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HYBRID FUNCTIONS APPROACH FOR SOLVING FREDHOLM AND VOLTERRA INTEGRAL EQUATIONS

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Abstract. This paper presents a computational technique for Fredholm and Volterra integral equations of the second kind. The method based upon hybrid functions approximation. The properties of hybrid functions consisting of block-pulse functions and legendre polynomials are presented. The operational matrices of integration and product are utilized to reduce the computation of integral equation into some algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Key words: hybrid, operational matrix, fredholm, volterra, integral equations.
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1. Introduction

There are three classes of sets of orthogonal functions which are widely used. The first includes sets of piecewise constant basis functions(e.g.,Walsh [1], block-pulse[2], Rationalized Haar[3], etc.). The second consists of sets of orthogonal polynomials(e.g., Laguerre [4], Legendre [5], Chebyshev [6], etc.). The third is the widely used sets of sine-cosine functions in Fourier series [7]. While orthogonal polynomials and sine-cosine functions together form a class of continuous basis functions, piecewise constant basis functions (PCBFs) have inherent discontinuities or jumps. It is worth noting that approximating a continuous function with PCBFs results in an approximation that is piecewise constant.

Orthogonal functions have been used to solve various problems of dynamic systems. The main characteristic of this technique is that it reduces these

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problems to those of solving a system of algebraic equations thus greatly simplifying the problem.

In the present paper we introduce a new numerical method to solve Fredholm integral equations of the second kind and Volterra integral equations of the second kind. This method consists of reducing the integral equation to a set of algebraic equations by expanding $y(t)$ as hybrid functions with unknown coefficients. These hybrid functions, which consist of block-pulse functions plus Legendre polynomials are given.

The operational matrix of integration is introduced, this matrix together with the product operational matrix are then utilized to evaluate the unknown coefficients.

The paper is organized as follows: In section 2 we describe the basic formulation of the hybrid functions of block-pulse and Legendre polynomials required for our subsequent development. Section 3 is devoted to the formulation of the Fredholm integral equations of the second kind and the proposed method is used to approximate the unknown function $y(t)$. In section 4 we describe the Volterra integral equations of the second kind and the proposed method is used to approximate the unknown function $y(t)$. In section 5, we report our numerical finding and demonstrate the accuracy of the proposed numerical scheme by considering two numerical examples.

2. Properties of Hybrid Functions

2.1. Hybrid Functions of Block Pulse and Legendre Polynomials. Hybrid functions $b_{nm}(t)$, $n = 1, \ldots, N$, $m = 0, 1, \ldots, M-1$, have three arguments; n and m are the order of block-pulse functions and Legendre polynomials respectively. They are defined on the interval $[0, t_f)$ as

$$b_{nm}(t) = \begin{cases} P_m(\frac{2N}{t_f}t - 2n + 1), & t \in \left[\frac{n-1}{N}t_f,\ \frac{n}{N}t_f\right) \\ \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Here, $P_m(t)$ are the well-known Legendre polynomials of order m which satisfy the following recursive formula on the interval $[-1, 1]$

$$P_0(t) = 1,\ P_1(t) = t \tag{2}$$

$$P_{m+1}(t) = \frac{2m+1}{m+1}tP_m(t) - \frac{m}{m+1}P_{m-1}(t), \qquad m = 1, 2, 3, \ldots \tag{3}$$

From the Eq. (1), it is clear that the set of hybrid functions are orthogonal.

2.2. **Function Approximation.** A function $f(t)$, defined over the interval 0 to t_f may be expanded as

$$f(t) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} b_{nm}(t) = C^T B(t), \tag{4}$$

where

$$C = [c_{10}, \ldots, c_{1(M-1)}, c_{20}, \ldots, c_{2(M-1)}, \ldots, c_{N0}, \ldots, c_{N(M-1)}]^T \tag{5}$$

and

$$B(t) = [B_1^T(t), B_2^T(t), \ldots, B_N^T(t)]^T \tag{6}$$

where

$$B_i(t) = [b_{i0}(t), b_{i1}(t), \ldots, b_{i(M-1)}(t)]^T, \qquad i = 1, 2, \ldots, N$$

In Eq. (4) c_{nm} is given as

$$c_{nm} = \frac{(f(t), b_{nm}(t))}{(b_{nm}(t), b_{nm}(t))} \tag{7}$$

In Eq. (7) (.,.) denotes the inner product and defined as

$$u(t), v(t)) = \int_0^{t_f} u(t)v(t)\mathrm{d}t.$$

We can also approximate the function $g(t, s) \in L^2([0, 1] \times [0, 1])$ as follows

$$g(t, s) = B^T(t)GB(s), \tag{8}$$

where $G = (g_{ij})$ is an $MN \times MN$ matrix such that

$$g_{ij} = \frac{(B_i(t), g(t, s), B_j(s))}{(B_i(t), B_i(t))\,(B_j(s), B_j(s))}, \qquad i = 1, \ldots, N, \quad j = 0, 1, \ldots, M-1 \tag{9}$$

2.3. **Operational Matrix of Integration.** The integration of the vector $B(t)$ defined in Eq.(6) can be approximated by

$$\int_0^t B(t^{'})dt^{'} \simeq PB(t), \tag{10}$$

where P is the $MN \times MN$ operational matrix for integration and is given [8] as:

$$P = \begin{pmatrix} E & H & H & \cdots & H \\ 0 & E & H & \cdots & H \\ 0 & 0 & E & \cdots & H \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & E \end{pmatrix} \tag{11}$$

where

$$H = \text{diagonal}\left[\frac{t_f}{N}, 0, \ldots, 0\right]$$

and

$$E = \frac{t_f}{2N}\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M-1} & 0 \end{pmatrix}$$

3. The Product Operational Matrix of The Hybrid of Block-Pulse and Legendre Polynomials

The following property of the product of two hybrid function vectors will also be used. Let

$$B(t)B^T(t)C \simeq \widetilde{C}B(t), \tag{12}$$

where C and $B(t)$ are given in Eqs. (5) and (6), respectively. Also $\widetilde{C}$ is a $MN \times MN$ product operational matrix as follows:

$$\widetilde{C} = \text{diagonal}\left[\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_N\right]. \tag{13}$$

In Eq. (13), 0 denotes $M \times M$-dimensional zero matrix and $\widetilde{C}_i$, $i = 1, 2, \ldots, N$ are $M \times M$ matrices depending on M. For example, if we choose $M = 3$, then

$\widetilde{C}_i$, $i = 1, \ldots, N$ are 3×3 matrices given by

$$\widetilde{C}_i = \begin{pmatrix} c_{i0} & c_{i1} & c_{i2} \\ \frac{1}{3}c_{i1} & c_{i0} + \frac{2}{5}c_{i2} & \frac{2}{3}c_{i1} \\ \frac{1}{5}c_{i2} & \frac{2}{5}c_{i1} & c_{i0} + \frac{2}{7}c_{i2} \end{pmatrix}.$$

Also, for $M = 4$ we have

$$\widetilde{C}_i = \begin{pmatrix} c_{i0} & c_{i1} & c_{i2} & c_{i3} \\ \frac{1}{3}c_{i1} & c_{i0} + \frac{2}{5}c_{i2} & \frac{2}{3}c_{i1} + \frac{3}{7}c_{i3} & 0 \\ \frac{1}{5}c_{i2} & \frac{2}{5}c_{i1} + \frac{9}{35}c_{i3} & c_{i0} + \frac{2}{7}c_{i2} & \frac{4}{15}c_{i3} \\ \frac{1}{7}c_{i3} & \frac{9}{35}c_{i2} & \frac{3}{7}c_{i1} + \frac{4}{21}c_{i3} & c_{i0} + \frac{4}{15}c_{i2} \end{pmatrix}, \qquad i = 1, \ldots, N.$$

Furthermore, the integration of cross-product of two hybrid functions vector is

$$L = \int_0^{t_f} B(t)B^T(t)dt. \tag{14}$$

In Eq. (14) L is a $MN \times MN$ diagonal matrix given by

$$L = \text{diagonal}\left[D, \ldots, D\right],$$

and D is a $M \times M$ diagonal matrix given by

$$D = \frac{t_f}{N}\text{diagonal}\left[1, \frac{1}{3}, \ldots, \frac{1}{2M-1}\right].$$

4. Fredholm Integral Equation of The Second Kind

Consider the following integral equation

$$y(t) = \int_0^1 k(t,s)y(s)ds + x(t), \tag{15}$$

where $x(t) \in L^2[0,1]$, $k(t,s) \in L^2([0,1] \times [0,1])$. The problem is to find an unknown function $y(t)$, satisfying Eq.(15). To solve for $y(t)$, we let

$$y(t) = Y^T B(t), \tag{16}$$

where $B(t)$ is given by Eq. (6) and Y is an unknown $MN \times 1$ vector. Furthermore, let

$$k(t,s) = B^T(t)KB(s), \tag{17}$$

$$x(t) = X^T B(t), \tag{18}$$

where $K \in R^{MN\times MN}$is the hybrid functions coefficient matrix given by Eq. (8) and X is a known $MN \times 1$ vector given by Eq. (4). By substituting Eqs. (16)-(18) in Eq. (15) we get

$$B^T(t)Y = \int_0^1 B^T(t)KB(s)B^T(s)Y ds + B^T(t)X. \tag{19}$$

Using Eqs. (14) and (19) we obtain

$$B^T(t)Y = B^T(t)KLY + B^T(t)X, \tag{20}$$

therefore

$$(I - KL)Y - X = 0, \tag{21}$$

where I is $MN \times MN$-dimensional identity matrix. Eq. (21) is a system of linear equations and can be solved for the unknown vector Y, easily.

5. Volterra Integral Equation of The Second Kind

Consider the following integral equation

$$y(t) = \int_0^t k(t,s)y(s)\mathrm{d}s + x(t), \tag{22}$$

where $x(t) \in L^2[0,1]$, $k(t,s) \in L^2([0,1]\times[0,1])$ and $y(t)$ is an unknown function. we expand $y(t)$ in hybrid functions as

$$y(t) = Y^T B(t), \tag{23}$$

where Y is an unknown vector of order $MN \times 1$ and $B(t)$ is given by Eq. (6). We also expand $k(t,s)$ and $x(t)$ in hybrid functions as Eqs. (17) and (18), respectively. By substituting Eqs. (17), (18) and (23) in Eq. (22) we get

$$Y^T B(t) = \int_0^t B^T(t)K\tilde{Y}B(s)\mathrm{d}s + X^T B(t) = B^T(t)K\tilde{Y}PB(t) + X^T B(t) \tag{24}$$

where $\tilde{Y}$ can be calculated similarly to matrix $\tilde{C}$ in Eq. (12) and P is given by Eq. (10). In order to construct the approximations for $y(t)$ we collocate Eq. (24) in MN points. For a suitable collocation points we choose Legendre polynomials nodes as $t_i,\ i = 1,2\ldots,MN$. These nodes are the roots of differential $(M-1)$th of hybrid functions $b_{nm}(t)$ for $n = 1,\ldots,N$, $m = 0,1,\ldots,M-1$. Furthermore, we define the square hybrid matrix as

$$H_{MN} = [B(t_1), B(t_2), \ldots, B(t_{MN})]. \tag{25}$$

In Eq. (25) H_{MN} is a $MN \times MN$-dimensional matrix given by

$$H_{MN} = \text{diagonal}[A, \ldots, A].$$

where 0 denotes $M \times M$-dimensional zero matrix and A is a $M \times M$ matrix depending on M. For example, if we choose $M = 3$, then A is a 3×3 matrix given by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -\frac{1}{2} & 1 \end{pmatrix}.$$

Furthermore, if we choose $M = 4$ then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -0.447214 & 0.447214 & 1 \\ 1 & -0.2 & -0.2 & 1 \\ -1 & 0.447214 & -0.447214 & 1 \end{pmatrix}.$$

With these comments and by using Eqs. (6) and (25) we have

$$B(t_i) = H_{MN} e_i, \qquad i = 1, 2, \ldots, MN \tag{26}$$

where

$$e_i = \Big(\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{MN-i}\Big)^T.$$

Now, equation (24) can be expressed as

$$Y^T H_{MN} e_i = e_i^T H_{MN}^T K \tilde{Y} P H_{MN} e_i + X^T H_{MN} e_i, \qquad i = 1, 2, \ldots, MN. \tag{27}$$

Obviously, Eq. (27) can be solved for the unknown vector Y.

6. Illustrative Examples

Two examples are given in this section. These examples were considered in [3] by using Haar Wavelets. Our methods differs from their approach and thus these examples could be used as a basis for comparison.

6.1. **Example 1.**

Consider the Fredholm integral equation of the second kind[3]

$$y(t) = \int_0^1 \left(-\frac{1}{3} e^{2t - \frac{5}{3}s}\right) y(s) \mathrm{d}s + e^{2t + \frac{1}{3}} \tag{28}$$

The analytic solution for $y(t)$ is [3]

$$y(t) = e^{2t}$$

By using the method in section 3, Eq. (28) is solved. In Table I, the results for $y(t)$ using hybrid functions of block pulse and Legendre polynomials together Rationalized Haar functions[3] are listed.

Table I.
Absolute error of the approximate solution of Example 1.

	Hybrid functions			Haar wavelets[3]
t	M=3, N=4	M=3, N=10	M=4, N=4	m=128
0.0	0.26E − 4	0.66E − 6	0.34E − 7	0.4E − 4
0.1	0.60E − 3	0.80E − 6	0.32E − 4	0.7E − 4
0.2	0.14E − 2	0.98E − 6	0.22E − 4	0.2E − 4
0.3	0.20E − 2	0.12E − 5	0.34E − 4	0.6E − 4
0.4	0.11E − 2	0.15E − 5	0.54E − 4	0.3E − 4
0.5	0.70E − 4	0.18E − 5	0.93E − 7	0.6E − 4
0.6	0.16E − 2	0.22E − 5	0.87E − 4	0.5E − 4
0.7	0.37E − 2	0.27E − 5	0.60E − 4	0.9E − 4
0.8	0.54E − 2	0.33E − 5	0.93E − 4	0.3E − 4
0.9	0.31E − 2	0.40E − 5	0.15E − 4	0.4E − 4
1.0	0.19E − 3	0.49E − 5	0.25E − 6	0.5E − 4

6.2. **Example 2.** Consider the following integral equation[3]

$$y(t) = \cos t - \int_0^t (t-s)\cos(t-s)\, y(s)\mathrm{d}s. \tag{29}$$

The exact solution for this problem is[3]

$$y(t) = \frac{1}{3}\left(2\cos\sqrt{3}t + 1\right)$$

We solve Eq. (29) using the method in section 4. In Table 2, a comparison is made between the computational results for hybrid functions together by rationalized Haar functions.

7. Conclusion

The hybrid functions operational matrices of integration and product are used to solve Fredholm and Volterra integral equations of the second kind. The problem has been reduced to solving a system of algebraic equations. The method can be implemented on a digital computer. It occupies less memory space and consumes less computer time than method in[3]. Illustrative examples are included to demonstrate the validity and applicability of the technique.

Table II.
Absolute error of the approximate solution of Example 2.

	Hybrid functions			Haar wavelets[3]
t	M=3, N=4	M=3, N=10	M=4, N=4	m=128
0.0	0.14E − 3	0.18E − 5	0.65E − 5	0.6E − 5
0.1	0.18E − 4	0.44E − 5	0.12E − 4	0.6E − 4
0.2	0.14E − 3	0.31E − 5	0.62E − 5	0.2E − 4
0.3	0.35E − 3	0.55E − 5	0.56E − 5	0.8E − 4
0.4	0.20E − 3	0.14E − 4	0.10E − 4	0.5E − 4
0.5	0.29E − 3	0.91E − 5	0.35E − 5	0.2E − 4
0.6	0.21E − 3	0.20E − 4	0.60E − 5	0.1E − 4
0.7	0.55E − 3	0.95E − 5	0.24E − 5	0.3E − 4
0.8	0.59E − 3	0.99E − 5	0.12E − 5	0.1E − 4
0.9	0.30E − 3	0.23E − 4	0.19E − 5	0.2E − 4
1.0	0.28E − 3	0.11E − 4	0.35E − 5	0.1E − 4

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