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ON F-DERIVATIONS OF BCI-ALGEBRAS

FARHAT NISAR[1]

Abstract. In this paper we introduce the notions of right F-derivation and left F-derivation of a BCI-algebra and some related properties are explored.

Key words: BCI-algebra, initial element, center, branch, derivation, f-derivation, F-derivation.
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1. Introduction

In [13], Y.B. Jun and X.L. Xin introduced the notion of derivation in BCI-algebras, which is defined in a way similar to the notion in ring theory (see [1, 2, 11, 14]), and investigated some properties related to this concept.In [16] J. Zhan and Y.L. Liu introduced the notion of f-derivation in BCI-algebras. In particular, they studied the regular f-derivations in detail and gave a characterization of regular f-derivations and characterized p-semisimple BCI-algebras using the notion of regular f-derivation. In this paper we introduce the notions of right F-derivation and left F-derivation of a BCI-algebra and some related properties are explored. We also investigated that the notion of left-right (resp. right-left) f-derivation of a p-semisimple BCI-algebra is a left (resp. right) F-derivation of X.

2. Preliminaries

Definition 1. *A BCI-algebra X is an abstract algebra* $(X, *, o)$ *of type (2, 0), satisfying the following conditions; for all* $x, y, z \in X$,

1.1 $((x * y) * (x * z)) * (z * y) = o$
1.2 $(x * (x * y)) * y = o$
1.3 $x * x = o$

[1]Department of Mathematics, Queen Mary College, Lahore, Pakistan. Email: fhtnr2003@yahoo.com.

1.4 $x * y = o = y * x \Rightarrow x = y$
1.5 $x * o = o \Rightarrow x = o$

where $x * y = o \Leftrightarrow x \leq y$
In a BCI-algebra X, the set $M = \{x \in X : o * x = o\}$ *is a subalgebra and is called the BCK-part of X. A BCI-algebra X is called proper if* $X - M \neq \phi$.

Moreover, the following properties hold in every BCK/BCI-algebra (see [9], [10]):

1.6 $x * o = x$
1.7 $(x * y) * z = (x * z) * y$
1.8 $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$
1.9 $(x * y) * (x * z) \leq x * z$

Definition 2. *Let X be a BCI-algebra. An element* $x_o \in X$ *is said to be an initial element of X, if* $x \leq x_o \Rightarrow x = x_o$. *[3]*

Definition 3. *Let* I_x *denote the set of all initial elements of X. We call it the center of X.[3] It is well known that the center* I_x *of a BCI-algebra X is p-semisimple. [4]*

Definition 4. *Let X be a BCI-algebra with* I_x *as its center. Let* $x_o \in I_x$*, then the set* $A(x_o) = \{x \in X : x_o \leq x\}$ *is known as the branch of X determined by* x_o. *[3]*

1.10 . Let X be a BCI-algebra. The following properties are equivalent for all x, $y \in X$:
(i) X is p-semi-simple.
(ii) o*(o*x) = x
(iii) $x * y = o \Rightarrow x = y$
(iv) $y * (y * x) = x$
for all x, y, $z \in X$.[6, 15]

1.11 Let X be a BCI-algebra. If $x \leq y$, then x, y are contained in the same branch of X [3].
1.12 Let X be a BCI-algebra and $A(x_o) \subseteq X$. Then $x, y \in A(x_o) \Rightarrow x * y, y * x \in M$.[3]
1.13 Let X be a BCI-algebra with I_x as its center. If $x \in A(x_o)$, $y \in A(y_o)$, then $x * y \in A(x_o * y_o)$, for $x_o, y_o \in I_x$. [7]
1.14 Let X be a BCI-algebra with I_x as its center. Let $x_o, y_o \in I_x$. Then for all $y \in A(y_o)$, $x_o * y = x_o * y_o$.[7]
1.15 Let X be a BCI-algebra. Then for $x \in X$, $o * (o * x) = x$ and $o * x \in I_x$. [8]

1.16 Let f be an endomorphism of a BCI-algebra X and I_x be its center. Then for any $x \in I_x$, $f(x) \in I_x$. [8]

1.17 Let f be an endomorphism of a BCI-algebra X with center I_x. Then for $x, y \in X$, following identities hold:
(i) $f_x * f_y \in I_x$.
(ii) $f_{x*y} = f_x * f_y$ [8]

1.18 Let f be an endomorphism of a BCI-algebra X. Then for all $x \in A(x_o)$, $f(x_o) = o * (o * f(x))$. [8]

Definition 5. *Let X be a BCI-algebra. By a left-right derivation (briefly, (l, r)-derivation) of X, a self map d of X satisfying the identity* $d(x * y) = (d(x) * y) \wedge (x * d(y))$*, for all x,* $y \in X$ *.If d satisfies the identity* $d(x * y) = (x * d(y)) \wedge (d(x) * y))$*, for all x,* $y \in X$ *then we say that d is a right-left f-derivation (briefly, (r, l)-derivation) of X. Moreover, if d is both (l, r)- and (r, l)-derivation, it is said that d is an derivation of X.(see [12])*

Definition 6. *A self-map of a BCI-algebra X is said to be regular if d (o) = o.(see [12])*

Definition 7. *Let X be a BCI-algebra. By a left-right f-derivation (briefly, (l, r)- derivation) of X, a self map* d_f *of X satisfying the identity* $d_f(x * y) = (d_f(x) * f(y)) \wedge (f(x) * d_f(y))$*, for all x,* $y \in X$ *is meant, where f is an endomorphism of X. If* d_f *satisfies the identity* $d_f(x * y) = (f(x) * d_f(y)) \wedge (d_f(x) * f(y)))$*, for all x,* $y \in X$ *then it is said that* d_f*is a right-left f-derivation (briefly, (r, l)-f- derivation) of X. Moreover, if d is both (l, r)- and (r, l)-f-derivation, it is said that* d_f *is an f-derivation of X.(see [16])*

Definition 8. *A mapping f of a BCI-algebra X into itself is called an endomorphism if* $f(x * y) = f(x) * f(y)$*. Note that* $f(o) = 0$*. (see [16])*

Definition 9. *A BCI-algebra X is said to be commutative if and only if* $x \leq y \Rightarrow y * (y * x) = x$*, for all* $x, y \in X$*. [12]*

3. F-Derivations

In this section we introduce the notion of right F-derivation and left F-derivation of a BCI-algebra and give some examples to explain the theory of derivation, f-derivation and F-derivation in BCI-algebras.

Definition 10. *Let X be a BCI-algebra. By a right F-derivation of X, we mean a self map* D_F *of X satisfying the identity*

$$D_F(x * y) = (F(x) * D_F(y)) \wedge (F(y) * D_F(x))$$

Fig1.bmp [width=2.3 in, height=.8 in]

for all $x, y \in X$, where F is an endomorphism of X.
If D_F satisfies the identity

$$D_F(x * y) = (D_F(x) * F(y)) \wedge (D_F(y) * F(x)))$$

for all $x, y \in X$, then it is said that D_F is a left F-derivation of X. Moreover, if D_F is both right and left F-derivation, then it is said that D_F is an F-derivation of X.

Definition 11. *An F-derivation D_F of a BCI-algebra X is said to be regular if $D_F(o) = o$. If $D_F(o) \neq o$, we call D_F an irregular F-derivation.*

3.3 Examples

Example 1
Let $X = \{o, a, b, c, d, e, f\}$ be a BCI-algebra with Hasse diagram and Cayley table defined as follows:

Table 1

*	o	a	b	c	d	e	f
o	o	o	b	b	d	d	f
a	a	o	b	b	d	d	f
b	b	b	o	o	f	f	d
c	c	b	a	o	f	f	d
d	d	d	f	f	o	o	b
e	e	d	f	f	a	o	b
f	f	f	d	d	b	b	o

Define a self map $D_F :\to X$ as follows:

$$D_F(x) = \begin{cases} b & \text{x = 0, a} \\ o & \text{x = b, c} \\ f & \text{x = d, e} \\ d & \text{x = f} \end{cases}$$

Define an endomorphism $F : X \to X$ as follows:

$$F(x) = \begin{cases} o & \text{x = 0, a} \\ b & \text{x = b, c} \\ d & \text{x = d, e} \\ f & \text{x = f} \end{cases}$$

It is easily checked that D_F is an irregular derivation, f-derivation and F-derivation of X.

Example 2 Let $X = \{o, a, b, c, d, e, f\}$ be a BCI-algebra in which * is defined as in Table 1. Define a self map $D_F : X \to X$ as follows:

$$D_F(x) = \begin{cases} b & \text{x = 0, a} \\ o & \text{x = b, c} \\ f & \text{x = d, e} \\ d & \text{x = f} \end{cases}$$

Define an endomorphism $F : X \to X$ as $F(x) = o$, for all $x \in X$.

Note that the self map D_F is a derivation of X but not a f-derivation of X as

$$D_F(b * d) = D_F(f) = d$$

but

$$(D_F(b)*F(d))\wedge(F(b)*D_F(d)) = (o*o)\wedge(o*f) = o\wedge f = f*(f*o) = f*f = o$$

Thus it follows $(D_F(a * b) = ((D_F(a) * b) \wedge ((D_F(b) * a)$.
Also note that the self map$D_F$is a regular f-derivation of X but not an F-derivation of X as $D_F(a) = (D_F(a * o) = a$ but

$$(D_F(a) * F(o) \wedge (D_F(o) * F(a)) = (a * o) \wedge (o * a) = a \wedge o = o$$

Thus it follows $D_F(a) = D_F(a * o) = (D_F(a) * F(o) \wedge (D_F(o) * F(a))$

In squeal, we will denote $o * (o * F(x)) = F_x$ and $x \wedge y = y * (y * x)$.

Theorem 1. *Let D_F be a F-derivation of a BCI-algebra X. Then*

(i) $D_F(o) \in I_x$

(ii) $D_F(x) \in I_x$, *for all* $x \in I_x$.

Proof *(i) Let D_F be a F-derivation a BCK-algebra X. Since D_F is a F-derivation, therefore it is right F-derivation as well as left F-derivation of X. When D_F is a right F-derivation, then*

$$D_F(o) = D_F(o * o) = (F(o) * D_F(o)) \wedge (F(o) * D_F(o)) = F(o) * D_F(o) \quad (1)$$

When D_F is a left F-derivation, then

$$D_F(o) = D_F(o*o) = (D_F(o)*F(o))\wedge(D_F(o)*F(o)) = D_F(o)*F(o) \quad (2)$$

Since D_F is a F-derivation, therefore from (1) and (2) it follows that $F(o)*D_F(o) = D_F(o) = D_F(o)*F(o) \Rightarrow o*D_F(o) = D_F(o)*o \Rightarrow o*D_F(o) = D_F(o)$
As D_F is a self map, so for $o \in X$, $D_F(o) \in X$ and because of 1.15, for $D_F(o) \in X$, $o * D_F(o) \in I_x$. Hence, $D_F(o) \in I_x$.
(ii) Let $x \in I_x$. Then $x = o * (o * x)$. Since D_F is a F-derivation of X, therefore it is both right as well as left F-derivation of X. When D_F is a right F-derivation, then

$$(D_F(x) = D_F(o*(o*x)) = (F(o)*D_F(o*x)) \wedge (F(o*x)*D_F(o))))$$
$$= (F(o*x)*D_F(o)))*((F(o*x)*D_F(o)))*(o*D_F(o*x))$$

$$(since F(o) = oandx \wedge y = y * (y * x))$$
$$\leq o * D_F(o * x) \qquad (using\ 1.2)$$

As D_F is a self map, so for $x \in I_x \subseteq X$, $D_F(x)$ and $D_F(o * x) \in X$. Because of 1.15, for $D_F(o * x) \in X$, $o * D_F(o * x) \in I_x$. So, $o * D_F(o * x)$ is an initial point. Thus $D_F(x) \leq o * D_F(o * x) \Rightarrow D_F(x) = o * D_F(o * x)$. Hence for $x \in I_x$, $D_F(x) \in I_x$.

When D_F is a left F-derivation, then

$$D_F(x) = D_F(o*(o*x)) = (D_F(o)*F(o*x)) \wedge (D_F(o*x)*F(o))$$
$$= (D_F(o) * F(o * x)) \wedge D_F(o * x)$$
$$= (D_F(o * x) * ((D_F(o * x) * (((D_F(o) * F(o * x)))$$
$$(since\ F(o) = o\ and\ x \wedge y = y * (y * x))$$

$$= D_F(o)*(o*F(x))$$

Using (i), $D_F(o) \in I_x$ and by 1.15, for $F(x) \in X$, $o * F(x) \in I_x$. As I_x is p-semisimple, so for $D_F(o)$, $o * F(x) \in I_x$, $D_F(o) * (o * F(x)) \in I_x$. So, $D_F(o) * (o * F(x))$ is an initial point. Thus above inequality implies $D_F(x) = D_F(o) * (o * F(x))$. Which implies $D_F(x) \in I_x$. This completes the proof.

Theorem 2. *Let D_F be a right F-derivation of a BCI-algebra X. Then $D_F(x) \in G(X)$, for all $x \in G(X)$.*
Proof: *Let $x \in G(X)$. Then $x = o * x$. So,*

$$D_F(x) = D_F(o * x) = (F(o) * D_F(x)) \wedge (F(x) * D_F(o)))$$
$$= (F(x)*D_F(o)))*((F(x)*D_F(o)))*(o*D_F(x)))$$

$$(since\ F(o) = o and\ x \wedge y = y * (y * x))$$

$$\leq o * D_F(x) \quad (using\ 1.2)$$

As D_F is a self map, so for $x \in G(X) \subseteq X$, $D_F(x) \in X$. Because of 1.15, for $D_F(x) \in X$, $o * D_F(x) \in I_x$. So, $o * D_F(x)$ is an initial point. Thus $D_F(x) \leq o * D_F(x) \Rightarrow D_F(x) = o * D_F(x) \Rightarrow D_F(x) \in G(X)$.

4. Irregular F-Ferivations

In this section we investigate some results on irregular F-derivations of BCI-algebras.

Theorem 3. *Let D_F be a F-derivation of a BCI-algebra X. If for distinct $x, y \in I_x$, $D_F(x) = F(y) \Rightarrow D_F(y) = F(x)$, then D_F is irregular.*

Proof: *Assume that for distinct $x, y \in I_x$,*

$$D_F(x) = F(y) \Rightarrow D_F(y) = F(x) \quad (1)$$

Since D_F is a F-derivation of X, therefore D_F is a left F-derivation as well as right F-derivation. When D_F is a left F-derivation, then

$$D_F(x * y) = (D_F(x) * F(y)) \wedge (D_F(y) * F(x))$$

$$= (D_F(y) * F(x)) * ((D_F(y) * F(x)) * (D_F(x) * F(y))) \quad (since\ x \wedge y = y * (y * x))$$

$$\leq D_F(x) * F(y) \quad (using\ 1.2)$$

$$= F(y) * F(y) = o \quad (using\ 1\ and\ 1.3)$$

$$\Rightarrow D_F(x * y) = o \quad (using\ 1.5)$$

Since for $x, y \in I_x$, $x * y \in I_x$ and $x * y \neq o$, otherwise by 1.10, (iii), $x * y = o \Rightarrow x = y$, a contradiction, therefore by our assumption

$$D_F(x * y) = o = F(o) \Rightarrow D_F(o) = F(x * y)$$

Thus it follows that $D_F(o) \neq o$ as $x * y \neq o$, so D_F is irregular.

Also, when D_F is a right F-derivation, then

$$D_F(x * y) = (F(x) * D_F(y)) \wedge (F(y) * D_F(y))$$

$$= (F(y) * D_F(y)) * ((F(y) * D_F(y)) * (F(x) * D_F(y))) \quad (since\ x \wedge y = y * (y * x))$$

$$\leq F(x) * D_F(y) \quad (using\ 1.2)$$

$$= F(x) * F(x) = o \quad (using\ D_F(x) = F(y)\ and\ 1.3)$$

$$\Rightarrow D_F(x * y) = o \quad (using\ 1.5)$$

Since for $x, y \in I_x$, $x * y \in I_x$ and $x * y \neq o$, otherwise by 1.10, (ii), $x * y = o \Rightarrow x = y$, a contradiction, therefore by our assumption

$$D_F(x * y) = o = F(o) \Rightarrow D_F(o) = F(x * y)$$

Thus it follows that $D_F(o) \neq o$ as $x * y \neq o$, so D_F is irregular. This completes the proof.

Theorem 4. *Let D_F be a F-derivation of a BCI-algebra X. If for all $x \in A(x_o)$, $F(x) \in A(x_o)$ and for $y \in A(y_o)$, $F(y) \in A(y_o)$, then $D_F(x) = y_o \Rightarrow D_F(y) \in A(x_o)$.*

Proof *Assume that for all $x \in A(x_o)$, $D_F(x) = y_o$. Since D_F is a F-derivation of BCI-algebra X, therefore D_F is a left as well as right F-derivation. When D_F is a left F-derivation, then for all $x \in A(x_o)$, $y \in A(y_o)$,*

$$D_F(x * y) = (D_F(x) * F(y)) \wedge (D_F(y) * F(x))$$

Since $F(o) = o$ *and* $x \wedge y = y * (y * x))$

$$D_F(x * y) = (D_F(y) * F(x)) * ((D_F(y) * F(x)) * (D_F(x) * F(y)))$$

$\leq D_F(x) * F(y)$ (*using* 1.2)

$\Rightarrow D_F(x*y) \leq y_o*F(y)$ (*since* $D_F(x) = y_o$)

According to given condition for $y \in A(y_o)$, $F(y) \in A(y_o)$ and by definition 2.4, $y \in A(y_o) \Rightarrow y_o \leq y$ which implies $F(y_o) \leq F(y)$.

And $F(y) \in A(y_o) \Rightarrow y_o \leq F(y) \Rightarrow y_o * F(y) = o$. So above inequality becomes $D_F(x * y) \leq o \Rightarrow D_F(x * y) = o$ (1).

As D_F is also a right F-derivation, So for all $x \in A(x_o)$, $y \in A(y_o)$,

$$D_F(x * y) = (F(x) * D_F(y)) \wedge (F(y) * D_F(x))$$

$$D_F(x * y) = (F(y) * D_F(x)) * ((F(y) * D_F(x)) * (F(x) * D_F(y)))$$

(*since* $x \wedge y = y * (y * x)$)

$\leq F(x) * D_F(y)$ (2) (*using* 1.2)

Since D_F is both right and left F-derivation. So from (1) and (2), it follows that

$o \leq F(x)*D_F(y)$

$\Rightarrow o * F(x) \leq (F(x) * D_F(y)) * F(x)$ (*using* 1.8)

$\Rightarrow o * F(x) \leq (F(x) * F(x)) * D_F(y)$ (*using* 1.7)

$\Rightarrow o * F(x) \leq o * D_F(y)$ (*using* 1.3)

$\Rightarrow o * (o * D_F(y)) \leq o * (o * F(x)) = F(x_o)$ (*using* 1.8 *and* 1.18)

Since $F(x_o) \in I_x$, therefore $F(x_o)$ is an initial point. Thus above inequality becomes

$o * (o * D_F(y)) = F(x_o) \Rightarrow F(x_o) \leq D_F(y)$ (*using* 1.2)

By 1.11 both $D_F(y)$ and $F(x_o)$ belong to the same branch of X. Since $F(x_o) \in A(x_o)$, therefore $D_F(y) \in A(x_o)$. This completes the proof.

5. Regular F-Derivations

In this section we characterize regular F-derivations of BCI-algebras.

Proposition 5. *Every F-derivation of a BCK-algebra is regular, where F is an endomorphism of X.*
Proof: *Let D_F be a F-derivation a BCK-algebra X. Since D_F is a F-derivation of X, therefore D_F is a left F-derivation as well as right F-derivation. When D_F is a right F-derivation, then*

$$D_F(o) = D_F(o * o) = (F(o) * D_F(o)) \wedge (F(o) * D_F(o)) = F(o) * D_F(o) \qquad (1)$$

When $D_F$is a left F-derivation, then

$$D_F(o) = D_F(o * o) = (D_F(o) * F(o)) \wedge (D_F(o) * F(o)) = D_F(o) * F(o) \qquad (2)$$

Since D_F is a F-derivation, therefore from (1) and (2) it follows that

$$F(o) * D_F(o) = D_F(o) = D_F(o) * F(o) \Rightarrow o * DF(o) = D_F(o) * o \Rightarrow o = D_F(o)$$

which implies D_F is regular. This completes the proof.

Theorem 6. *Let D_F be a regular F-derivation of a BCI-algebra X. Then $D_F(x)$ and $F(x)$ belong to the same branch of X and $D_F(x) = F(x)$.*
Proof: *Since D_F be a regular F-derivation, therefore $D_F(o) = o$. Since D_F is a F-derivation of X, therefore D_F is a left F-derivation as well as right F-derivation. When D_F is a right F-derivation,*

$$o = D_F(o) = D_F(x*x) = (F(x)*D_F(x)) \wedge (F(x)*D_F(x)) = F(x)*D_F(x)$$

$$\Rightarrow F(x) \leq D_F(x) \qquad (1)$$

Also when D_F is a left F-derivation,

$$o = D_F(o) = D_F(x*x) = (D_F(x)*F(x)) \wedge (D_F(x)*F(x)) = D_F(x)*F(x)$$

$$\Rightarrow D_F(x) \leq F(x) \qquad (2)$$

Because of 1.11, from inequalities (1) and (2), it follows $D_F(x)$ and $F(x)$ belong to the same branch of X. Since D_F is F-derivation, therefore it is left as well as right F-derivation, therefore by property 1.4, from (1) and (2), it follows: $D_F(x) = F(x)$.

Note that the converse of above theorem is not true (see example 3).

Theorem 7. *Let D_F be a self map and $A(x_o)$ be any branch of a BCI-algebra X. If for any $x \in A(x_o)$, $D_F(x) = F(x_o)$, then D_F is a regular left F-derivation.*

Proof *Let D_F be a self map and $A(x_o)$ be any branch of a BCI-algebra X. According to given condition for any $x \in A(x_o)$, $D_F(x)) = F(x_o)$* (1)

For $x \in A(x_o)$, $x_o \leq x \Rightarrow x_o * x = o$. By 1.12 for x_o, $x \in A(x_o)$, $x_o * x = o$ and $x * x_o \in M$. So, for some $m \neq o \in M = A(o)$, $x * x_o = m$, otherwise, $x_o * x = o = x * x_o \Rightarrow x = x_o$, a contradiction. Also by 1.14 for o, $x_o \in I_x$ and $m \neq o \in M = A(o)$, $x_o * m = x_o * o = x_o$

Now for $x, y \in X$ following two cases arise:

Case 1:*Both x and y belongs to the same branch of X.*
Case 2:*x and y belongs to different branches of X.*

Case 1: *Let $x, y \in A(x_o)$. So, $x_o \leq x$ and $x_o \leq y$. Then by 1.13, $x * y \in A(o * o) = A(o) = M$. So using (1),*

$$D_F(x * y) = F(o) = o \tag{2}$$

Also $x_o \leq y \Rightarrow x_o * y = o$ (3)
Further $o = F(o) = F(x_o * x) = F(x_o) * F(x)$
And $o = F(o) = F(x_o * y) = F(x_o) * F(y)$
Now

$$(D_F(x) * F(y)) \wedge (D_F(y) * F(x)) = (F(x_o) * F(y)) \wedge (F(x_o) * F(x)) \ (using\ 1)$$

$$= o \wedge o = o$$

i.e

$$(D_F(x)*F(y)) \wedge (D_F(y)*F(x)) = o = D_F(x*y)(x_o*y_o) \qquad (using\ 2)$$

which implies D_F is a left derivation.

Case 2:*Let $x \in A(x_o)$ and $y \in A(y_o)$. Then by 1.15, $x * y \in A(x_o * y_o)$. So, using (1)*

$$D_F(x*y) = F(x_o*y_o) \tag{4}$$

Now

$$(D_F(x) * F(y)) \wedge (D_F(y) * F(x)) = (F(x_o) * F(y)) \wedge (F(y_o) * F(x))$$

$$= (F(y_o)*F(x))*((F(y_o)*F(x))*(F(x_o)*F(y)))$$

$$\leq F(x_o) * F(y) = F(xo * y) \qquad (Since\ F\ is\ an\ endomorphism)$$

$$= F(x_o * y_o) \qquad (using\ 1.14)$$

Since for $x_o, y_o \in I_x, x_o * y_o \in I_x$, *therefore by 1.16,* $F(x_o, y_o) \in I_x$. *So* $F(x_o, y_o)$ *is an initial element. Thus it follows that*

$$(D_F(x) * F(y)) \wedge (D_F(y) * F(x)) = F(xo * yo)$$

$$\Rightarrow (D_F(x) * F(y)) \wedge (D_F(y) * F(x)) = D_F(x * y) \qquad (using\ 4)$$

which implies D_F *is a left F-derivation. Obviously* D_F *is regular. This completes the proof.*

Proposition 8. *Let* D_F *be a self map of a BCI-algebra X. Then the following hold:*

(i) *If* D_F *is a right F-derivation, then* $F(x) * D_F(x) = F(y) * D_F(y)$
(ii) *If* D_F *is a left F-derivation, then* $D_F(x) * F(x) = D_F(y) \wedge F(y)$.

Proof: *(i) Let* $x, y \in X$. *Then*

$$D_F(o) = DF(x * x) = (F(x) * DF(x)) \wedge (F(x) * D_F(x)) = F(x) * D_F(x)$$

Similarly, $D_F(o) = F(y) * D_F(y)$. *Thus it follows* $F(x) * D_F(x) = F(y) * D_F(y)$.

(ii) Let $x, y \in X$. *Then*

$$D_F(o) = D_F(x * x) = (D_F(x) * F(x)) \wedge (D_F(x) * F(x)) = D_F(x) * F(x)$$

Similarly, $D_F(o) = D_F(y) * F(y)$. *Thus it follows* $D_F(x) * F(x) = D_F(y) \wedge F(y)$.

Proposition 9. *Let* D_F *be a self map of a BCI-algebra X. Then the following hold:*

(i) *If* D_F *is a right F-derivation, then* $D_F(x) = D_F(x) \wedge F(x)$.
(ii) *If* D_F *is a left F-derivation, then* $D_F(x) = D_F(x) \wedge D_F(o)$.

Proof: *(i) Let* $x \in X$. *Then*

$$D_F(x) = D_F(x * o) = (F(x) * D_F(o)) \wedge (F(o) * D_F(x))$$

$$= (F(o) * D_F(x)) * ((F(o) * D_F(x)) * (F(x) * D_F(o)))$$

$$\leq F(x) * D_F(o) = F(x) * (F(x) * D_F(x)) \leq D_F(x)$$

Because of property 1.4, $D_F(x) = F(x) * (F(x) * D_F(x))$ *which implies that* $D_F(x) = D_F(x)$.

(ii)) Let $x \in X$. Then

$D_F(x) = D_F(x{*}o) = (D_F(x){*}F(o)){\wedge}(D_F(o){*}F(x))$

$= D_F(x) \wedge (D_F(o) * F(x))$

$= (D_F(o){*}F(x)){*}((D_F(o){*}F(x)){*}D_F(x))$

$= (D_F(o){*}F(x)){*}((D_F(o){*}D_F(x))){*}F(x))$

$\leq D_F(o){*}(D_F(o){*}D_F(x)) \leq D_F(x)$ $\qquad$ $(using\ 1.9)$

Because of property (4), $D_F(x) = D_F(o) * (D_F(o) * D_F(x))$ which implies that $D_F(x) = D_F(x) \wedge D_F(o)$.

Proposition 10. *Let D_F be a right F-derivation of a BCI-algebra X. Then the following hold:*

(i) *If D_F is a right F-derivation, then $D_F(x) = F(x)$.*
(ii) *If D_F is a regular left F-derivation, then $D_F(x) \in I_x$.*

Proof: *(i) Using proposition 9,(i),*

$$D_F((x) = D_F((x) \wedge F(x) = F(x) * (F(x) * D_F((x)) = F(x) * D_F(o) = F(x)$$

(ii) Using proposition 9, (ii),

$$D_F(x) = D_F(x) \wedge D_F(o) = D_F(x) \wedge o = o * (o * D_F(x)) \Rightarrow D_F(x) \in I_x$$

Theorem 11. *Let D_F be a right F-derivation of a BCI-algebra X. Then the following hold:*

(i) *$D_F(x) \in I_x$, for all $x \in X$.*
(ii) *$F(y) * (F(y) * D_F(x)) = D_F(x)$, for all $x \in X$.*
(iii) *$D_F(x) * F(y) = o * (F(y) * D_F(x))$, for all $x, y \in X$.*
(iv) *$D_F(x) * F(y) \in I_x$, for all $x, y \in X$.*

Proof: *(i) Let $x \in X$. Then*

$D_F(x) = D_F(x{*}o) = (F(x){*}D_F(o)){\wedge}(F(o){*}D_F(x))$

$= (F(x) * D_F(o)) \wedge (o * D_F(x))$

$= (o * D_F(x)) * (o * D_F(x)) * (F(x) * D_F(o))$

$= (o * D_F(x)) * ((o * (F(x) * D_F(o))) * D_F(x))$ $\qquad$ $(using\ 1.7)$

$\leq o * (o * (F(x) * D_F(o)))$ $\qquad$ $(using\ 1.9)$

Because of 1.15, $o*(o*(F(x)*D_F(o))) \in I_x$, therefore $o*(o*(F(x)*D_F(o)))$ is an initial point so above inequality becomes $D_F(x) = o*(o*(F(x)*D_F(o)))$.

Thus it follows $D_F(x) \in I_x$.

(ii) Because of property (1.2), $F(y) * (F(y) * D_F(x)) \leq D_F(x)$. *From (i), it follows* $D_F(x) \in I_x$, *therefore* $D_F(x)$ *is an initial element. So, above inequality implies* $F(y) * (F(y) * D_F(x)) = D_F(x)$.
(iii) Using (ii),

$$F(y) * (F(y) * D_F(x)) = D_F(x)$$

$$\Rightarrow (F(y)*(F(y)*D_F(x))*F(y) = D_F(x)*F(y)$$

$$\Rightarrow (F(y)*F(y))*(F(y)*D_F(x)) = D_F(x)*F(y) \qquad (using\ 1.7)$$

$$\Rightarrow o * (F(y) * D_F(x)) = D_F(x) * F(y)$$

(iv) Since $F(y) * D_F(x)) \in X$, *therefore by 1.15,* $o * (F(y) * D_F(x)) \in I_x$. *Hence* $D_F(x) * F(y) \in I_x$.

Theorem 12. *A self map* D_F *of a BCI-algebra X defined as* $D_F(x) = o * (o * F(x)) = F_x$, *for all* $x \in X$, *is a left F- derivation of X, where F is an endomorphism of X.*
Proof: *Let* D_F *be a self map of a BCI-algebra X, where F is an endomorphism of X. defined as follows:*

$$D_F(x) = o*(o*F(x)) = F_x \qquad (1)$$

for all $x \in X$. *As for* $x, y \in X$, $x * y \in X$, *therefore* $D_F(x * y) = o * (o * F(x * y)) = F_{x*y}$ *which implies that*

$$D_F(x*y) = Fx*Fy \qquad (2) \qquad (using\ 1.17, (ii))$$

Now

$$(D_F(x) * F(y)) \wedge (D_F(y) * F(x))$$

$$= (F_x * F(y)) \wedge (F_y * F(x)) \qquad (using\ 1)$$

$$= (F_y * F(x)) * ((F_y * F(x)) * (F_x * F(y)) \qquad (since\ x \wedge y = y * (y * x))$$

$$\leq F_x * F(y) \qquad (using\ 1.2)$$

$$\Rightarrow (F_x * F(y)) \wedge (F_y * F(x)) * (F_x * Fy) \leq (Fx * F(y)) * (Fx * Fy) \quad (using\ 1.8)$$

$$\leq F_y * F(y) \qquad (using\ 1.1)$$

$$\Rightarrow (F_x * F(y)) \wedge (F_y * F(x)) * (F_x * F_y) \leq o \qquad (since\ Fy * F(y) = o)$$

$$\Rightarrow (Fx * F(y)) \wedge (F_y * F(x)) * (F_x * Fy) = o \qquad (using\ 1.5)$$

$$\Rightarrow (F_x * F(y)) \wedge (F_y * F(x)) * \leq F_x * F_y$$

Because of 1.17, (i) $F_x * F_y \in I_x$, therefore $F_x * F_y$ is an initial element. Thus it follows that

$$(F_x * F(y)) \wedge (F_y * F(x)) = F_x * F_y$$

$$\Rightarrow (D_F(x) * F(y)) \wedge (D_F(y) * F(x)) = F_x * F_y = D_F(x * y) \qquad (using\ 2)$$

which implies that D_F is a left F-derivation.

Theorem 13. *Let D_F be a F-derivation of a commutative BCI-algebra X, where F is an endomorphism of X. Then $x \leq y$ implies $D_F(x)$ and $D_F(y)$ belong to the same branch of X.*

Proof: *Let D_F be a F-derivation of a commutative BCI-algebra X, where F is an endomorphism of X. Since X is a commutative BCI-algebra, therefore $x \leq y \Rightarrow y * (y * x) = x$. So, when D_F is a left F-derivation,*

$$D_F(x) = D_F(y*(y*x)) = (D_F(y)*F(y*x)) \wedge (D_F(y*x)*F(y))$$
$$= (D_F(y * x) * F(y)) * ((D_F(y * x) * F(y)) * (D_F(y) * F(y * x))$$
$$\leq (D_F(y) * F(y * x)) \Rightarrow D_F(x) \leq D_F(y) * (F(y) * F(x)) \qquad (A)$$
$$(since\ F\ is\ an\ endomorphism)$$

Since $x \leq y \Rightarrow x * y = o$, therefore $o = F(o) = F(x * y) = F(x) * F(y) \Rightarrow F(x) \leq F(y)$. By 1.12 $F(x) * F(y)$ and $F(y) * F(x) \in M$. As $F(x) * F(y) = o$, so $F(y) * F(x) \neq o$, otherwise because of property 1.4, $F(x) * F(y) = o = F(y) * F(x) \Rightarrow F(x) = F(y)$, a contradiction. Thus there exists some $m \neq o \in M$ such that $F(y) * F(x) = m$. Hence inequality (A) becomes

$$D_F(x) \leq D_F(y) * m \Rightarrow D_F(x) * D_F(y) \leq (D_F(y) * m) * D_F(y) \qquad (using\ 1.8)$$
$$\Rightarrow D_F(x) * D_F(y) \leq (D_F(y) * D_F(y)) * m \qquad (using\ 1.7)$$
$$\Rightarrow D_F(x)*D_F(y) \leq o*m \qquad (using\ 1.3)$$
$$\Rightarrow D_F(x)*D_F(y) \leq o \qquad (since\ m \in M)$$
$$\Rightarrow D_F(x)*D_F(y) = o \qquad (using\ 1.5)$$

which shows that $D_F(x) \leq D_F(y)$. Because of 1.11, it follows $D_F(x)$ and $D_F(y)$ belong to the same branch of X. This completes the proof.

Definition: Let X be a BCI-algebra. Define Ker $D_F = \{x \in X : D_F(x) = o, forall F - derivations D_F\}$.

Proposition 14. *Let D_F be a F-derivation of a BCI-algebra X, where F is an endomorphism of X. Then Ker D_F is a subalgebra of X.*
Proof:: *Let $x, y \in KerD_F$. Then $D_F(x) = o$ and $D_F(y) = o$. As D_F is a F-derivation of a BCI-algebra X. Therefore D_F is both right as well as left F-derivation of X. When D_F is a right F-derivation of X, then*

$$D_F(x * y) = (F(x) * D_F(y)) \wedge (F(y) * D_F(x)) = (F(x) * o) \wedge (F(y) * o)$$

$$= F(x) \wedge (F(y) = F(y) * ((F(y) * F(x)) \leq F(x)$$

As D_F is a regular F-derivation therefore by theorem 6, $F(x) = D_F(x)$. But $D_F(x) = o$. So above inequality becomes $D_F(x * y) \leq o$. So $D_F(x * y) = o$. Thus it follows $x * y \in KerD_F$. Also when D_F is a left F-derivation of X, then

$$D_F(x * y) = (D_F(x) * F(y)) \wedge (D_F(y) * F(x)) = (o * F(y)) \wedge (o * F(x))$$

$$= (o * F(x)) * ((o * F(x)) * (o * F(y))) \leq o * F(y)$$

As D_F is a regular F-derivation therefore by theorem 5.1, $F(y) = D_F(y)$. But $D_F(y) = o$. So above inequality becomes $D_F(x * y) \leq o$. So $D_F(x * y) = o$. Thus it follows $x * y \in KerD_F$. Hence $KerD_F$ is a subalgebra of X. This completes the proof.

Proposition 15. *The left-right F-derivation (briefly (l, r)-F-derivation) of a p-semisimple BCI-algebra is a left F-derivation of X, where F is an endomorphism of X.*
Proof: *Let $x, y \in X$ and D_F be its left-right f-derivation. Then*

$$D_F(x * y) = (D_F(x) * F(y)) \wedge (F(y) * D_F(y))$$
$$= (F(y) * D_F(y)) * ((F(y) * D_F(y)) * (D_F(x) * F(y)))$$
$$\leq D_F(x) * F(y)$$
$$D_F(x * y) = D_F(x) * F(y) \qquad (using\ 1.10, iii)$$
$$= (D_F(y) * F(x)) * ((D_F(y) * F(x)) * (D_F(x) * F(y))) \qquad (using\ 1.10, iv)$$
$$= (D_F(x) * F(y)) \wedge (D_F(y) * F(x))$$

which implies D_F is a left F-derivation of X.

Proposition 16. *The right-left F-derivation (briefly (r, l)-F-derivation) of a p-semisimple BCI-algebra is a right F-derivation of X, where F is an endomorphism of X.*
Proof: *Let $x, y \in X$ and D_F be its right-left f-derivation. Then*

$$D_F(x * y) = (F(x) * D_F(y)) \wedge (D_F(x) * F(y))$$
$$= (D_F(x) * F(y)) * ((D_F(x) * F(y)) * (F(x) * D_F(y)))$$
$$\leq F(x) * D_F(y) \qquad (using\ 1.2)$$

$D_F(x*y) = F(x)*D_F(y)$ $(using\ 1.10, iii)$
$= (F(y) * D_F(x)) * ((F(y) * D_F(x)) * (F(x) * D_F(y)))$ $(using\ 1.10, iv)$
$= (F(x)*D_F y))\wedge(F(y)*D_F(x))$
which implies D_F is a right F-derivation of X.

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