\sum_{i=0}^{2}\binom{k-2}{j}\binom{k-j-2}{i}$ representations which appear for major as well as minor vertices for vertices in class other than S_1.

Theorem 17. *Let $n \geq 3$ and k denote the partition dimension of H_n. Then* $2n+1 < 2^{k-1}+\sum_{i=0}^{3}2^{k-i-1}\binom{k-1}{i}(k-i)+\sum_{j=0}^{1}\sum_{i=0}^{2}2^{k-i-j-2}\binom{k-1}{i,j}(k-i-j+1)$.

Proof. If $c \in S_1$, then by Lemmas 12-14 and Observation 1, we deduce that

$$|S_1| < 2^{k-1} + \sum_{i=0}^{3} \binom{k-1}{i} 2^{k-i-1} + \sum_{j=0}^{1} \sum_{i=0}^{2} \binom{k-1}{j} \binom{k-j-1}{i} 2^{k-i-j-1}$$

for every $k \geq 3$. If $l \neq 1$ then by Lemmas 15, 16 and Observation 2, we get

$$|S_l| < \sum_{i=0}^{3} \binom{k-2}{i} 2^{k-i-2} + \sum_{j=0}^{1} \sum_{i=0}^{2} \binom{k-2}{j} \binom{k-j-2}{i} 2^{k-i-j-2} + \sum_{i=0}^{2} \binom{k-2}{i} 2^{k-i-2}$$

for every $k \geq 3$. This implies that

$$\begin{aligned} 2n+1 &= \sum_{l=1}^{k} |S_l| \\ &< 2^{k-1} + \sum_{i=0}^{3} 2^{k-i-1} \binom{k-1}{i} (k-i) \\ &+ \sum_{j=0}^{1} \sum_{i=0}^{2} 2^{k-i-j-2} \binom{k-1}{i,j} (k-i-j+1) \end{aligned}$$

for every $k \geq 3$. Then it is easy to see that $pd(H_n) \geq 3$ for every $n \geq 3$, which implies that $2n+1 < 2^{k-1} + \sum_{i=0}^{3} 2^{k-i-1} \binom{k-1}{i} (k-i) + \sum_{j=0}^{1} \sum_{i=0}^{2} 2^{k-i-j-2}$ $\binom{k-1}{i,j} (k-i-j+1)$, where $k = pd(H_n)$ for every $n \geq 3$. □

4. Partition Dimension of Sunflower

Sunflower graph SF_n is defined as follows: consider a wheel with central vertex c and an n-cycle $v_0, v_1, v_2, \ldots, v_{n-1}$ and additional n vertices $w_0, w_1, w_2, \ldots, w_{n-1}$ where w_i is joined by edges to v_i, v_{i+1} for $i = 0, 1, 2, \ldots, n-1$ where $i+1$ is taken modulo n. SF_n has order $2n+1$ and size $4n$.

The vertices of $SF_n \backslash \{c\}$ are of two kinds: vertices of degree five and two, respectively. The vertices of degree two will be referred to as minor vertices and vertices of degree five to as major vertices. It is not difficult to see that $pd(SF_3) = 4$ ($S_1 = \{c, v_0, w_0\}$, $S_2 = \{v_1, w_1\}$, $S_3 = \{v_2\}$ and $S_4 = \{w_2\}$), $pd(SF_4) = 3$ ($S_1 = \{c, v_0, w_0, v_1\}$, $S_2 = \{w_1, v_2, w_2\}$ and $S_3 = \{v_3, w_3\}$) and $pd(SF_5) = 3$ with minimal resolving partition consisting of

$S_1 = \{c, v_0, w_0, v_1, w_1, v_2\}$, $S_2 = \{w_2, v_3, w_3\}$ and $S_3 = \{v_4, w_4\}$. Now we find an upper bound for the cardinality of $V(SF_n)$ for a given k such that $pd(SF_n) = k$. Suppose that $c \in S_1$.

Claim 1. There can be at most four 1's in the vector representation of a major vertex other than the first position.
Claim 2. There can be at most two 1's in the vector representation of a minor vertex.
Claim 3. There can be at most four 2's in the vector representation of a minor vertex other than the first position.
Claim 4. The greatest number in the vector representation of a major vertex is 3.
Claim 5. The greatest number in the vector representation of a minor vertex is 4.

Lemma 18. *The number of distinct representations of the center c of sunflower with respect to partitions of $V(SF_n)$ is 2^{k-1}.*

Lemma 19. *The number of distinct representations of the major vertices in S_1 containing the center c of sunflower with respect to partitions of $V(SF_n)$ is $\sum_{i=0}^{4} \binom{k-1}{i} 2^{k-i-1}$.*

Lemma 20. *The number of distinct representations of the minor vertices in S_1 containing the center c of sunflower with respect to partitions of $V(SF_n)$ is $\sum_{j=0}^{2} \sum_{i=0}^{4} \binom{k-1}{j} \binom{k-j-1}{i} 2^{k-i-j-1}$.*

Observation 1: There are $\sum_{j=0}^{2} \sum_{i=0}^{4} \binom{k-1}{j} \binom{k-j-1}{i}$ representations which appear for major as well as minor vertices in class S_1.

Lemma 21. *The number of distinct representations of the major vertices in other classes than S_1 containing the center c of sunflower with respect to partitions of $V(SF_n)$ is $\sum_{i=0}^{4} \binom{k-2}{i} 2^{k-i-2}$.*

Lemma 22. *The number of distinct representations of the minor vertices in other classes than S_1 containing the center c of sunflower with respect to partitions of $V(SF_n)$ is $2\sum_{j=0}^{2} \sum_{i=0}^{4} \binom{k-2}{j} \binom{k-j-2}{i} 2^{k-i-j-2} - \sum_{i=0}^{4} \binom{k-2}{2} \binom{k-4}{i} 2^{k-i-4}$.*

Proofs of Lemmas 18–22 are analogous to proofs of lemmas proved for helm in the last section.
Observation 2: There are $\sum_{j=0}^{2} \sum_{i=0}^{4} \binom{k-2}{j} \binom{k-j-2}{i}$ representations which appear for major as well as minor vertices in classes other than S_1.

Theorem 23. *Let $n \geq 3$ and k denote the partition dimension of SF_n. Then $2n + 1 < 2^{k-1} + \sum_{i=0}^{4} 2^{k-i-2} \binom{k-1}{i} (k - i + 1) + \sum_{j=0}^{2} \sum_{i=0}^{4} 2^{k-i-j-1} \binom{k-1}{i,j} (k - i - j)$.*

Proof. If $c \in S_1$, then by Lemmas 18-20 and Observation 1, we deduce that

$$|S_1| \;<\; 2^{k-1} + \sum_{i=0}^{4} \binom{k-1}{i} 2^{k-i-1} + \sum_{j=0}^{2}\sum_{i=0}^{4} \binom{k-1}{j}\binom{k-j-1}{i} 2^{k-i-j-1}$$

for every $k \geq 3$. If $l \neq 1$ then by Lemmas 21, 22 and Observation 2, we get

$$\begin{aligned} |S_l| \;<\; & \sum_{i=0}^{4} \binom{k-2}{i} 2^{k-i-2} \;+\; 2\sum_{j=0}^{2}\sum_{i=0}^{4} \binom{k-2}{j}\binom{k-j-2}{i} 2^{k-i-j-2} \\ & -\sum_{i=0}^{4} \binom{k-2}{2}\binom{k-4}{i} 2^{k-i-4} \end{aligned}$$

for every $k \geq 3$. This implies that

$$\begin{aligned} 2n+1 &= \sum_{l=1}^{k} |S_l| \\ &< 2^{k-1} \;+\; \sum_{i=0}^{4} 2^{k-i-2}\binom{k-1}{i}(k-i+1) \\ &\quad + \sum_{j=0}^{2}\sum_{i=0}^{4} 2^{k-i-j-1}\binom{k-1}{i,j}(k-i-j) \end{aligned}$$

for every $k \geq 3$. Then it is easy to see that $pd(SF_n) \geq 3$ for every $n \geq 3$, which implies that $2n+1 < 2^{k-1} + \sum_{i=0}^{4} 2^{k-i-2}\binom{k-1}{i}(k-i+1) + \sum_{j=0}^{2}\sum_{i=0}^{4} 2^{k-i-j-1}$ $\binom{k-1}{i,j}(k-i-j)$, where $k = pd(SF_n)$ for every $n \geq 3$. □

5. Partition Dimension of Friendship Graph

Friendship graph f_n is collection of n triangles with a common point. Friendship graph can also be obtained from a wheel W_{2n} with cycle C_{2n} by deleting alternate edges of the cycle. Another way of obtaining friendship graph is addition of K_1 and n copies of K_2. We present a theorem for the cardinality of friendship graph with partition dimension k.

Theorem 24. *Let $n \geq 2$ and $pd(f_n) = k$ then $n \leq \binom{k}{2}$.*

Proof. Let $pd(f_n) = k$ then there exist a resolving partition $\Pi = \{S_1, S_2, \ldots, S_k\}$ where $S_{i(1\leq i\leq k)} \subset V(f_n)$. Since both vertices in a copy of K_2 can not belong to same class by Lemma 1. So one class can contain at most $k-1$ vertices other than center. As there are k classes and center can belong to only one class so

$$\begin{aligned} V(f_n) = 2n \quad +1 &\leq k(k-1)+1 \\ 2n &\leq k(k-1) \\ n &\leq \binom{k}{2}. \end{aligned}$$

□

This bound is attainable for every value of k. Hence for every $k \geq 2$, there exists a friendship graph such that $pd(f_n) = k$.

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