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ON THE RAMSEY NUMBER FOR PATHS AND BEADED WHEELS

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Abstract. For given graphs G and H, the *Ramsey number* $R(G, H)$ is the least natural number n such that for every graph F of order n the following condition holds: either F contains G or the complement of F contains H. Beaded wheel $BW_{2,m}$ is a graph of order $2m + 1$ which is obtained by inserting a new vertex in each spoke of the wheel W_m. In this paper, we determine the Ramsey number of paths versus Beaded wheels: $R(P_n, BW_{2,m}) = 2n - 1$ or $2n$ if $m \geq 3$ is even or odd, respectively, provided $n \geq 2m^2 - 5m + 4$.

Key words: ramsey number, path, beaded wheel.
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1. Introduction

Let $G(V, E)$ be a graph with the vertex-set $V(G)$ and edge-set $E(G)$. If $xy \in E(G)$ then x is called *adjacent* to y, and y is a *neighbor* of x and vice versa. For any $A \subseteq V(G)$, we use $N_A(x)$ to denote the set of all neighbors of x in A, namely $N_A(x) = \{y \in A | xy \in E(G)\}$. Let P_n be a path with n vertices, C_n be a cycle with n vertices, W_k be a wheel with $k+1$ vertices, i.e., a graph consisting of a cycle C_k with one additional vertex adjacent to all vertices of C_k. For $m \geq 3$, the *Beaded wheel* $BW_{2,m}$ is a graph with $2m+1$ vertices which is obtained by inserting one vertex in each spoke of the wheel W_m. The hub of W_m is also called the hub of $BW_{2,m}$. For example, Figure 1 shows Beaded

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wheel $BW_{2,12}$.

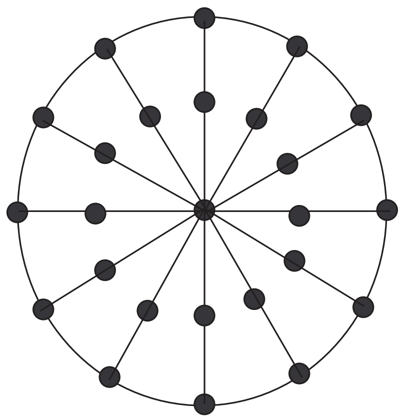

FIGURE 1. Beaded wheel $BW_{2,12}$

Surahmat and Baskoro [1] determined the Ramsey number of a combination of P_n versus a wheel W_k, as follows.

Theorem A [1].

$$R(P_n, W_k) = \begin{cases} 2n-1 & \textit{if } k \geq 4 \textit{ is even and } n \geq \frac{k}{2}(k-2), \\ 3n-2 & \textit{if } k \geq 5 \textit{ is odd and } n \geq \frac{k-1}{2}(k-3). \end{cases}$$

Other papers concerning Ramsey numbers of paths versus wheel related graphs are [2-4,6,7]; a nice survey paper on Ramsey numbers is [5].

In this paper, we determine the Ramsey numbers involving paths and Beaded wheels $BW_{2,m}$ as follows.

Theorem 1

$$R(P_n, BW_{2,m}) = \begin{cases} 2n-1 & \textit{if } m \geq 4 \textit{ is even and } n \geq 2m^2-5m+4, \\ 2n & \textit{if } m \geq 3 \textit{ is odd and } n \geq 2m^2-5m+3. \end{cases}$$

2. The Proof of Theorem 1.

First we shall show that $R(P_n, BW_{2,m}) \geq 2n - 1$ if $m \geq 4$ is even and $R(P_n, BW_{2,m}) \geq 2n$ if $m \geq 3$ is odd.
[1]

Consider the graph $F_1 = 2K_{n-1}$. It is clear that $F_1 \not\supseteq P_n$ nor $\overline{F_1} \supseteq BW_{2,m}$ since $\overline{F_1} \cong K_{n-1,n-1}$ is bipartite but $BW_{2,m}$ is not, which implies $R(P_n, BW_{2,m}) \geq 2n - 1$. By taking $F_2 = K_1 \cup 2K_{n-1}$ we also deduce that $F_2 \not\supseteq P_n$. $\overline{F_2}$ is the join $K_1 + K_{n-1,n-1}$ and every odd cycle of $\overline{F_2}$ contains the vertex of K_1. If m is odd, $m \geq 3$, for every vertex $x \in V(BW_{2,m})$ there exists an odd cycle in $BW_{2,m} - x$, hence $K_1 + K_{n-1,n-1} \not\supseteq BW_{2,m}$. It follows that if $m \geq 3$ is odd then $R(P_n, BW_{2,m}) \geq 2n$.

We will prove the opposite inequalities in the following cases: A. $m \geq 4$ is even and $n \geq 2m^2 - 5m + 4$ and B. $m \geq 3$ is odd and $n \geq 2m^2 - 5m + 3$.

A. Let F be a graph on $2n-1$ vertices containing no P_n. Let $L_1 = (l_{1,1}, l_{1,2}, \cdots, l_{1,k-1}, l_{1,k})$ be a longest path in F and so $k \leq n - 1$. If $k = 1$ we have $\overline{F} \cong K_{2n-1}$ which contains $BW_{2,m}$. Suppose that $k \geq 2$. We shall prove that $\overline{F}$ contains $BW_{2,m}$. Obviously, for each $z \in V_1$, where $V_1 = V(F) \backslash V(L_1)$, $zl_{1,1}, zl_{1,k} \notin E(F)$. Let $L_2 = (l_{2,1}, l_{2,2}, \cdots, l_{2,t-1}, l_{2,t})$ be a longest path in $F[V_1]$. It is clear that $1 \leq t \leq k$. Let $V_2 = V(F) \backslash (V(L_1) \cup V(L_2))$. Since $|V(F)| = 2n - 1$, there exists at least one vertex $x \in V_2$, which is not adjacent to any endpoint $l_{1,1}, l_{1,k}, l_{2,1}, l_{2,t}$. We distinguish three cases.

Case A1: $k < 2m - 2$. If $t = 1$ then the vertices in V_1 induce a subgraph having only isolated vertices. In this case we shall add an edge uv to F, where $u, v \in V_1$ and denote $L_2 = u, v$. In this way we can define inductively a system of paths $L_1, L_2, \cdots, L_m$ such that L_i is a longest path in $F[V_{i-1}]$, where $V_{i-1} = V(F) \setminus \bigcup_{j=1}^{i-1} V(L_j)$ or an edge added to F as above. If F_1 denotes the graph F or the graph F plus some edges added in the process of defining the system of paths, it follows that the endpoints of these L_j $(1 \leq j \leq m)$ induce a complete graph K_{2m} minus a matching with at most m edges in $\overline{F_1}$ if some of the endpoints of the same L_j are adjacent in F_1. If Y denotes the set of the remaining vertices, we have $|V(Y)| \geq 2n - 1 - m(2m - 3) > 1$. Let $x \in Y$ be a vertex which is not adjacent to any endpoint of these L_j for $1 \leq j \leq m$. It is easy to see that x together with all endpoints of paths L_j contains a $BW_{2,m} \subseteq \overline{F_1} \subseteq \overline{F}$ having the hub x.

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Case A2 : $k \geq 2m-2$ and $t \geq 2m-2$. For $1 \leq i \leq m-2$ we define the couples A_i in path L_1 as follows:

$$A_i = \begin{cases} \{l_{1,i+1}, l_{1,i+2}\} & \text{for } i \text{ odd,} \\ \{l_{1,k-i}, l_{1,k-i+1}\} & \text{for } i \text{ even }. \end{cases}$$

In a similar way let

$$B_i = \begin{cases} \{l_{2,i+1}, l_{2,i+2}\} & \text{for } i \text{ odd,} \\ \{l_{2,t-i}, l_{2,t-i+1}\} & \text{for } i \text{ even} \end{cases}$$

for the path L_2.

Since $t \leq k \leq n-1$ and $|V(F)| = 2n-1$, we have seen that there exists at least one vertex x which is not in $L_1 \cup L_2$. L_1 being a longest path in F, there exists one vertex of A_i for each i, say a_i, which is not adjacent with x. Similarly, there must be one vertex, say b_i in the couple B_i which is not adjacent to x for each i.
By the maximality of the path L_1 we have that a_ib_i for $1 \leq i \leq m-2$, b_ia_{i+1} for $1 \leq i \leq m-3$ and $a_ib_{i+\frac{m}{2}-1}$, $b_ia_{i+\frac{m}{2}-1}$ for $1 \leq i \leq \frac{m}{2}-1$ are not in $E(F)$.

If $m = 4$ then $C = \{l_{2,1}, l_{1,1}, l_{2,t}, l_{1,k}\}$ induces a cycle C_4 in $\overline{F}$. We define the set of inserted vertices $I = \{a_1, b_1, a_2, b_2\}$. Since $a_1l_{2,1}, b_1l_{1,1}$, $a_2l_{2,t}, b_2l_{1,k} \notin E(F)$ and x is not adjacent to any vertex from $C \cup I$ it follows that $C \cup I \cup \{x\}$ induces in $\overline{F}$ a subgraph containing $BW_{2,4}$ with the hub x.

Let $m \geq 6$.
In this case $C = \{l_{2,1}, l_{1,1}, l_{2,t}, a_{\frac{m}{2}+1}, b_{\frac{m}{2}+1}, \cdots, a_{m-2}, b_{m-2}, l_{1,k}\}$ will form in this order a cycle C_m in $\overline{F}$ and we define the set of inserted vertices as $I = \{a_1, b_1, a_2, b_2, \cdots, a_{\frac{m}{2}}, b_{\frac{m}{2}}\}$. Since x is not adjacent with any vertex of C and I, it follows that C, I together with x gives us a subgraph in $\overline{F}$ which contains $BW_{2,m}$ with the hub x, so $BW_{2,m} \subseteq \overline{F}$.

Case A3: $k \geq 2m-2$ and $t < 2m-2$. Since F has no P_n it follows that $k \leq n-1$, hence V_1 will have at least n vertices. Then we can define the same process as in case A1. We obtain a system of paths $L_2, \cdots, L_m$ in the subgraph induced by V_1 such that the endpoints of $L_1, \cdots, L_m$ induce in $\overline{F_1}$ a complete graph K_{2m} minus a matching having at most m edges. We get in this case $|V(Y)| \geq n-(m-1)(2m-3) \geq 1$ and the proof is similar to the case A1.

B. Let F be a graph on $2n$ vertices containing no P_n. With the same notation and reasoning as in the case A, since $|V(F)| = 2n$, there exist at least two vertices $x_1, x_2 \in V_2$ which are not adjacent to any endpoint $l_{1,1}, l_{1,k}, l_{2,1}, l_{2,t}$ of L_1 and L_2. We shall consider three cases.

Case B1: $k < 2m-2$. This case can be treated exactly in the same way as the case A1.

Case B2: $k \geq 2m-2$ and $t \geq 2m-2$. As for the case A2 for $i = 1, 2, \cdots, m-2$ we can define the couples A_i in L_1 and B_i in L_2. Since $t \leq k \leq n-1$ and $|V(F)| = 2n$, we have seen that there exist at least two vertices $x_1, x_2 \notin V(L_1) \cup V(L_2)$ which are not adjacent to any endpoint of L_1 or L_2.
In a similar manner to the case A2 we get vertices $a_i \in A_i$ and $b_i \in B_i$ for $1 \leq i \leq m-2$ which are not adjacent to x_1 for each i.

Let $m = 3$. In this case $C = \{x_2, l_{1,1}, l_{2,1}\}$ induces a cycle C_3 in $\overline{F}$. The set of inserted vertices is defined to be $I = \{l_{1,k}, b_1, a_1\}$ since by the maximality of L_1 we have $x_2 l_{1,k}, l_{1,1} b_1, l_{2,1} a_1 \notin E(F)$. Since x_1 is not adjacent to any vertex from I, it follows that $C \cup I \cup \{x_1\}$ induces in $\overline{F}$ a subgraph containing $BW_{2,3}$ with the hub x_1.

Let $m = 5$. In this case $C = \{l_{1,1}, x_2, l_{2,t}, a_2, b_3\}$ induces a cycle C_5 in $\overline{F}$. The set of inserted vertices is $I = \{l_{2,1}, y, a_3, b_2, l_{1,k}\}$, where y is a_1 or b_1: if $x_2 a_1 \notin E(F)$ then $y = a_1$ and otherwise, by the maximality of L_1, it follows that $x_2 b_1 \notin E(F)$ and we define $y = b_1$. Since $l_{1,1} l_{2,1}, x_2 y, l_{2,t} a_3, a_2 b_2, b_3 l_{1,k} \notin E(F)$ it follows that $C \cup I \cup \{x_1\}$ induces in $\overline{F}$ a subgraph containing $BW_{2,5}$ with the hub x_1.

If $m \geq 7$ by the maximality of L_1 we deduce that $a_i b_i$ for $1 \leq i \leq m-2$, $b_i a_{i+1}$ for $1 \leq i \leq m-3$ and $a_i b_{i+\frac{m-3}{2}}$, $b_i a_{i+\frac{m-3}{2}}$ for $1 \leq i \leq \frac{m+1}{2} - 1$ are not in $E(F)$.
Thus $C = \{l_{2,1}, x_2, l_{1,1}, l_{2,t}, a_{\frac{m+1}{2}+1}, b_{\frac{m+1}{2}+1}, \cdots, a_{m-2}, b_{m-2}, l_{1,k}\}$ induces in this order a cycle C_m in $\overline{F}$. It is clear that from the maximality of L_1 it follows that x_2 cannot be adjacent to both a_1 and b_1. If $x_2 a_1 \notin E(F)$ then we shall define the set of inserted vertices as $I = \{a_1, b_1, a_2, b_2, \cdots, a_{\frac{m+1}{2}-1}, b_{\frac{m+1}{2}-1}, a_{\frac{m+1}{2}}\}$.
If $x_2 a_1 \in E(F)$ then $x_2 b_1 \notin E(F)$ and we shall define $I = \{a_1, b_1, a_2, b_2, \cdots, a_{\frac{m+1}{2}-1}, b_{\frac{m+1}{2}-1}, b_{\frac{m+1}{2}}\}$.
In both cases we have $a_i b_{i+\frac{m-3}{2}}, b_i a_{i+\frac{m-3}{2}} \notin E(F)$ for $3 \leq i \leq \frac{m+1}{2} - 1$; $l_{1,k} b_2, l_{2,t} a_2 \notin E(F)$ by the maximality of L_1. In the first case ($x_2 a_1 \notin E(F)$) we also have $l_{2,1} a_{\frac{m+1}{2}} \notin E(F)$ since L_1 is maximal and $m \geq 7$ and $l_{1,1} b_1 \notin E(F)$. In the second case ($x_2 b_1 \notin E(F)$) we also have $l_{2,1} a_1, l_{1,1} b_{\frac{m+1}{2}} \notin E(F)$.
Since x_1 is not adjacent to any vertex of I, as in the case A2 we have obtained

a subgraph in $\overline{F}$ which contains $BW_{2,m}$ with the hub x_1, so $BW_{2,m} \subseteq \overline{F}$.

Case B3: $k \geq 2m-2$ and $t < 2m-2$. We deduce that $|V_1| \geq n+1$. As in the case A3 we get $|V(Y)| \geq n+1-(m-1)(2m-3) \geq 1$ and the remaining proof is analogous to the case B1. $\square$

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