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THE CONNECTED VERTEX GEODOMINATION NUMBER OF A GRAPH

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Abstract. For a connected graph G of order $p \geq 2$, a set $S \subseteq V(G)$ is an x-geodominating set of G if each vertex $v \in V(G)$ lies on an x-y geodesic for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the x-geodomination number of G, denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set of G. A connected x-geodominating set of G is an x-geodominating set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-geodominating set of G is defined as the connected x-geodomination number of G and is denoted by $cg_x(G)$. A connected x-geodominating set of cardinality $cg_x(G)$ is called a cg_x-set of G. We determine bounds for it and find the same for some special classes of graphs. If p, a and b are positive integers such that $2 \leq a \leq b \leq p-1$, then there exists a connected graph G of order p, $g_x(G) = a$ and $cg_x(G) = b$ for some vertex x in G. Also, if p, d and n are integers such that $2 \leq d \leq p-2$ and $1 \leq n \leq p$, then there exists a connected graph G of order p, diameter d and $cg_x(G) = n$ for some vertex x in G. For positive integers r, d and n with $r \leq d \leq 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $cg_x(G) = n$ for some vertex x in G.

Key words: geodesic, vertex geodomination number, connected vertex geodomination number.
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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [4]. For

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vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an *x-y geodesic*. A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. The *diameter diam* G of a connected graph G is the length of any longest geodesic. For any vertex u of G, the eccentricity of u is $e(u) = max\{d(u, v) : v \in V\}$. A vertex v of G such that $d(u, v) = e(u)$ is called an *eccentric vertex* of u. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is a *simplicial vertex* if the subgraph induced by its neighborhood $N(v)$ is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$,

$$I[S] = \bigcup_{x,y \in S} I[x, y].$$

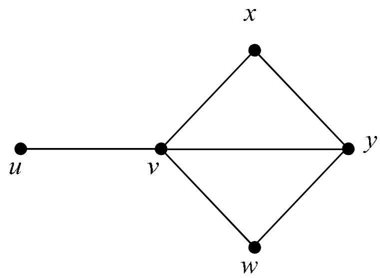
A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G. The geodetic number of a graph was introduced in [1, 5] and further studied in [2]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem. Geodetic concepts were first studied from the point of view of domination by Chartrand, Harary, Swart and Zhang in [3], where a pair x, y of vertices in a nontrivial connected graph G is said to *geodominate a vertex* v of G if $v \in I[x, y]$, that is, v lies on an x-y geodesic of G. In [3], geodetic sets and the geodetic number were referred to as *geodominating sets* and the *geodomination number* respectively and it is this terminology that we adopt in this paper.

The concept of vertex geodomination number was introduced by Santhakumaran and Titus in [7] and further studied in [8]. A vertex y in a connected graph G is said to *x-geodominate* a vertex u if u lies on an x-y geodesic. A set S of vertices of G is an *x-geodominating set* if each vertex $v \in V(G)$ is x-geodominated by some element of S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a *g_x-set*.

Every vertex of an x-y geodesic is x-geodominated by the vertex y. Since, by definition, a g_x-set is minimum, the vertex x and also the internal vertices of an x-y geodesic do not belong to a g_x-set. For the graph G given in Figure 1.1, $g_u(G) = 3$, $g_v(G) = 4$, $g_w(G) = 2$, $g_x(G) = 2$ and $g_y(G) = 3$ with minimum vertex geodominating sets $\{x, y, w\}, \{x, y, u, w\}, \{x, u\}, \{u, w\}$ and $\{x, u, w\}$ respectively.

It is proved in [7] that for any vertex x in G, g_x-set is unique and $1 \leq g_x(G) \leq p - 1$ for any vertex x in G. An elaborate study of results in vertex geodomination with several interesting applications is given in [7, 8].

The following theorems will be used in the sequel.



G
Figure 1.1

Theorem 1.1. [4] *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) *v is a cut vertex of G.*
(ii) *There exist vertices u and w distinct from v such that v is on every u-w path.*
(iii) *There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every u-w path.*

Theorem 1.2. [7] *Let G be a connected graph.*

(i) *Every simplicial vertex of G other than the vertex x (whether x is simplicial or not) belongs to the g_x-set for any vertex x in G.*
(ii) *For any vertex x, eccentric vertices of x belong to the g_x-set.*
(iii) *No cut vertex of G belongs to any g_x-set.*

Theorem 1.3. [7] *Let T be a tree with number of end vertices t. Then $g_x(T) = t - 1$ or t according as x is an end vertex or a cut vertex.*

Throughout the following G denotes a connected graph with at least two vertices.

2. Connected Vertex Geodomination Number

Definition 2.1. *Let x be any vertex of a connected graph G. A connected x-geodominating set of G is an x-geodominating set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-geodominating set of G is the connected x-geodomination number of G and is denoted by $cg_x(G)$. A connected x-geodominating set of cardinality $cg_x(G)$ is called a cg_x-set of G.*

Example 2.2. *For the graph G given in Figure 2.1, the minimum vertex geodominating sets, the vertex geodomination numbers, the minimum connected vertex geodominating sets and the connected vertex geodomination numbers are given in Table 2.1.*

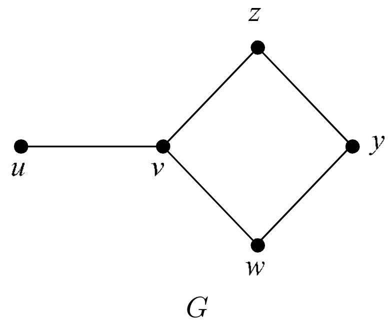


Figure 2.1

Vertex x	g_x*-set*	$g_x(G)$	cg_x*-sets*	$cg_x(G)$
u	$\{y\}$	*1*	$\{y\}$	*1*
v	$\{u, y\}$	*2*	$\{u, v, z, y\}, \{u, v, w, y\}$	*4*
w	$\{u, z\}$	*2*	$\{u, v, z\}$	*3*
y	$\{u\}$	*1*	$\{u\}$	*1*
z	$\{u, w\}$	*2*	$\{u, v, w\}$	*3*

Table 2.1.

It is proved in [7] that for any vertex x in G, g_x-set of G with respect to x is unique. However, we observe that in the case of connected x-geodominating sets, there can be more than one minimum connected x-geodominating set. For the vertex v of the graph G in Figure 2.1, $\{u, v, z, y\}$ and $\{u, v, w, y\}$ are two distinct cg_v- sets of G. It is observed in [7] that x is not an element of the g_x-set of G, where as x may belong to a cg_x-set of G. For the graph G given in Figure 2.1, the vertex v is an element of a cg_v-set.

In the following theorem we establish the relationship between the g_x-set and a connected x-geodominating set of G.

Theorem 2.3. *For any vertex x in G, the g_x-set is contained in every connected x-geodominating set of G.*

Proof. Let S be the g_x-set of G and let $y \in S$. Since S is minimum, y is not x-geodominated by any other vertex of G. If there exists a connected x-geodominating set, say S', with $y \notin S'$, then y lies on an x-v geodesic for some $v \in S'$ and hence y is x-geodominated by the vertex v in G, which is a contradiction. □

Remark 2.4. *In the proof of Theorem 2.3 the connectedness property of x-geodominating set is not used. This shows that the same result is true for any property.*

Corollary 2.5. *For any vertex x in G, $g_x(G) \leq cg_x(G)$.*

Proof. This follows from Theorem 2.3. □

Theorem 2.6. *Let x be any vertex of a connected graph G.*

(i) *If $y \neq x$ is a simplicial vertex of G, then y belongs to every connected x-geodominating set of G.*
(ii) *The eccentric vertices of x belong to every connected x-geodominating set of G.*

Proof. This follows from Theorem 1.2 and Theorem 2.3. □

Theorem 2.7.

(i) *For the complete graph $K_p, cg_x(K_p) = p - 1$ for any vertex x in K_p.*
(ii) *For any vertex x in a cycle $C_p, cg_x(C_p) = 1$ or 2 according as p is even or odd.*
(iii) *For the wheel $W_p = K_1 + C_{p-1} (p \geq 5), cg_x(W_p) = p - 1$ or $p - 4$ according as x is K_1 or x is in C_{p-1}.*

Proof. (i) For any vertex x in K_p, let $S = V(K_p) - \{x\}$. Since each vertex in S is an eccentric vertex of x, it follows from Theorem 2.6(ii) that $cg_x(K_p) \geq |S| = p - 1$. It is clear that S is a connected x-geodominating set of G so that $cg_x(K_p) = p - 1$.

(ii) Let C_p be an even cycle. For any vertex x in C_p, let y be the eccentric vertex of x. Clearly every vertex of C_p lies on an x-y geodesic and so $\{y\}$ is a connected x-geodominating set of C_p so that $cg_x(C_p) = 1$.

Let C_p be an odd cycle. For any vertex x in C_p, let $S = \{y, z\}$ be the set of eccentric vertices of x. By Theorem 2.6(ii), $cg_x(C_p) \geq |S| = 2$. Clearly S is an x-geodominating set and the induced subgraph $G[S]$ is connected so that $cg_x(C_p) = 2$.

(iii) Let x be the vertex of K_1. Clearly $S = V(C_{p-1})$ is the set of all eccentric vertices of x. By Theorem 2.6(ii), $cg_x(W_p) \geq p - 1$. Since S is a connected x-geodominating set, $cg_x(W_p) = p - 1$.

Let $C_{p-1} : u_1, u_2, \ldots, u_{p-1}, u_1$ be the cycle of W_p. Let x be any vertex in C_{p-1}. Assume that $x = u_1$. Since the diameter *diam* $W_p = 2$, $S = \{u_3, u_4, \ldots, u_{p-2}\}$ is the set of all eccentric vertices of x. By Theorem 2.6(ii), $cg_x(W_p) \geq p - 4$. Let K_1 be z. Then the vertices u_2, z and u_{p-1} lie on the geodesics x, u_2, u_3; x, z, u_3; and x, u_{p-1}, u_{p-2} respectively and hence S is an x-geodominating set of W_p. Clearly the induced subgraph $G[S]$ is connected and so $cg_x(W_p) = p - 4$. □

Theorem 2.8. *Let $K_{m,n} (2 \leq m \leq n)$ be the complete bipartite graph with bipartition (V_1, V_2). Then*

(i) $cg_x(K_{2,2}) = 1$ *for any vertex x*
(ii) $cg_x(K_{2,n}) = \begin{cases} 1 & \text{if } x \in V_1 \\ n & \text{if } x \in V_2 \text{ for } n \geq 3 \end{cases}$

(iii) $cg_x(K_{m,n}) = \begin{cases} m & \text{if } x \in V_1 \\ n & \text{if } x \in V_2 \text{ for } m, n \geq 3. \end{cases}$

Proof. (i) By Theorem 2.7(ii), $cg_x(K_{2,2}) = 1$ for any vertex x.

(ii) Let $x \in V_1$ be any vertex. Let y be the other vertex of V_1. Then any vertex v of V_2 lies on an x-y geodesic x, v, y and so $\{y\}$ is a connected x-geodominating set of $K_{2,n}$. Thus $cg_x(K_{2,n}) = 1$.

Let $x \in V_2$ be any vertex. Clearly $S = V_2 - \{x\}$ is the set of all eccentric vertices of x. By Theorem 2.6(ii), $cg_x(K_{2,n}) \geq n - 1$. Then any vertex v of V_1 lies on the geodesic x, v, u where $u \in S$ so that S is an x-geodominating set of $K_{2,n}$. Since $n \geq 3$, the induced subgraph $G[S]$ is disconnected so that $cg_x(K_{2,n}) > n - 1$. Now, the induced subgraph $G[S \cup \{w\}]$ is connected for any vertex w in V_1 and so $cg_x(K_{2,n}) = n$.

(iii) The proof is similar to the second part of the proof of (ii) □

Theorem 2.9.

(i) *If T is any tree of order p, then $cg_x(T) = p$ for any cut vertex x of T.*
(ii) *If T is any tree of order p which is not a path, then for an end vertex x, $cg_x(T) = p - d(x, y)$, where y is the vertex of T with deg $y \geq 3$ such that $d(x, y)$ is minimum.*
(iii) *If T is a path, then $cg_x(T) = 1$ for any end vertex x of T.*

Proof. (i) Let x be a cut vertex of T and let S be any connected x-geodominating set of T. By Theorem 2.6(i), every connected x-geodominating set of T contains all simplicial vertices. If $S \neq V(T)$, there exists a cut vertex v of T such that $v \notin S$. Let u and w be two end vertices belonging to different components of $T - \{v\}$. Since v lies on the unique path joining u and w, it follows that the subgraph $G[S]$ induced by S is disconnected, which is a contradiction. Hence $cg_x(T) = p$.

(ii) Let T be a tree which is not a path and x an end vertex of T. Let $S = (V(T) - I[x, y]) \cup \{y\}$. Clearly S is a connected x-geodominating set of T and so $cg_x(T) \leq |S| = p - d(x, y)$. We claim that $cg_x(T) = p - d(x, y)$. Otherwise, there is a connected x-geodominating set M of T with $|M| < p - d(x, y)$. By Theorem 2.6(i), every connected x-geodominating set of T contains all simplicial vertices except possibly x and hence there exists a cut vertex v of T such that $v \in S$ and $v \notin M$. Let $B_1, B_2, \ldots, B_m (m \geq 3)$ be the components of $T - \{y\}$. Assume that x belongs to B_1.

Case 1. Suppose $v = y$. Let $z \in B_2$ and $w \in B_3$ be two end vertices of T. By Theorem 1.1, v lies on the unique z-w geodesic. Since z and w belong to M and $v \notin M, G[M]$ is not connected, which is a contradiction.

Case 2. Suppose $v \neq y$. Let $v \in B_i (i \neq 1)$. Now, choose an end vertex $u \in B_i$ such that v lies on the y-u geodesic. Let $a \in B_j (j \neq i, 1)$ be an end vertex of T. By Theorem 1.1, y lies on the u-a geodesic. Hence it follows that

v lies on the u-a geodesic. Since u and a belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.

(iii) Let T be a path. For an end vertex x in T, let y be the eccentric vertex of x. Clearly every vertex of T lies on the x-y geodesic and so $\{y\}$ is a connected x-geodominating set of T so that $cg_x(T) = 1$. □

Corollary 2.10. *For any tree $T, cg_x(T) = p$ if and only if x is a cut vertex of T.*

Proof. This follows from Theorem 2.9. □

Theorem 2.11. *For any vertex x in a connected graph G, $1 \leq cg_x(G) \leq p$.*

Proof. Since $V(G)$ induces a connected x-geodominating set of G, it follows that $cg_x(G) \leq p$. Also it is clear that $cg_x(G) \geq 1$ and so $1 \leq cg_x(G) \leq p$. □

Remark 2.12. *The bounds for $cg_x(G)$ in Theorem 2.11 are sharp. For the even cycle C_{2n}, $cg_x(C_{2n}) = 1$ for any vertex x. Also, for any non-trivial path P_n, $cg_x(P_n) = 1$ for any end vertex x. For any path $P_n(n \geq 3)$, $cg_x(P_n) = n$ for any cut vertex x.*

Theorem 2.13. *Let x be any vertex of a connected graph G. Then the following are equivalent:*

(i) $cg_x(G) = 1$
(ii) $g_x(G) = 1$
(iii) *There exists a vertex y such that every vertex of G is on a diametral path joining x and y.*

Proof. (i) ⇒ (ii) Let $cg_x(G) = 1$. By Corollary 2.5, $g_x(G) \leq cg_x(G) = 1$ and so $g_x(G) = 1$.

(ii) ⇒ (iii) Let $g_x(G) = 1$. Let $S = \{y\}$ be the g_x-set of G. If $d(x, y) < d(G)$, then there exist vertices u and v on distinct geodesics joining x and y such that $d(u, v) = d(G)$. Thus $d(x, y) < d(u, v)$. Hence we see that

$$d(x, y) = d(x, u) + d(u, y) \quad (1)$$
$$d(x, y) = d(x, v) + d(v, y) \quad (2)$$

By triangle inequality,

$d(u, v) \leq d(u, x) + d(x, v)$ and $d(u, v) \leq d(u, y) + d(y, v)$ (3)

From (1) and (3), $d(u, y) = d(x, y) - d(x, u)$
$< d(u, v) - d(x, u)$
$\leq d(x, v)$

Thus $d(u, y) < d(x, v)$ (4)

Now from (2),(3) and (4), we see that $d(u, v) < d(x, v) + d(y, v)$
$= d(x, v) + d(v, y)$
$= d(x, y)$

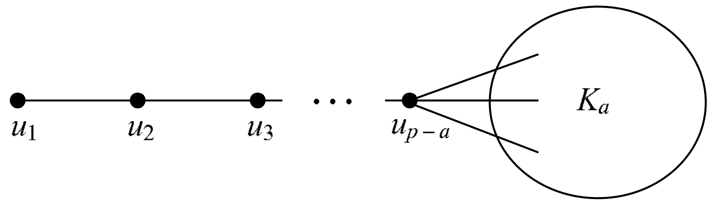
Thus $d(u, v) < d(x, y)$, which is a contradiction. Hence $d(x, y) = d(G)$ and each vertex of G is on a diametral path joining x and y.

(iii)⇒ (i) Let y be a vertex of G such that every vertex of G is on a diametral path joining x and y. Then $\{y\}$ is a connected x-geodominating set of G so that $cg_x(G) = 1$. □

We proved (Corollary 2.5) that $g_x(G) \leq cg_x(G)$ for any vertex x in G. The following theorem gives a realization for these parameters when $2 \leq a \leq b \leq p-1$.

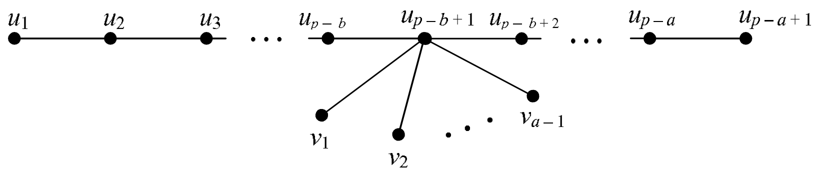
Theorem 2.14. *If p, a and b are positive integers such that $2 \leq a \leq b \leq p-1$, then there exists a connected graph G of order p, $g_x(G) = a$ and $cg_x(G) = b$ for some vertex x in G.*

Proof. We prove this theorem by considering two cases.
Case 1. $2 \leq a = b \leq p-1$. Let $P_{p-a} : u_1, u_2, \ldots, u_{p-a}$ be a path of order $p-a$ and K_a be the complete graph of order a. Let G be the graph obtained by joining u_{p-a} to every vertex of K_a and it is shown in Figure 2.2.



G
Figure 2.2

Then G is of order p and has $a+1$ simplicial vertices $\{u_1\} \cup V(K_a)$. By Theorem 1.2(i), the g_x-set of G contains $V(K_a)$ for $x = u_1$ and hence $g_x(G) \geq a$. Now, every vertex $u_i (1 \leq i \leq p-a)$ lies on the x-v geodesic for some $v \in V(K_a)$, it follows that $V(K_a)$ is an x-geodominating set of G and so $g_x(G) = a$. Also, since K_a is connected, $cg_x(G) = a$.



G
Figure 2.3

Case 2. $2 \leq a < b \leq p-1$. Let $P_{p-a+1} : u_1, u_2, \ldots, u_{p-a+1}$ be a path of order $p-a+1$. Add $a-1$ new vertices $v_1, v_2, \ldots, v_{a-1}$ to P_{p-a+1} and join these to u_{p-b+1}, there by producing the tree G of Figure 2.3. Then G is of order p with $a+1$ pendant vertices. For the vertex $x=u_1$, $g_x(G)=a$ by Theorem 1.3 and $cg_x(G)=b$ by Theorem 2.9(ii). □

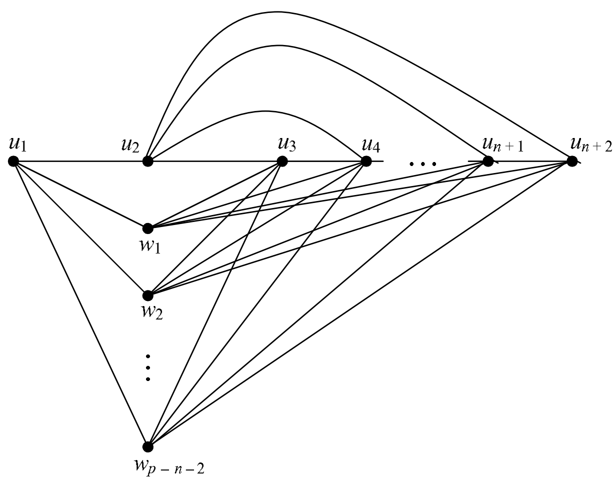
In the following, we construct a graph of prescribed order, diameter and connected vertex geodomination number under some conditions.

Theorem 2.15. *If p, d and n are positive integers such that $2 \leq d \leq p-2$ and $1 \leq n \leq p$, then there exists a connected graph G of order p, diameter d and $cg_x(G)=n$ for some vertex x in G.*

Proof. We prove this theorem by considering two cases.
Case 1. Let $d=2$. If $n=p-1$ or p, then take $G=K_{1,p-1}$. By Theorem 2.9, $cg_x(G)=p-1$ or p according as x is an end vertex or the cut vertex. Now we consider two cases. First let $n=1$. Let G be the complete bipartite graph $K_{2,p-2}$ with partite sets $X=\{u_1,u_2\}$ and $Y=\{w_1,w_2,\ldots,w_{p-2}\}$. Then G has order p and diameter $d=2$. For the vertex $x=u_1$, clearly $\{u_2\}$ is a connected x-geodominating set of G so that $cg_x(G)=1$.

Now let $2 \leq n \leq p-2$. Let $P_{n+2}: u_1,u_2,\ldots,u_{n+2}$ be the path of order $n+2$. Join u_2 with $u_4,u_5,\ldots,u_{n+2}$. Now add $p-n-2$ new vertices $w_1,w_2,\ldots,w_{p-n-2}$ to P_{n+2}. Let G be the graph obtained by joining each $w_i (1 \leq i \leq p-n-2)$ to $u_i (i=1,3,4,\ldots,n+2)$. The graph G is shown in Figure 2.4. Then G has order p and diameter $d=2$.

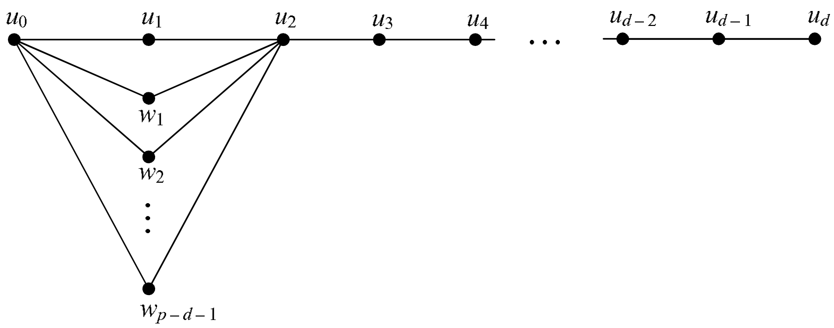
Let $x=u_1$ and let $S=\{u_3,u_4,\ldots,u_{n+2}\}$. Then S is the set of all eccentric vertices of x in G. By Theorem 2.6(ii), S is a subset of every connected x-geodominating set of G and so $g_x(G) \geq |S|=n$. Clearly the induced subgraph $G[S]$ is connected and so $cg_x(G)=|S|=n$.



G
Figure 2.4

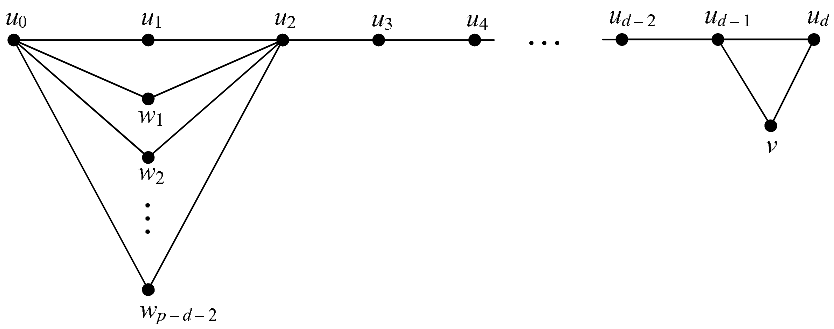
Case 2. Let $3 \leq d \leq p-2$. Let $P_{d+1} : u_0, u_1, u_2, \ldots, u_d$ be a path of length d.

Subcase 1. Let $n = 1$. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-1}$ to P_{d+1} and join these to both u_0 and u_2, there by producing the graph G of Figure 2.5. Then G has order p and diameter d. For the vertex $x = u_0$, clearly $\{u_d\}$ is a connected x-geodominating set of G so that $cg_x(G) = 1$.



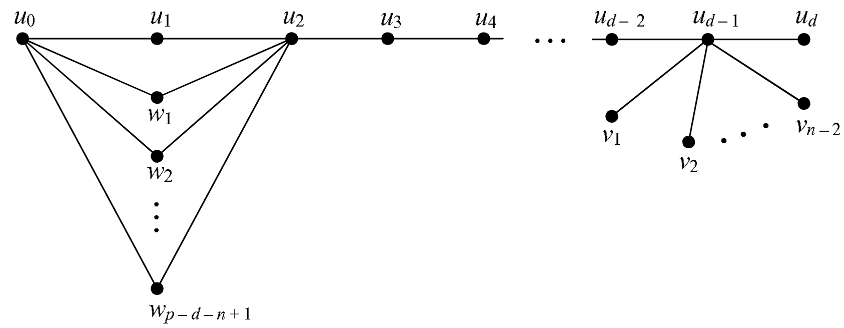
G
Figure 2.5

Subcase 2. Let $n = 2$. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-2}, v$ to P_{d+1} and join $w_1, w_2, \ldots, w_{p-d-2}$ to both u_0 and u_2 and join v to both u_{d-1} and u_d, there by producing the graph G of Figure 2.6. Then G has order p and diameter d. For the vertex $x = u_0$, clearly $\{u_d, v\}$ is the cg_x-set so that $cg_x(G) = 2$.



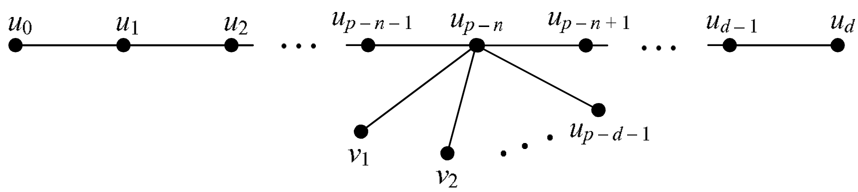
G
Figure 2.6

Subcase 3. Let $3 \leq n \leq p-1$. We consider two cases. If $n \leq p-d$, then add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-n+1}, v_1, v_2, \ldots, v_{n-2}$ to P_{d+1} and join $w_1, w_2, \ldots, w_{p-d-n+1}$ to both u_0 and u_2 and join $v_1, v_2, \ldots, v_{n-2}$ to u_{d-1}, there by producing the graph G of Figure 2.7. Then G has order p and diameter d. Clearly $S = \{u_d, v_1, v_2, \ldots, v_{n-2}\}$ is the set of all simplicial vertices of G. Let $x = u_0$. By Theorem 2.6(i), $cg_x(G) \geq |S| = n-1$. Since the induced subgraph $G[S]$ is not connected, $cg_x(G) > |S| = n-1$. Let $S' = S \cup \{u_{d-1}\}$. Then S' is an x-geodominating set of G and $G[S']$ is connected so that $cg_x(G) = |S'| = n$.



G
Figure 2.7

If $n > p-d$, then add $p-d-1$ new vertices $v_1, v_2, \ldots, v_{p-d-1}$ to P_{d+1} and join each $v_i (1 \leq i \leq p-d-1)$ to u_{p-n}, there by producing the graph G of Figure 2.8. Since G is a tree, by Theorem 2.9(ii), $cg_x(G) = p-(p-n) = n$ for the vertex $x = u_0$.



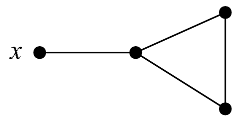
G
Figure 2.8

Subcase 4. Let $n = p$. Let G be any tree of order p and diameter d. Then for any cut vertex x in G, $cg_x(G) = p$, by Theorem 2.9(i). □

For every connected graph G, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [6] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Ostrand's theorem can be extended so that the connected vertex geodomination number can also be prescribed.

Theorem 2.16. *For positive integers r, d and n with $r \leq d \leq 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $cg_x(G) = n$ for some vertex x in G.*

Proof. If $r = 1$, then $d = 1$ or 2. If $d = 1$, let $G = K_{n+1}$. Then by Theorem 2.7(i), $cg_x(G) = n$ for any vertex x in G. Let $d = 2$. For $n = 1$, take $G = P_3$ so that for an end vertex x, $cg_x(G) = 1$. For $n = 2$, consider the graph G given in Figure 2.9. Then $cg_x(G) = 2$ for the vertex x in G. For $n \geq 3$, let $G = K_{1,n}$. Then by Theorem 2.9(ii), $cg_x(G) = n$ for an end vertex x in G.



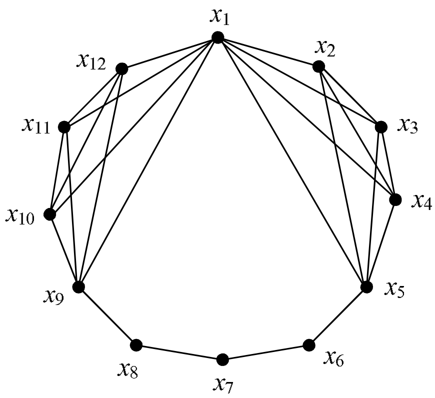
G
Figure 2.9

Now, let $r \geq 2$. We construct a graph G with the desired properties as follows:

Case 1. Suppose $r = d$. For $n = 1$, let $G = C_{2r}$. Then $r = d$ and by Theorem 2.7(ii), $cg_x(G) = 1$ for any vertex x in G. For $n = 2$, let $G = C_{2r+1}$. Then $r = d$ and by Theorem 2.7(ii), $cg_x(G) = 2$ for any vertex x in G. Now, let $n \geq 3$. Let $l = 2\left\lceil \frac{n}{2} \right\rceil - 1$ and $p = l + 2r - 1$. Then l is odd, p is even and $3 \leq l \leq p - 3$. Let $m = (l+3)/2, k = p - (l-1)/2$ and $s = p/2 + 1$. Then it is clear that $2 < m < s < k < p$. Let $C : x_1, x_2, \ldots, x_m, \ldots, x_s, \ldots, x_k, \ldots, x_p, x_1$ be an even cycle. Let G be the graph obtained from C by joining every pair of vertices of $\{x_1, x_2, \ldots, x_m\}$ and also every pair of vertices of $\{x_k, x_{k+1} \ldots, x_p, x_1\}$. The graph G is shown in Figure 2.10 for $n = 7$ and $r = 3$. It is to be noted that for any fixed $r \geq 2$, when $n \geq 3$ and n is odd the graph G constructed as above is same for $n, n+1$. Then $S = \{x_2, x_3, \ldots, x_{m-1}, x_{k+1}, x_{k+2}, \ldots, x_p\}$ is the set of all simplicial vertices of G with $|S| = l - 1$. It is easily verified that the eccentricity of each vertex of G is r so that *rad* $G =$ *diam* $G = r$. Now, we shall consider two subcases.

Subcase 1. Suppose n is odd. Then $l = n$. Let $x = x_s$. Clearly $d(x, x_1) = r = e(x)$ and hence x_1 is an eccentric vertex of x in G. Let $T = S \cup \{x_1\}$. By Theorem 2.6, $cg_x(G) \geq |S| + 1 = l$. It is clear that $x = x_s, x_{s-1}, \ldots, x_m, x_1$ is an x-x_1 geodesic and also $x = x_s, x_{s+1}, \ldots, x_k, x_1$ is an x-x_1 geodesic so that T is an x-geodominating set of G. Clearly the induced subgraph $G[T]$ is connected and so $cg_x(G) = l = n$.

Subcase 2. Suppose n is even. Then $l = n - 1$. Let $x = x_{s+1}$. Clearly $d(x, x_m) = r = e(x)$ and hence x_m is an eccentric vertex of x in G. Let $T = S \cup \{x_m\}$. By Theorem 2.6, $cg_x(G) \geq |S| + 1 = l$. It is clear that $x = x_{s+1}, x_s, \ldots, x_m$ is an x-x_m geodesic and also $x = x_{s+1}, x_{s+2}, \ldots, x_k, x_1, x_m$ is



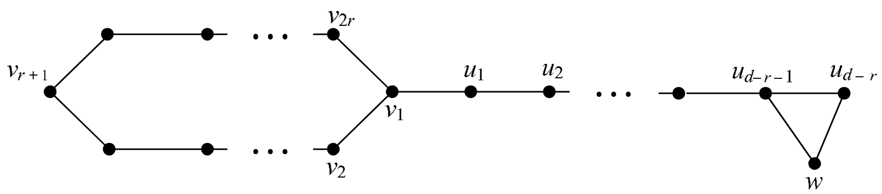
G
Figure 2.10

an x-x_m geodesic so that T is an x-geodominating set of G. Since the induced subgraph $G[T]$ is disconnected we see that $cg_x(G) > l$. Let $T' = T \cup \{x_1\}$. Then T' is a connected x-geodominating set of G and so $cg_x(G) = l + 1 = n$.

Case 2. Suppose $r < d \leq 2r$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}.

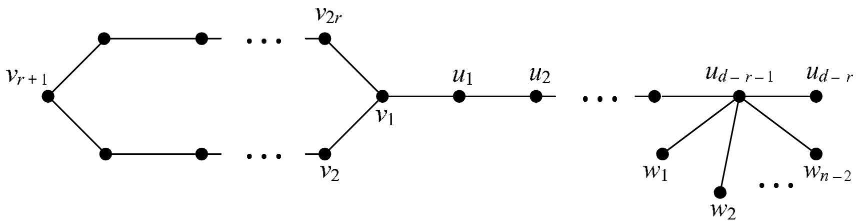
For $n = 1$, let $G = H$. Then for $x = v_{r+1}$, the set $\{u_{d-r}\}$ is a connected x-geodominating set of G so that $cg_x(G) = 1$.

For $n = 2$, we add a new vertex w to H and join w to both u_{d-r-1} and u_{d-r} and obtain the graph G of Figure 2.11. Then $rad\ G = r$ and $diam\ G = d$. The set $S = \{w, u_{d-r}\}$ is the set of all simplicial vertices of G. For the vertex $x = v_{r+1}$, it is clear that S is a connected x-geodominating set of G and so by Theorem 2.6(i), $cg_x(G) = 2$.



G
Figure 2.11

For $n \geq 3$, we add $n - 2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to H and join each vertex $w_i (1 \leq i \leq n - 2)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 2.12.



G
Figure 2.12

Then $rad\ G = r$ and $diam\ G = d$. The set $S = \{u_{d-r}, w_1, w_2, \ldots, w_{n-2}\}$ is the set of all simplicial vertices of G. For the vertex $x = v_{r+1}$, it is clear that S is an x-geodominating set of G. Since the induced subgraph $G[S]$ is disconnected we see that $cg_x(G) > |S| = n - 1$. Let $T = S \cup \{u_{d-r-1}\}$. Then T is a connected x-geodominating set of G so that $cg_x(G) = n$. □

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