Journal of Prime Research in Mathematics Vol. **5***(2009), 139-148*

EXACT SOLUTIONS OF GENERALIZED OLDROYD-B FLUID SUBJECT TO A TIME-DEPENDENT SHEAR STRESS IN A PIPE

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Abstract. The velocity field and the shear stress corresponding to the unsteady flow of a generalized Oldroyd-B fluid in an infinite circular cylinder subject to a longitudinal time-dependent shear stress are determined by means of Hankel and Laplace transforms. The exact solutions, written in terms of the generalized G-functions, satisfy all imposed initial and boundary conditions. The similar solutions for ordinary Oldroyd-B, ordinary and generalized Maxwell, ordinary and generalized second grade as well as for Newtonian fluids are obtained as limiting cases of our general solutions.

Key words: exact solutions, generalized Oldroyd-B fluid, longitudinal shear stress.
AMS SUBJECT: 76A05, 76A10

1. Introduction

Numerous models have been proposed to describe the response of fluids that cannot be characterized by the classical Navier-Stokes fluid model. The simplest constitutive equation for a fluid is a Newtonian one and the classical Navier-Stokes theory is based on this equation. The mechanical behavior of many fluids is well enough described by this theory. However, there are many rheologically complicated fluids such as polymer solutions, blood and certain oils, suspensions, liquid crystals in industrial processes with non-linear viscoelastic behavior that can not be described by a Newtonian constitutive equation, as it does not reflect any relaxation and retardation phenomena. For

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this reason, many models have been proposed. Among them, the models of differential type [1] and those of rate type [2] have received much attention. One of the rate type models, in which besides the upper-convected derivative of the stress tensor, also the upper-convected derivative of the rate of strain tensor is included, is the Oldroyd-B model [3]. This model has become very popular among rheologists in modeling the response of dilute polymeric solutions [4, 5]. It can describe many of the non-Newtonian characteristics exhibited by polymeric materials such as stress-relaxation, normal stress differences in simple shear flows and creep. Moreover, it is amenable to analysis and more importantly experimental.

Recently, the fractional calculus has encountered much success in the description of viscoelasticity. Especially, the rheological constitutive equations with fractional derivatives play an important role in the description of the behavior of the polymer solutions and melts. The constitutive equations corresponding to the generalized non-Newtonian fluids are obtained from those for non-Newtonian fluids by replacing the inner time derivatives of an integer order by the so called Riemann-Liouville fractional operators. More exactly, the ordinary derivatives of first, second or higher orders are replaced by fractional derivatives of non-integer order [6, 9-16].

The aim of this paper is to establish exact solutions for the unsteady flow of an incompressible generalized Oldroyd-B fluid (GOF) in an infinite circular cylinder subject to a longitudinal time-dependent shear stress. These solutions, obtained by means of Hankel and Laplace transforms, are presented under integral and series form in terms of the generalized G-functions. Similar solutions for the flow of Maxwell and second grade fluids with fractional derivative models as well as those for the ordinary models are obtained as limiting cases of our general solutions. Moreover, respective solutions for the flow of ordinary Oldroyd-B and Newtonian fluids are also achieved.

2. Governing Equation

The type of flows to be here considered has the velocity $\mathbf{V}$ and the extra-stress $\mathbf{S}$ of the form [7]

$$\mathbf{V} = \mathbf{V}(r,t) = v(r,t)\mathbf{e}_z, \qquad \mathbf{S} = \mathbf{S}(r,t), \tag{1}$$

where $\mathbf{e}_z$ is the unit vector along the z-direction of the cylindrical coordinate system r, θ and z. For such flows the constraint of incompressibility is automatically satisfied. Furthermore, if the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{V}(r,0) = \mathbf{0}, \quad \mathbf{S}(r,0) = \mathbf{0}, \tag{2}$$

The constitutive equations corresponding to incompressible Oldroyd-B fluids and the balance of linear momentum, in the absence of body forces and pressure gradient in the flow direction, lead to $S_{rr} = S_{r\theta} = S_{\theta\theta} = S_{\theta z} = 0$ and the relevant equations [7]

$$(1 + \lambda\partial_t)\tau(r,t) = \mu(1 + \lambda_r\partial_t)\partial_r v(r,t), \qquad \rho\partial_t v(r,t) = \left(\partial_r + \frac{1}{r}\right)\tau(r,t), \quad (3)$$

where μ is the dynamic viscosity, ρ is the constant density of the fluid, λ and λ_r are relaxation and retardation times and $\tau(r,t) = S_{rz}(r,t)$ is the tangential stress, which is different of zero. Eliminating $\tau(r,t)$ between Eqs. (3) we attain to the governing equation (cf. [7, Eq. (2.5)] or [8, Eq. (4.1)])

$$(1 + \lambda\partial_t)\partial_t v(r,t) = \nu(1 + \lambda_r\partial_t)\left(\partial_r + \frac{1}{r}\right)\partial_r v(r,t), \quad (4)$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

The governing equations corresponding to an incompressible generalized Oldroyd-B fluid (GOF) are obtained from Eqs. $(3)_1$ and (4) by replacing the inner derivatives with respect to t by the fractional differential operators D_t^γ and D_t^β $(\beta \geq \gamma)$, defined by [6]

$$D_t^p f(t) = \begin{cases} \frac{1}{\Gamma(1-p)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^p}d\tau; & 0 < p < 1, \\ \frac{d}{dt}f(t); & p = 1, \end{cases} \quad (5)$$

where $\Gamma(\cdot)$ is the Gamma function. More exactly, the governing equations to be used here are (cf. [9, Eqs. (7) and (10)]) or [10, Eqs. (2) and (23)]

$$(1 + \lambda D_t^\gamma)\tau(r,t) = \mu(1 + \lambda_r D_t^\beta)\partial_r v(r,t), \quad (6)$$

$$(1 + \lambda D_t^\gamma)\partial_t v(r,t) = \nu(1 + \lambda_r D_t^\beta)\left(\partial_r + \frac{1}{r}\right)\partial_r v(r,t), \quad (7)$$

where the new material constants λ and λ_r have the dimensions of t^γ and t^β, respectively. In some recent papers (see [11], for instance), the authors use λ^γ and λ_r^β instead of λ and λ_r into their constitutive equations. However, for simplicity, we are keeping the same notations although these material constants have different significations in Eqs. $(3)_1$, (4) and (6), (7), respectively.

3. Axial Couette Flow Through an Infinite Circular Cylinder

Let us consider an incompressible GOF at rest in an infinite circular cylinder of radius R. At time $t = 0^+$, a time-dependent shear stress

$$\tau(R,t) = \frac{f}{\lambda}G_{\gamma,-2,1}\left(-\frac{1}{\lambda}, t\right); \quad t > 0, \quad (8)$$

is created at the boundary of the cylinder, where f is a constant and [20, Eq. (101)]

$$G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{d^j\beta(c+j)}{\beta(c)\beta(j+1)}\frac{t^{(c+j)a-b-1}}{\beta[(c+j)a-b]}. \tag{9}$$

Owing to the shear, the fluid is gradually moved, its velocity being of the form of Eq. $(1)_1$ and the governing equation is (7). The appropriate initial and boundary conditions are

$$v(r,0)=\partial_t v(r,0)=\tau(r,0)=0\,;\qquad r\in[0,R), \tag{10}$$

$$(1+\lambda D_t^{\gamma})\tau(R,t)=\mu(1+\lambda_r D_t^{\beta})\partial_r v(R,t)=ft;\quad t\geq 0\,. \tag{11}$$

Of course, $\tau(R,t)$ given by Eq. (8) is just the solution of the fractional differential equation $(11)_1$. To solve this problem we shall use as in [9, 10, 18, 19] the Laplace and Hankel transforms.

3.1. **Calculation of the Velocity Field.** Having in mind the initial conditions (10), apply the Laplace transform to Eqs. (7) and $(11)_2$ and using the Laplace transform formula for fractional derivatives [6], we obtain

$$(q+\lambda q^{\gamma+1})\overline{v}(r,q)=\nu(1+\lambda_r q^{\beta})\left(\partial_r+\frac{1}{r}\right)\partial_r\overline{v}(r,q);\quad r\in(0,R), \tag{12}$$

$$\partial_r\overline{v}(R,q)=\frac{f}{\mu q^2(1+\lambda_r q^{\beta})}, \tag{13}$$

where $\overline{v}(r,q)=\int_0^{\infty}v(r,t)\exp(-qt)dt$ is the Laplace transform of $v(r,t)$ and q is the transform parameter. In the following we denote by [17]

$$\overline{v}_H(r_n,q)=\int_0^R r\overline{v}(r,q)J_0(rr_n)dr\,,$$

the finite Hankel transform of $\overline{v}(r,q)$, where $J_0(\cdot)$ is the Bessel function of first kind of order zero and $r_n,\ n=1,2,3,\cdots$ are the positive roots of the transcendental equation $J_1(Rr)=0$.

Multiplying both sides of Eq. (12) by $rJ_0(rr_n)$, integrating with respect to r from 0 to R and taking into account the condition (13) and using the relation [17, Eq. (13.4.31)]

$$\int_0^R r\left[\partial_r^2\overline{v}(r,q)+\frac{1}{r}\partial_r\overline{v}(r,q)\right]J_0(rr_n)dr$$
$$=RJ_0(Rr_n)\partial_r\overline{v}(R,q)-r_n^2\overline{v}_H(r_n,q)\,, \tag{14}$$

we find that

$$\overline{v}_H(r_n,q)=\frac{RfJ_0(Rr_n)}{\rho}\frac{1}{q^2[q+\lambda\, q^{\gamma+1}+\nu r_n^2(1+\lambda_r q^\beta)]}\,. \tag{15}$$

Now, for a suitable presentation of the final results, we rewrite $\overline{v}_H(r_n,q)$ in the following equivalent form

$$\overline{v}_H(r_n,q)=\frac{RfJ_0(Rr_n)}{\mu r_n^2}\frac{1}{q^2(1+\lambda_r q^\beta)}$$

$$-\frac{RfJ_0(Rr_n)}{\mu r_n^2}\frac{1+\lambda q^\gamma}{q(1+\lambda_r q^\beta)[q+\lambda\, q^{\gamma+1}+\nu r_n^2(1+\lambda_r q^\beta)]}. \tag{16}$$

Applying the inverse Hankel transform [17, Eq. (13.4.30)] to Eq. (16) and taking into consideration the fact that the finite Hankel transform of r^2 is $2R^2J_0(Rr_n)/r_n^2$, we find that

$$\overline{v}(r,q)=\frac{fr^2}{2\mu R}\,\frac{1}{q^2(1+\lambda_r q^\beta)}-\frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2J_0(Rr_n)}$$

$$\times\frac{q^{-1}}{1+\lambda_r q^\beta}\ \frac{1+\lambda q^\gamma}{q+\lambda\, q^{\gamma+1}+\nu r_n^2(1+\lambda_r q^\beta)}. \tag{17}$$

In order to obtain the velocity field $v(r,t)=L^{-1}\{\overline{v}(r,q)\}$ and to avoid the burdensome calculations of residues and contour integrals, we apply the discrete Laplace transform method [9-16]. However, we firstly rewrite the last factor of Eq. (17) in the following equivalent form

$$\frac{(1+\lambda q^\gamma)}{q+\lambda\, q^{\gamma+1}+\nu r_n^2(1+\lambda_r q^\beta)}$$

$$=\frac{1}{\lambda}\sum_{k=0}^{\infty}\sum_{m=0}^{k}C_k^m\lambda_r^m\left(\frac{-\nu r_n^2}{\lambda}\right)^k\ \frac{q^{\beta m-k-1}+\lambda\, q^{\gamma+\beta m-k-1}}{\left(q^\gamma+\frac{1}{\lambda}\right)^{k+1}}, \tag{18}$$

where $C_k^m=k!/\left(m!(k-m)!\right)$ is the binomial coefficient. Introducing (18) into (17), applying the discrete inverse Laplace transform and using the convolution theorem of Laplace transform as well as the formula [20]

$$L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\}=G_{a,b,c}(d,t);\ \ \mathrm{Re}\,(ac-b)>0,\ \mathrm{Re}\,(q)>0,\ \left|\frac{d}{q^a}\right|<1, \tag{19}$$

we find for $v(r,t)$ the expression

$$v(r,t) = \frac{fr^2}{2\mu\lambda_r R} G_{\beta,-2,1}\left(-\frac{1}{\lambda_r}, t\right) - \frac{2f}{\mu\lambda\lambda_r R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}$$
$$\times \sum_{k=0}^{\infty}\sum_{m=0}^{k} C_k^m \lambda_r^m \left(\frac{-\nu r_n^2}{\lambda}\right)^k \int_0^t G_{\beta,-1,1}\left(-\frac{1}{\lambda_r}, t-\tau\right)$$
$$\times \left[G_{\gamma,\beta m-k-1,k+1}\left(-\frac{1}{\lambda},\tau\right) + \lambda G_{\gamma,\gamma+\beta m-k-1,k+1}\left(-\frac{1}{\lambda},\tau\right)\right] d\tau. \quad (20)$$

3.2. **Calculation of the Shear Stress.** Applying the Laplace transform to Eq. (6), it results that

$$\overline{\tau}(r,q) = \mu\frac{(1+\lambda_r q^\beta)}{(1+\lambda q^\gamma)}\partial_r \overline{v}(r,q). \quad (21)$$

Differentiating Eq. (17) with respect to r, we find that

$$\partial_r \overline{v}(r,q) = \frac{fr}{\mu R}\frac{1}{q^2(1+\lambda_r q^\beta)} + \frac{2f}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}$$
$$\times \frac{1}{1+\lambda_r q^\beta}\;\frac{1+\lambda q^\gamma}{q[q+\lambda\, q^{\gamma+1}+\nu r_n^2(1+\lambda_r q^\beta)]}. \quad (22)$$

Introducing (22) into (21), we get

$$\overline{\tau}(r,q) = \frac{fr}{R}\frac{1}{q^2(1+\lambda q^\gamma)} + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}$$
$$\times \frac{1}{q[q+\lambda\, q^{\gamma+1}+\nu r_n^2(1+\lambda_r q^\beta)]}, \quad (23)$$

Applying again the discrete inverse Laplace transform to Eq. (23), we find the shear stress $\tau(r,t)$ under the form

$$\tau(r,t) = \frac{fr}{\lambda R} G_{\gamma,-2,1}\left(-\frac{1}{\lambda}, t\right) + \frac{2f}{\lambda R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}$$
$$\times \sum_{k=0}^{\infty}\sum_{m=0}^{k} C_k^m \lambda_r^m \left(-\frac{\nu r_n^2}{\lambda}\right)^k G_{\gamma,\beta m-k-2,k+1}\left(-\frac{1}{\lambda}, t\right). \quad (24)$$

Of course, making $r = R$ into (24), we recover Eq. (8).

4. Limiting Cases

1. Making γ and $\beta \to 1$ into Eqs. (20), (24) and using the following results

$$G_{1,-2,1}\left(-\frac{1}{\lambda_r}, t\right) = \lambda_r \left[t - \lambda_r \left(1 - e^{-t/\lambda_r}\right)\right], \tag{25}$$

and

$$G_{1,-1,1}\left(-\frac{1}{\lambda_r}, t\right) = \lambda_r \left(1 - e^{-t/\lambda_r}\right), \tag{26}$$

we obtain the velocity field

$$\begin{aligned} v(r,t) = {} & \frac{fr^2}{2\mu R}\left[t - \lambda_r \left(1 - e^{-t/\lambda_r}\right)\right] - \frac{2f}{\mu\lambda R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)} \\ & \times \sum_{k=0}^{\infty}\sum_{m=0}^{k} C_k^m \lambda_r^m \left(\frac{-\nu r_n^2}{\lambda}\right)^k \int_0^t \left(1 - e^{(\tau - t)/\lambda_r}\right) \\ & \times \left[G_{1,m-k-1,k+1}\left(-\frac{1}{\lambda}, \tau\right) + \lambda G_{1,m-k,k+1}\left(-\frac{1}{\lambda}, \tau\right)\right] d\tau, \end{aligned} \tag{27}$$

and the shear stress

$$\begin{aligned} \tau(r,t) = {} & \frac{rf}{R}\left[t - \lambda \left(1 - e^{-t/\lambda}\right)\right] + \frac{2f}{\lambda R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \\ & \times \sum_{k=0}^{\infty}\sum_{m=0}^{k} C_k^m \lambda_r^m \left(-\frac{\nu r_n^2}{\lambda}\right)^k G_{1,m-k-2,k+1}\left(-\frac{1}{\lambda}, t\right), \end{aligned} \tag{28}$$

corresponding to ordinary Oldroyd-B fluids performing the same motion.

2. Making $\beta \to 1$ and $\lambda_r \to 0$ into Eqs. (20) and (24) and using the following limit

$$\lim_{\eta \to 0} \frac{1}{\eta^k} G_{a,b,k}\left(-\frac{1}{\eta}, t\right) = \frac{t^{-b-1}}{\Gamma(-b)}; \qquad b < 0, \tag{29}$$

the velocity field

$$\begin{aligned} v(r,t) = {} & \frac{fr^2}{2\mu R}t - \frac{2f}{\mu\lambda R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}\sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k \\ & \times \left[G_{\gamma,-k-2,k+1}\left(-\frac{1}{\lambda}, \tau\right) + \lambda G_{\gamma,\gamma-k-2,k+1}\left(-\frac{1}{\lambda}, \tau\right)\right] d\tau. \end{aligned} \tag{30}$$

and the associated shear stress

$$\tau(r,t) = \frac{rf}{\lambda R} G_{\gamma,-2,1}\left(-\frac{1}{\lambda}, t\right) + \frac{2f}{\lambda R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \times \sum_{k=0}^{\infty}\left(\frac{-\nu r_n^2}{\lambda}\right)^k G_{\gamma,-k-2,k+1}\left(-\frac{1}{\lambda}, t\right), \tag{31}$$

for a generalized Maxwell fluid are obtained. By making $\gamma \to 1$ in (30) and (31), the respective solutions for the flow of ordinary Maxwell fluid can be obtained.

3. Now, making $\gamma \to 1$ and $\lambda \to 0$ into Eqs. (20) and (24) and again using the limit (29) and the following equalities

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k} C_k^m \lambda_r^m \big(-\nu r_n^2\big)^k \int_0^t G_{\beta,-1,1}\left(-\frac{1}{\lambda_r}, t-\tau\right)\frac{\tau^{-\beta m+k}}{\Gamma(-\beta m+k+1)}\, d\tau$$
$$= \sum_{k=0}^{\infty}(-\nu r_n^2)^k \int_0^t G_{\beta,0,1}\left(-\frac{1}{\lambda_r}, \tau\right) G_{1-\beta,-\beta k-\beta-1,k+1}\left(-\nu\lambda_r r_n^2,\, t-\tau\right) d\tau, \tag{32}$$

$$\sum_{k=0}^{\infty}\sum_{m=0}^{k} C_k^m \lambda_r^m \left(-\nu r_n^2\right)^k \frac{t^{-\beta m+k+1}}{\Gamma(-\beta m+k+2)}$$
$$= \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{1-\beta,-\beta k-\beta-1,k+1}\left(-\nu\lambda_r r_n^2,\, t\right), \tag{33}$$

the known solutions of velocity field and shear stress

$$v(r,t) = \frac{fr^2}{2\alpha_1 R} G_{\beta,-2,1}\left(-\frac{\mu}{\alpha_1}, t\right) - \frac{2f}{\alpha_1 R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}\sum_{k=0}^{\infty}\big(-\nu r_n^2\big)^k \times \int_0^t G_{\beta,0,1}\left(-\frac{\mu}{\alpha_1}, \tau\right) G_{1-\beta,\beta k-\beta-1,k+1}\left(-\alpha r_n^2,\, t-\tau\right) d\tau, \tag{34}$$

$$\tau(r,t) = \frac{fr}{R} t + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)} \times \sum_{k=0}^{\infty}\left(-\nu r_n^2\right)^k G_{1-\beta,-\beta k-\beta-1,k+1}\left(-\alpha r_n^2,\, t\right). \tag{35}$$

corresponding to the similar flow of generalized second grade fluid are recovered [18, Eqs. (21) and (24)], where in (34) and (35), $\alpha_1 = \mu\lambda_r$ and

$\alpha = \alpha_1/\rho$. Of course, by making $\beta \to 1$ in (34) and (35), the solutions for ordinary second grade fluid can be recovered.

4. Finally, by making $\gamma \to 1$ and $\lambda \to 0$ into Eqs. (30) and (31) $\big($or by making $\beta \to 1$ and $\lambda_r \to 0$ into Eqs. (34) and (35)$\big)$, the known solutions [7, 18, 19]

$$v(r,t) = \frac{fr^2 t}{2\mu R} - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}\left[1 - e^{-\nu r_n^2 t}\right], \tag{36}$$

$$\tau(r,t) = \frac{frt}{R} + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}\left[1 - e^{-\nu r_n^2 t}\right], \tag{37}$$

corresponding to the similar flow of Newtonian fluid are recovered.

5. Concluding Remarks

The aim of this note is to provide exact solutions for the velocity field $v(r,t)$ and the shear stress $\tau(r,t)$ corresponding to the unsteady flow of a generalized Oldroyd-B fluid due to an infinite circular cylinder subject to a longitudinal time-dependent shear stress. The solutions have been obtained by using Hankel and Laplace transforms and they are written under integral and series form in terms of the generalized G-function. Furthermore, these solutions satisfy the governing equation of motion and all imposed initial and boundary conditions. In the special cases, the similar solutions for Ordinary Oldroyd-B, generalized and ordinary Maxwell, generalized and ordinary second grade as well as Newtonian fluids are recovered from literature by taking into consideration suitable limits.

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