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C^2 RATIONAL QUINTIC FUNCTION

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Abstract. A two-parameter family of piecewise C^2 rational quintic functions is presented along with an error investigation for the approximation of an arbitrary C^3 function. The two parameters have a direct geometric interpretation making their use straightforward. Illustrations of their effect on the shape of the rational function are given. The relaxed continuity constraints and the increased flexibility via the two parameters make the proposed function a suitable candidate for interactive CAD.

Key words: C^2 rational quintic function, free parameters, peono kernel.
AMS SUBJECT: 68U05, 65D05, 65D07, 65D18.

1. Introduction

Existing rational functions are either limited in the order of continuity or restrictive to the form of the data they can be applied to. To overcome some of the limitations of existing rational forms, the aim of this paper is to construct a family of piecewise C^2 rational quintic functions that can be applied to a range of data and which possess parameters for shape refinement. The shape control given by the C^2 continuity across piecewise segments is useful in the field of CADCAM and the flexibility of the two free parameters in each piecewise segment makes the proposed form suitable for the construction of curves and the description of geometric shapes within a computer-based environment.

Rational functions have become more popular in the construction of curves and surfaces in CADCAM because of their increased flexibility and their shape

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preserving ability [1-9, 11]. Duan *et. al.* [2] developed a piecewise rational cubic spline with constraints on first derivatives at knots to preserve C^2 continuity. Sarfraz *et. al.* [6] used the piecewise C^1 rational cubic function developed by Delbourgo and Gregory [1] to preserve the shape of positive data. Since their rational function has only a single parameter there is no freedom for curve modification and hence the method is not suitable for interactive curve design. Sarfraz [7] developed a rational cubic function with quadratic denominator that involves two free parameters. A general solution was obtained for points in N-space. The rational function in [7] attained C^2 continuity by imposing constraints on first derivatives at the knots and is unable to interpolate the data with specified derivatives. Sarfraz *et. al.* [9] developed a rational cubic function with cubic denominator which involves four free parameters but the order of continuity achieved was only C^1 making it unsuitable for CAD applications that require G^2 continuity.

The paper is organised as follows: The piecewise rational quintic function is developed and illustrated in Section 2. An investigation of the approximation error to an arbitrary C^3 function is presented in Section 3 by using the technique of [2]. Finally, the characteristics of the proposed C^2 rational function and its suitability for use in interactive curve and surface construction are discussed in the conclusions.

2. C^2 Rational Quintic Function

Let $(t_i,\ f_i),\ i = 0, 1, 2, \ldots, n$ be the given set of data points, $t_0 < t_1 < t_2 < \cdots < t_n$ and $t_i, f_i \in \Re$. The general form of piecewise rational quintic function is:

$$P(t)|_{[t_i,\ t_{i+1}]} = \frac{p_i(\theta)}{q_i(\theta)}, \tag{1}$$

with

$$\begin{aligned} p_i(\theta) &= (1-\theta)^5 U_i + (1-\theta)^4 \theta V_i + (1-\theta)^3 \theta^2 W_i + (1-\theta)^2 \theta^3 X_i \\ &\quad + (1-\theta)\theta^4 Y_i + \theta^5 Z_i, \\ q_i(\theta) &= \alpha_i (1-\theta) + \beta_i \theta, \end{aligned}$$

where $\alpha_i,\ \beta_i \in \Re$ are scalar parameters and $\theta = \frac{t-t_i}{h_i}$, $h_i = t_{i+1} - t_i$.
To ensure the rational quintic function (1) is C^2, the following interpolation properties are imposed:

$$\begin{aligned} P(t_i) &= f_i, & P(t_{i+1}) &= f_{i+1}, \\ P'(t_i) &= d_i, & P'(t_{i+1}) &= d_{i+1}, \\ P''(t_i) &= D_i, & P''(t_{i+1}) &= D_{i+1}, \end{aligned}$$

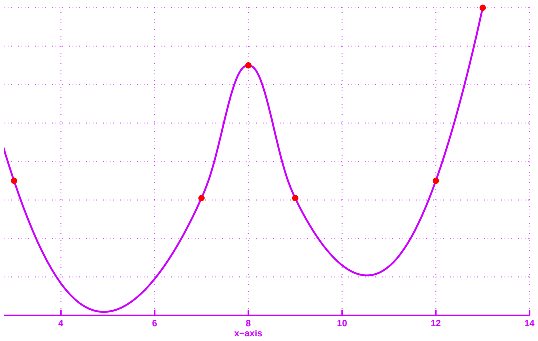

FIGURE 1. Quintic polynomial ($\alpha_i = \beta_i = 1.0$).

resulting in:

$$\begin{aligned}
U_i &= \alpha_i f_i, \\
V_i &= (4\alpha_i + \beta_i)f_i + h_i\alpha_i d_i, \\
W_i &= (6\alpha_i + 4\beta_i)f_i + (3\alpha_i + \beta_i)h_i d_i + \alpha_i \frac{h_i^2}{2} D_i, \\
X_i &= (4\alpha_i + 6\beta_i)f_{i+1} - (\alpha_i + 3\beta_i)h_i d_{i+1} + \beta_i \frac{h_i^2}{2} D_{i+1}, \\
Y_i &= (\alpha_i + 4\beta_i)f_{i+1} - h_i\beta_i d_{i+1}, \\
Z_i &= \beta_i f_{i+1},
\end{aligned}$$

where d_i and D_i denote first and second derivatives $w.r.t\ t$ at knot t_i.

Some Observations.

(1) When $\alpha_i = \beta_i = 1$, (1) reduces to a non-rational quintic polynomial.
(2) As $\alpha_i \to \infty$ or $\beta_i \to \infty$, (1) reduces (continuously) to an interpolating C^1 quartic polynomial , which is independent α_i and β_i.
(3) As $\alpha_i \to 0$ ($\beta_i \to 0$), (1) reduces to a non-rational interpolating quartic polynomial with one free parameter $\beta_i(\alpha_i)$.

To illustrate these observations consider the following data pairs taken from [8] with slight change:
$\{(t_i, f_i) : (2, 10), (3, 1), (7, 0.1), (8, 7), (9, 0.1), (12, 1), (13, 10)\}$.

Fig. 1 is generated using (1) with $\alpha_i = \beta_i = 1$. In this case the rational quintic function reduces to a non-rational quintic polynomial. As can be seen from Fig. 1, the resulting interpolation is visually smooth with no unnecessary undulations. Fig. 2 is generated using $\alpha_i = 0,\ \beta_i = 0.05$, resulting in a non-rational quartic polynomial with one free parameter β_i. Again the resultant curve is visually pleasing. Fig. 3 is generated with $\alpha_i = 50,\ \beta_i = 0$, resulting in

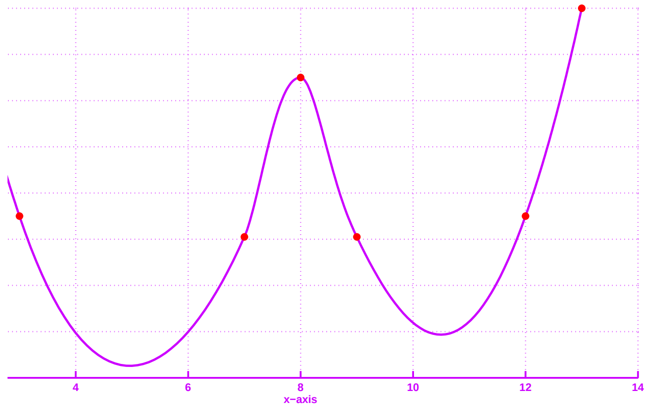

FIGURE 2. Quartic polynomial ($\alpha_i = 0.0$).

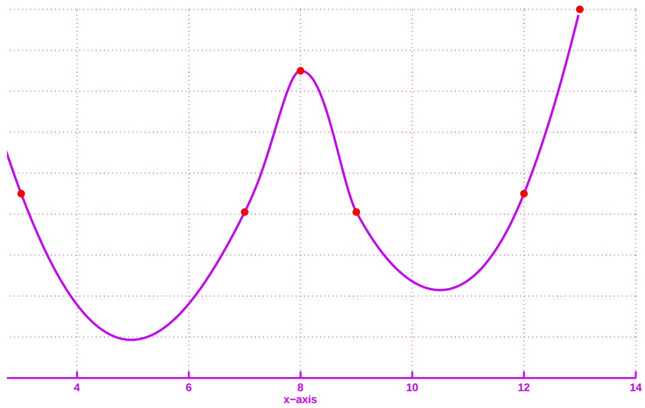

FIGURE 3. Quartic polynomial ($\beta_i = 0.0$).

a non-rational quartic polynomial with one free parameter α_i. Its shape control behaviour is similar to Fig. 2. The smooth curve in Fig. 4 is generated by (1) with $\alpha_i = 0.1,\ \beta_i = 1000$. For these values, (1) reduces to a non-rational quartic polynomial where the influence of the parameters is negligible and the functions behaves as if independent of α_i and β_i. Fig. 5 is generated using (1) with $\alpha_i = 4$ and $\beta_i = 3$, resulting in a rational quintic function which is smooth and exhibits no unnecessary undulations.

It is observed that the rational quintic function does not exhibit local or global tension behaviour.

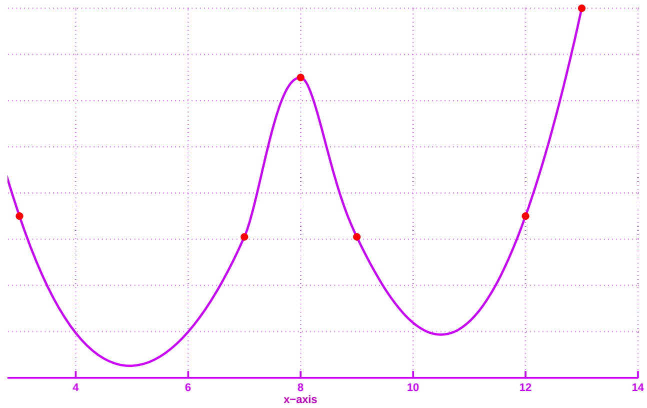

FIGURE 4. Quartic polynomial ($\alpha_i = 0.1,\ \beta_i = 1000.0$).

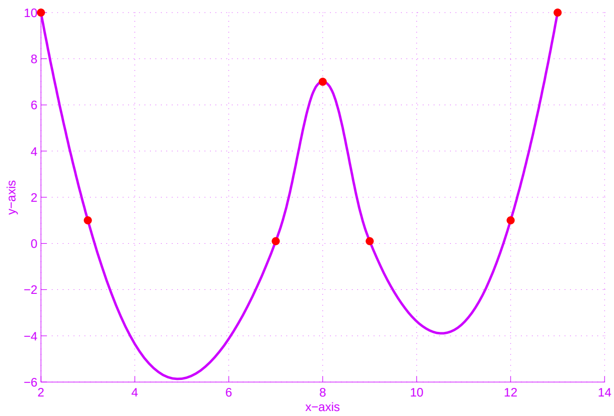

FIGURE 5. Rational quintic function ($\alpha_i = 4,\ \beta_i = 3$).

3. ERROR ESTIMATION OF INTERPOLATION

This section investigates the estimation of the approximation error incurred when the rational quintic function (1) is used to interpolate data from an arbitrary function that is $C^3[t_0,\ t_n]$. Locality of the interpolation allows error estimation in each subinterval $[t_i, t_{i+1}]$ without loss of generality.

The Peano Kernel theorem [10] is used to estimate the error in each subinterval $[t_i,\ t_{i+1}]$ as:

$$R[f] = f(t) - P(t) \quad = \quad \frac{1}{2}\int_{t_i}^{t_{i+1}} f^{(3)}(\tau)R_t[(t-\tau)_+^2]d\tau. \tag{2}$$

Using the uniform norm, (2) becomes:

$$|f(t) - P(t)| \le \frac{1}{2}\|f^{(3)}(\tau)\| \int_{t_i}^{t_{i+1}} |R_t[(t-\tau)_+^2]|d\tau, \tag{3}$$

where

$$R_t[(t-\tau)_+^2] = \left\{ \begin{array}{ll} r(\tau,t), & t_i < \tau < t, \\ s(\tau,t), & t < \tau < t_{i+1}, \end{array} \right\},$$

$$\begin{aligned} r(\tau,t) \;=\;& [(t-\tau)^2[(1-\theta)^2\{\alpha_i(1+\theta+\theta^2) + \beta_i\theta(1+\theta) + \beta_i\theta^2 - 3(\alpha_i+\beta_i)\theta^3\}] \\ & -2h_i(t-\tau)(1-\theta)^3\theta^3(3\alpha_i+\beta_i) - h_i^2(1-\theta)^3\theta^3[(2\alpha_i+\beta_i)\theta \\ & -3(\alpha_i+\beta_i)]]/q_i(\theta),\ t_i < \tau < t, \end{aligned}$$

and

$$\begin{aligned} s(\tau,t) \;=\;& \frac{-\theta^3}{q_i(\theta)}[[(1-\theta)^2(4\alpha_i+6\beta_i) + \theta(1-\theta)(\alpha_i+4\beta_i) + \theta^2\beta_i](t_{i+1}-\tau)^2 \\ & -2h_i(t_{i+1}-\tau)[(1-\theta)^2(\alpha_i+3\beta_i) + \beta_i(1-\theta)\theta] + h_i^2\beta_i(1-\theta)^2], \\ & t < \tau < t_{i+1}, \end{aligned}$$

$R_t[(t-\tau)_+^2]$ is called Peano Kernel. In (3), $\int_{t_i}^{t_{i+1}} |R_t[(t-\tau)_+^2]|d\tau$ can be expressed as:

$$\int_{t_i}^{t_{i+1}} |R_t[(t-\tau)_+^2]|d\tau \quad = \quad \int_{t_i}^{t} |r(\tau,t)|d\tau + \int_{t}^{t_{i+1}} |s(\tau,t)|d\tau.$$

The roots of $r(t,t) = 0$ and $s(t,t) = 0$ are: $0,\ 1,\ \frac{2\alpha_i+\beta_i}{3(\alpha_i+\beta_i)}$. These roots lie in $[0,1]$, $\forall\ \alpha_i > 0$ and $\beta_i > 0$.

The roots of $r(\tau,t) = 0$ are

$$\tau_i \quad = \quad t - \frac{h_i\theta(G + (-1)^{i+1}H)}{E}, i = 1, 2,$$

where

$$\begin{aligned} G &= 3(\alpha_i+\beta_i)\theta^2, \\ H &= \sqrt{\alpha_i^2\theta + (\alpha_i\beta_i + \alpha_i)(\theta+\theta^2) + \theta^2\beta_i^2} \\ E &= \alpha_i + (2\alpha_i+\beta_i)\theta + G. \end{aligned}$$

The roots of $s(\tau,t) = 0$ are

$$\tau_i \quad = \quad t - \frac{h_i(1-\theta)(K + (-1)^{i+1}L)}{F}, i = 3, 4,$$

where

$$
\begin{aligned}
K &= (\alpha_i + 3\beta_i)(1-\theta) + \beta_i\theta, \\
L &= \sqrt{(1-\theta)\{(1-\theta)(\alpha_i^2 + 3\beta_i^2 + 2\alpha_i\beta_i) + \theta(2\beta_i^2 + \alpha_i\beta_i)\}}, \\
F &= K + (3\alpha_i + 3\beta_i)(1-\theta)^2.
\end{aligned}
$$

Case 1: $0 \le \theta \le \frac{2\alpha_i+\beta_i}{3(\alpha_i+\beta_i)}$, (3) takes the form

$$
\begin{aligned}
|f(t) - P(t)| &\le \frac{1}{2}\|f^{(3)}(\tau)\| h_i^3 \omega_1(\alpha_i, \beta_i, \theta), \\
\omega_1(\alpha_i, \beta_i, \theta) &= \int_{t_i}^{t} |r(\tau, t)| d\tau + \int_{t}^{t_{i+1}} |s(\tau, t)| d\tau = -\int_{t_i}^{\tau_1} r(\tau, t) d\tau + \int_{\tau_1}^{\tau_2} r(\tau, t) d\tau \\
&\quad - \int_{\tau_2}^{t} r(\tau, t) d\tau - \int_{t}^{\tau_3} s(\tau, t) d\tau + \int_{\tau_3}^{\tau_4} s(\tau, t) d\tau - \int_{\tau_4}^{t_{i+1}} s(\tau, t) d\tau
\end{aligned}
$$

Direct integration yields:

$$
\omega_1(\alpha_i, \beta_i, \theta) = \frac{h_i^3 (1-\theta)^3 \theta^3}{(1-\theta)\alpha_i + \theta\beta_i} \sum_{i=0}^{7} U_i, \qquad (4)
$$

with

$$
\begin{aligned}
U_0 &= \frac{(12G^2H + 4H^3)}{3E^2}, \\
U_1 &= \frac{8(3\alpha_i + 3\beta_i)\theta^2 GH}{E^2}, \\
U_2 &= \frac{-4h_i H\theta((2\alpha_i + \beta_i) - \theta(3\alpha_i + 3\beta_i))}{E}, \\
U_3 &= -\frac{E}{3} + \theta(2\alpha_i + \beta_i) + \theta(\theta - 1)(3\alpha_i + 3\beta_i), \\
U_4 &= \frac{F}{3} - (\alpha_i + 2\beta_i)(1-\theta), \\
U_5 &= \frac{-(12K^2L + 4L^3)}{F^2}, \\
U_6 &= \frac{8KL((\alpha_i + 3\beta_i) - \theta(\alpha_i + 2\beta_i))}{F^2}, \\
U_7 &= \frac{-4\beta_i L}{F^2}.
\end{aligned}
$$

Case 2: $\frac{2\alpha_i+\beta_i}{3(\alpha_i+\beta_i)} \le \theta \le 1$, (3) takes the form

$$
|f(t) - P(t)| \le \frac{1}{2}\|f^{(3)}(\tau)\| h_i^3 \omega_2(\alpha_i, \beta_i, \theta),
$$

where

$$\begin{aligned}\omega_2(\alpha_i,\beta_i,\theta) &= \int_{t_i}^{t} |r(\tau,t)|d\tau + \int_{t}^{t_{i+1}} |s(\tau,t)|d\tau = \int_{t_i}^{\tau_1} r(\tau,t)d\tau - \int_{\tau_1}^{\tau_2} r(\tau,t)d\tau \\ &+ \int_{\tau_2}^{t} r(\tau,t)d\tau + \int_{t}^{\tau_3} s(\tau,t)d\tau - \int_{\tau_3}^{\tau_4} s(\tau,t)d\tau + \int_{\tau_4}^{t_{i+1}} s(\tau,t)d\tau\end{aligned}$$

Again, direct integration yields:

$$\omega_2(\alpha_i,\beta_i,\theta) = -\omega_1(\alpha_i,\beta_i,\theta). \quad (5)$$

The above can be summarized as:

Theorem 1. *The error of rational quintic function defined in (1) for* $f(t) \in C^3[t_0,t_n]$*, in each subinterval* $[t_i,t_{i+1}]$ *is*

$$|f(t) - P(t)| \le \|f^{(3)}(\tau)\| h_i^3 c_i,$$

$$c_i = \max_{0\le\theta\le 1} \omega(\alpha_i,\beta_i,\theta),$$

$$\omega(\alpha_i,\beta_i,\theta) = \left\{ \begin{array}{ll} \max \omega_1(\alpha_i,\beta_i,\theta), & 0 \le \theta \le \frac{2\alpha_i+\beta_i}{3(\alpha_i+\beta_i)}, \\ \max \omega_2(\alpha_i,\beta_i,\theta), & \frac{2\alpha_i+\beta_i}{3(\alpha_i+\beta_i)} \le \theta \le 1, \end{array} \right\}$$

for all $\alpha_i,\ \beta_i > 0$*, where* $\omega_1(\alpha_i,\beta_i,\theta)$ *and* $\omega_2(\alpha_i,\beta_i,\theta)$ *are defined in (4) and (5) respectively.*

4. Conclusion and Future Work

A C^2 piecewise rational quintic function with two free parameters (α_i,β_i) has been developed where the parameters can be used for shape refinement. The effect of free parameters on the shape of curve is illustrated in Figs. 1-5. For very high and low values of (α_i,β_i) the rational quintic function (1) reduces to a C^2 smooth non-rational polynomial curve. Existing rational schemes [2, 7] preserved C^2 continuity by imposing constraints on the first derivatives at the knots and hence are not applicable to data with derivatives, for example when interpolating numerical solutions of differential equations. The proposed scheme is equally applicable to data when first as well as second derivatives are prescribed. The approximation order of rational quintic function was investigated and found to be $O(h_i^3)$. The proposed rational function has been shown to produce visually acceptable interpolatory curves and is worthy of further consideration. Extension of the proposed function to be shape preserving is currently under investigation.

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