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EXACT ANALYTIC SOLUTIONS FOR THE FLOW OF SECOND GRADE FLUID BETWEEN TWO LONGITUDINALLY OSCILLATING CYLINDERS

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Abstract. The velocity field and associated shear stress corresponding to the longitudinal oscillatory flow of a second grade fluid, between two infinite coaxial circular cylinders, are determined by means of Laplace and Hankel transforms. The flow is due to both of the cylinders that at $t = 0^+$ suddenly begin to oscillate along their common axis with different angular frequencies of their respective velocities. The solutions for the motion between the cylinders, when one of them is at rest, can be obtained from our general solutions. Furthermore, the corresponding solutions for the similar flow of Newtonian fluid are also obtained as limiting case. The flows of second grade and Newtonian fluids are compared graphically by plotting their velocity profiles.

Key words: second grade fluid, shear stress, longitudinal oscillatory flow, Hankel transform.
AMS SUBJECT: 76A05, 76A10.

1. Introduction

From the engineering and industrial point of view, the flows in the neighborhood of spinning or oscillating bodies are of interest to both academic workers and industry. Among them, the flows between oscillating cylinders are some of the most important and interesting problems of motion. As early as 1886,

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Stokes [1] established an exact solution for the rotational oscillations of an infinite rod immersed in a classical linearly viscous fluid. Casarella and Laura [2] obtained an exact solution for the motion of the same fluid due to both longitudinal and torsional oscillations of the rod. Later, Rajagopal [3] found two simple but elegant solutions for the flow of a second grade fluid induced by the longitudinal and torsional oscillations of an infinite rod. These solutions have been already extended to Oldroyd-B fluids by Rajagopal and Bhatnagar [4]. More interesting results have been obtained by Hayat *et al* [5].

Recently, Erdogan [6] provided two starting solutions for the motion of a linearly viscous fluid due to the cosine and sine oscillations of a flat plate. He used the Laplace transform and found that the steady-state flows are set up with the same frequency as the boundary velocity. The extension of these solutions to non-Newtonian fluids has been achieved by Fetecau and Fetecau [7], Hayat *et al* [8] and Aksel *et al* [9]. Exact solutions for the motion of second grade fluid and a Maxwell fluid due to the longitudinal and torsional oscillations of a single circular cylinder have been recently obtained by Fetecau and Fetecau [10] and Vieru *et al* [11], while exact analytic solutions of velocity field and associated shear stress for the motion of fractional Maxwell fluid between two cylinders, when both of them are oscillating longitudinally, have been obtained by Mahmood *at al* [16].

As far as the knowledge of author is concerned, no attempt is made to achieve exact analytic solutions corresponding to the longitudinal oscillatory flow of second grade fluid in the annular region between two cylinders, when both of them are oscillating sinusoidally along their common axis. So the aim of this note is to consider the longitudinal oscillatory motion of second grade fluid between two infinite coaxial circular cylinders, oscillating along their common axis with given constant angular frequencies Ω_1 and Ω_2. Velocity field and associated tangential stress of the motion are determined by using Laplace and Hankel transforms. The solutions that have been obtained satisfy the governing differential equation and all imposed initial and boundary conditions as well. The solutions corresponding to the Newtonian fluid, performing the same motion, are also determined as special case of our general solutions. The respective solutions for the oscillatory motion of the second grade fluid between cylinders, when one of them is at rest, are also obtained from our general solutions. Furthermore, at the end of this note, the velocity profiles as well as the time variation of velocity corresponding to the oscillatory flow of second grade fluid and that of Newtonian fluid are plotted and discussed.

2. Longitudinal Oscillatory Flow between Two Coaxial Cylinders

Among the many constitutive assumptions that have been employed to study non-Newtonian fluid behavior, one class that has gained support from both the experimentalists and the theoreticians is that of Rivlin-Ericksen fluids of second grade. The Cauchy stress tensor $\mathbf{T}$ for such fluids is given by [12, 13].

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2\,, \tag{1}$$

where $-p$ is the pressure, $\mathbf{I}$ is the unit tensor, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined through

$$\mathbf{A}_1 = \text{grad}\mathbf{v} + (\text{grad}\mathbf{v})^T\,, \quad \mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1(\text{grad}\mathbf{v}) + (\text{grad}\mathbf{v})^T\mathbf{A}_1\,. \tag{2}$$

In the above relations, $\mathbf{v}$ is the velocity, d/dt denotes the material time derivative and grad denotes the gradient operator. Since the fluid is incompressible, it can undergo only isochoric motions and hence

$$div\,\mathbf{v} = tr\mathbf{A}_1 = 0\,. \tag{3}$$

If this model is required to be compatible with thermodynamics, then the material moduli must meet the following restrictions

$$\mu \geq 0, \quad \alpha_1 \geq 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0\,. \tag{4}$$

The sign of the material moduli α_1 and α_2 has been the subject of much controversy. A comprehensive discussion on the restrictions given in (4), as well as a critical review on the fluids of differential type, can be found in the extensive work of Dunn and Rajagopal [14].

2.1. **Mathematical Formulation and Governing Equation of Problem.** Suppose that an incompressible second grade fluid is situated in the annular region of two infinite coaxial circular cylinders of radii R_1 and $R_2(> R_1)$. At time $t = 0$, the fluid and cylinders are at rest. At time $t = 0^+$, the cylinders suddenly begin to oscillate along their common axis ($r = 0$) with the velocities $V_1\sin(\Omega_1 t)$ and $V_2\sin(\Omega_2 t)$, where Ω_1 is angular frequency of velocity of the inner cylinder and Ω_2 is that of the outer cylinder. Owing to the shear, the fluid between cylinders is gradually moved, its velocity being of the form

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\,\mathbf{e}_z\,, \tag{5}$$

where $\mathbf{e}_z$ is the unit vector along z-axis. For such flows the constraint of incompressibility is automatically satisfied.

Introducing (5) into the constitutive equation (1), we find that

$$\tau(r,t) = (\mu + \alpha_1\frac{\partial}{\partial t})\frac{\partial v(r,t)}{\partial r}\,, \tag{6}$$

where $\tau(r,t) = S_{rz}(r,t)$ is the shear stress. In the absence of body forces and a pressure gradient in the axial direction, the balance of the linear momentum leads to the relevant equation

$$\rho\frac{\partial v(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\tau(r,t)\,. \tag{7}$$

Eliminating $\tau(r,t)$ between Eqs. (6) and (7), we get the governing differential equation of our problem, as follows

$$\frac{\partial v(r,t)}{\partial t} = (\nu + \alpha\frac{\partial}{\partial t})\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t)\,; \quad r \in (R_1, R_2)\,, \;\; t > 0\,, \tag{8}$$

where $\alpha = \alpha_1/\rho$ and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid (ρ being its constant density).

The appropriate initial and boundary conditions are

$$v(r,0) = 0\,; \qquad r \in (R_1, R_2)\,, \tag{9}$$

$$v(R_1,t) = V_1\sin(\Omega_1 t)\,, \qquad v(R_2,t) = V_2\sin(\Omega_2 t) \quad \text{for} \quad t > 0\,, \tag{10}$$

where V_1, V_2, Ω_1 and Ω_2 are constants. To solve this problem, we shall use as in [16 - 18], the Laplace and Hankel transforms.

2.2. **Calculation of the Velocity Field.** Applying the Laplace transform to Eqs. (8) - (10), we obtain the ordinary differential equation

$$\frac{\partial^2\overline{v}(r,q)}{\partial r^2} + \frac{1}{r}\frac{\partial\overline{v}(r,q)}{\partial r} - \frac{q}{\alpha q + \nu}\overline{v}(r,q) = 0\,; \qquad r \in (R_1, R_2)\,, \tag{11}$$

where q is the parameter of Laplace transform and the image function $\overline{v}(r,q)$ of $v(r,t)$ has to satisfy the conditions

$$\overline{v}(R_1,q) = \frac{V_1\Omega_1}{q^2 + \Omega_1^2}\,, \qquad \overline{v}(R_2,q) = \frac{V_2\Omega_2}{q^2 + \Omega_2^2}\,. \tag{12}$$

In the following, let us denote by

$$\overline{v}_n(q) = \int_{R_1}^{R_2} r\overline{v}(r,q)B_0(r\,r_n)dr\,; \quad n = 1,2,3,\cdots, \tag{13}$$

the finite Hankel transforms of $\overline{v}(r,q)$, where r_n are the positive roots of the transcendental equation $B_0(R_1\,r) = 0$ and

$$B_0(r\,r_n) = J_0(r\,r_n)Y_0(R_2\,r_n) - J_0(R_2\,r_n)Y_0(r\,r_n)\,. \tag{14}$$

In the above relation, $J_0(\cdot)$ and $Y_0(\cdot)$ are the Bessel functions of order *zero* of first and second kind, respectively. Applying the finite Hankel transform (13) to Eq. (11) and taking into account the conditions (12), we find that

$$\frac{2V_2\Omega_2}{\pi(q^2+\Omega_2^2)} - \frac{2V_1\Omega_1}{\pi(q^2+\Omega_1^2)}\frac{J_0(R_2 r_n)}{J_0(R_1 r_n)} - r_n^2\overline{v}_n(q) - \frac{q}{\alpha q+\nu}\overline{v}_n(q) = 0\,, \tag{15}$$

or equivalently,

$$\overline{v}_n(q) = \frac{2V_2\Omega_2(\alpha q+\nu)}{\pi(q^2+\Omega_2^2)(\alpha r_n^2 q + q + \nu r_n^2)} - \frac{2V_1\Omega_1(\alpha q+\nu)}{\pi(q^2+\Omega_1^2)(\alpha r_n^2 q + q + \nu r_n^2)}\frac{J_0(R_2 r_n)}{J_0(R_1 r_n)}\,. \tag{16}$$

In order to determine $\overline{v}(r,q)$, we firstly write $\overline{v}_n(q)$ under the suitable form

$$\overline{v}_n(q) = \frac{2V_2\Omega_2}{\pi r_n^2(q^2+\Omega_2^2)} - \frac{2V_1\Omega_1}{\pi r_n^2(q^2+\Omega_1^2)}\frac{J_0(R_2 r_n)}{J_0(R_1 r_n)} - \frac{2V_2\Omega_2 q}{\pi r_n^2(q^2+\Omega_2^2)(\alpha r_n^2 q + q + \nu r_n^2)} + \frac{2V_1\Omega_1 q}{\pi r_n^2(q^2+\Omega_1^2)(\alpha r_n^2 q + q + \nu r_n^2)}\frac{J_0(R_2 r_n)}{J_0(R_1 r_n)}\,, \tag{17}$$

and use the inverse Hankel transform formula [15]

$$\overline{v}(r,q) = \frac{\pi^2}{2}\sum_{n=1}^{\infty}\frac{r_n^2 J_0^2(R_1\,r_n)B_0(r\,r_n)}{J_0^2(R_1\,r_n) - J_0^2(R_2\,r_n)}\overline{v}_n(q)\,, \tag{18}$$

and (A1) from appendix, we get $\overline{v}(r,q)$ in the following form

$$\overline{v}(r,q) = \frac{\frac{V_1\Omega_1}{q^2+\Omega_1^2}\ln(R_2/r) + \frac{V_2\Omega_2}{q^2+\Omega_2^2}\ln(r/R_1)}{\ln(R_2/R_1)} - \pi\sum_{n=1}^{\infty}\frac{J_0(R_1 r_n)B_0(r r_n)}{J_0^2(R_1 r_n) - J_0^2(R_2 r_n)} \times\left[V_2\Omega_2 J_0(R_1 r_n)\frac{q}{(q^2+\Omega_2^2)(\alpha r_n^2 q + q + \nu r_n^2)} - V_1\Omega_1 J_0(R_2 r_n)\frac{q}{(q^2+\Omega_1^2)(\alpha r_n^2 q + q + \nu r_n^2)}\right]. \tag{19}$$

Finally, by applying inverse Laplace transform to $\overline{v}(r,q)$ and using (A2), we get the velocity field in the following form

$$
\begin{aligned}
v(r,t) = {} & \frac{V_1 \ln(R_2/r)\sin(\Omega_1 t) + V_2 \ln(r/R_1)\sin(\Omega_2 t)}{\ln(R_2/R_1)} \\
& - \pi \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) B_0(r r_n)}{(1+\alpha r_n^2)[J_0^2(R_1 r_n) - J_0^2(R_2 r_n)]} \\
& \times \left[V_2 \Omega_2 J_0(R_1 r_n) \int_0^t \cos \Omega_2 (t-\tau) \exp\left(\frac{-\nu r_n^2 \tau}{1+\alpha r_n^2} \right) d\tau \right. \\
& \left. - V_1 \Omega_1 J_0(R_2 r_n) \int_0^t \cos \Omega_1 (t-\tau) \exp\left(\frac{-\nu r_n^2 \tau}{1+\alpha r_n^2} \right) d\tau \right],
\end{aligned} \tag{20}
$$

or equivalently,

$$
\begin{aligned}
v(r,t) = {} & \frac{V_1 \ln(R_2/r)\sin(\Omega_1 t) + V_2 \ln(r/R_1)\sin(\Omega_2 t)}{\ln(R_2/R_1)} \\
& - \pi \sum_{n=1}^{\infty} \frac{J_0(R_1 r_n) B_0(r r_n)}{J_0^2(R_1 r_n) - J_0^2(R_2 r_n)} \\
& \times \left[\frac{V_2 \Omega_2 J_0(R_1 r_n)}{\nu^2 r_n^4 + \Omega_2^2 (1+\alpha r_n^2)^2} \right. \\
& \times \left\{ \nu r_n^2 \left(\cos(\Omega_2 t) - \exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right) \right) + \Omega_2 (1+\alpha r_n^2) \sin(\Omega_2 t) \right\} \\
& - \frac{V_1 \Omega_1 J_0(R_2 r_n)}{\nu^2 r_n^4 + \Omega_1^2 (1+\alpha r_n^2)^2} \\
& \left. \times \left\{ \nu r_n^2 \left(\cos(\Omega_1 t) - \exp\left(\frac{-\nu r_n^2 t}{1+\alpha r_n^2}\right) \right) + \Omega_1 (1+\alpha r_n^2) \sin(\Omega_1 t) \right\} \right].
\end{aligned} \tag{21}
$$

2.3. **Shear Stress Distribution.** Applying the Laplace transform to Eq. (6), we find that

$$
\overline{\tau}(r,q) = (\mu + \alpha_1 q) \frac{\partial \overline{v}(r,q)}{\partial r}, \tag{22}
$$

where,

$$
\begin{aligned}
\frac{\partial \overline{v}(r,q)}{\partial r} = {} & \frac{1}{r \ln(R_2/R_1)} \left(\frac{V_2 \Omega_2}{q^2 + \Omega_2^2} - \frac{V_1 \Omega_1}{q^2 + \Omega_1^2} \right) \\
& + \pi \sum_{n=1}^{\infty} \frac{r_n J_0(R_1 r_n) \widetilde{B}_0(r r_n)}{J_0^2(R_1 r_n) - J_0^2(R_2 r_n)}
\end{aligned}
$$

$$\times\Bigg[\frac{V_2\Omega_2 J_0(R_1r_n)}{\nu^2r_n^4+\Omega_2^2(1+\alpha r_n^2)^2}\Bigg\{\nu r_n^2\Bigg(\frac{\Omega_2}{q^2+\Omega_2^2}-\frac{1}{q+\frac{\nu r_n^2}{1+\alpha r_n^2}}\Bigg)+\frac{\Omega_2^2(1+\alpha r_n^2)}{q^2+\Omega_2^2}\Bigg\}$$

$$-\frac{V_1\Omega_1 J_0(R_2r_n)}{\nu^2r_n^4+\Omega_1^2(1+\alpha r_n^2)^2}\Bigg\{\nu r_n^2\Bigg(\frac{\Omega_1}{q^2+\Omega_1^2}-\frac{1}{q+\frac{\nu r_n^2}{1+\alpha r_n^2}}\Bigg)+\frac{\Omega_1^2(1+\alpha r_n^2)}{q^2+\Omega_1^2}\Bigg\}\Bigg]. \quad (23)$$

has been obtained from (21) and (A3), where in above relation

$$\widetilde{B}_0(rr_n)=J_1(rr_n)Y_0(R_2r_n)-J_0(R_2r_n)Y_1(rr_n)\,.$$

Introducing (23) into (22), applying again the inverse Laplace transform to the obtained result, we find for the shear stress the expression

$$\tau(r,t)=\frac{1}{r\ln(R_2/R_1)}\Big\{\mu\big[V_2\sin(\Omega_2t)-V_1\sin(\Omega_1t)\big]$$
$$+\alpha_1\big[V_2\Omega_2\cos(\Omega_2t)-V_1\Omega_1\cos(\Omega_1t)\big]\Big\}$$
$$+\pi\sum_{n=1}^{\infty}\frac{r_nJ_0(R_1r_n)\widetilde{B}_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}\Big(J_0(R_1r_n)g_2-J_0(R_2r_n)g_1\Big), \quad (24)$$

where, in above Eq. (24)

$$g_m=\frac{V_m\Omega_m}{\nu^2r_n^4+\Omega_m^2(1+\alpha r_n^2)^2}\Big\{\big[\mu\nu r_n^2+\alpha_1\Omega_m^2(1+\alpha r_n^2)\big]\cos(\Omega_mt)$$
$$+\big[\mu\Omega_m(1+\alpha r_n^2)-\alpha_1\nu\Omega_mr_n^2\big]\sin(\Omega_mt)$$
$$+\big[\frac{\alpha_1\nu^2r_n^4}{1+\alpha r_n^2}-\mu\nu r_n^2\big]\exp\big(\frac{-\nu r_n^2t}{1+\alpha r_n^2}\big)\Big\};\quad m=1,2.$$

3. Limiting Case

Making $\alpha\to0$ (equivalently $\alpha_1\to0$) into Eqs. (21) and (24), we obtain the velocity field

$$v_N(r,t)=\frac{V_1\ln(R_2/r)\sin(\Omega_1t)+V_2\ln(r/R_1)\sin(\Omega_2t)}{\ln(R_2/R_1)}$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_0(R_1r_n)B_0(rr_n)}{J_0^2(R_1r_n)-J_0^2(R_2r_n)}$$

$$\times\left[\frac{V_2\Omega_2 J_0(R_1 r_n)}{\nu^2 r_n^4+\Omega_2^2}\left\{\nu r_n^2\Big(\cos(\Omega_2 t)-\exp(-\nu r_n^2 t)\Big)+\Omega_2\sin(\Omega_2 t)\right\}\right.$$
$$\left.-\frac{V_1\Omega_1 J_0(R_2 r_n)}{\nu^2 r_n^4+\Omega_1^2}\left\{\nu r_n^2\Big(\cos(\Omega_1 t)-\exp(-\nu r_n^2 t)\Big)+\Omega_1\sin(\Omega_1 t)\right\}\right], \qquad (25)$$

and associated shear stress

$$\tau_N(r,t)=\frac{\mu\big[V_2\sin(\Omega_2 t)-V_1\sin(\Omega_1 t)\big]}{r\ln(R_2/R_1)}$$
$$+\mu\pi\sum_{n=1}^{\infty}\frac{r_n J_0(R_1 r_n)\widetilde{B}_0(r r_n)}{J_0^2(R_1 r_n)-J_0^2(R_2 r_n)}$$
$$\times\left[\frac{V_2\Omega_2 J_0(R_1 r_n)}{\nu^2 r_n^4+\Omega_2^2}\left\{\nu r_n^2\Big(\cos(\Omega_2 t)-\exp(-\nu r_n^2 t)\Big)+\Omega_2\sin(\Omega_2 t)\right\}\right.$$
$$\left.-\frac{V_1\Omega_1 J_0(R_2 r_n)}{\nu^2 r_n^4+\Omega_1^2}\left\{\nu r_n^2\Big(\cos(\Omega_1 t)-\exp(-\nu r_n^2 t)\Big)+\Omega_1\sin(\Omega_1 t)\right\}\right], \qquad (26)$$

corresponding to the Newtonian fluid, performing the same motion.

4. Concluding Remarks

Our purpose, in this paper, was to establish exact solutions for the velocity field and associated shear stress corresponding to the flow of a second grade fluid between two infinite coaxial circular cylinders, by using Laplace and Hankel transforms. The motion of fluid was due to the simple harmonic sine oscillations of both cylinders along their common axis, with different angular frequencies Ω_1 and Ω_2 of their velocities. It is important to point out that the velocity field and the shear stress for the oscillatory motion between the cylinders, when one of them is at rest, can be obtained from our general solutions by making $V_1 = 0$, $V_2 = V$ and $\Omega_2 = \Omega$ (when inner cylinder is at rest) or $V_1 = V$, $V_2 = 0$ and $\Omega_1 = \Omega$ (when outer cylinder is at rest). For instance, the velocity field for the flow of second grade fluid, when inner cylinder is at rest and outer cylinder is oscillating longitudinally, is given by (from Eq. (21))

$$v(r,t)=\frac{V\ln(r/R_1)\sin\Omega t}{\ln(R_2/R_1)}-\pi V\Omega\sum_{n=1}^{\infty}\frac{J_0^2(R_1 r_n)B_0(r r_n)}{J_0^2(R_1 r_n)-J_0^2(R_2 r_n)}$$
$$\times\frac{\nu r_n^2\big(\cos\Omega t-\exp(\frac{-\nu r_n^2 t}{1+\alpha r_n^2})\big)+\Omega(1+\alpha r_n^2)\sin\Omega t}{\nu^2 r_n^4+\Omega^2(1+\alpha r_n^2)^2}, \qquad (27)$$

The solutions, that have been obtained, satisfy the governing equation and all imposed initial and boundary conditions and for $\alpha \to 0$ (equivalently $\alpha_1 \to 0$) reduce to the similar solutions for Newtonian fluid.

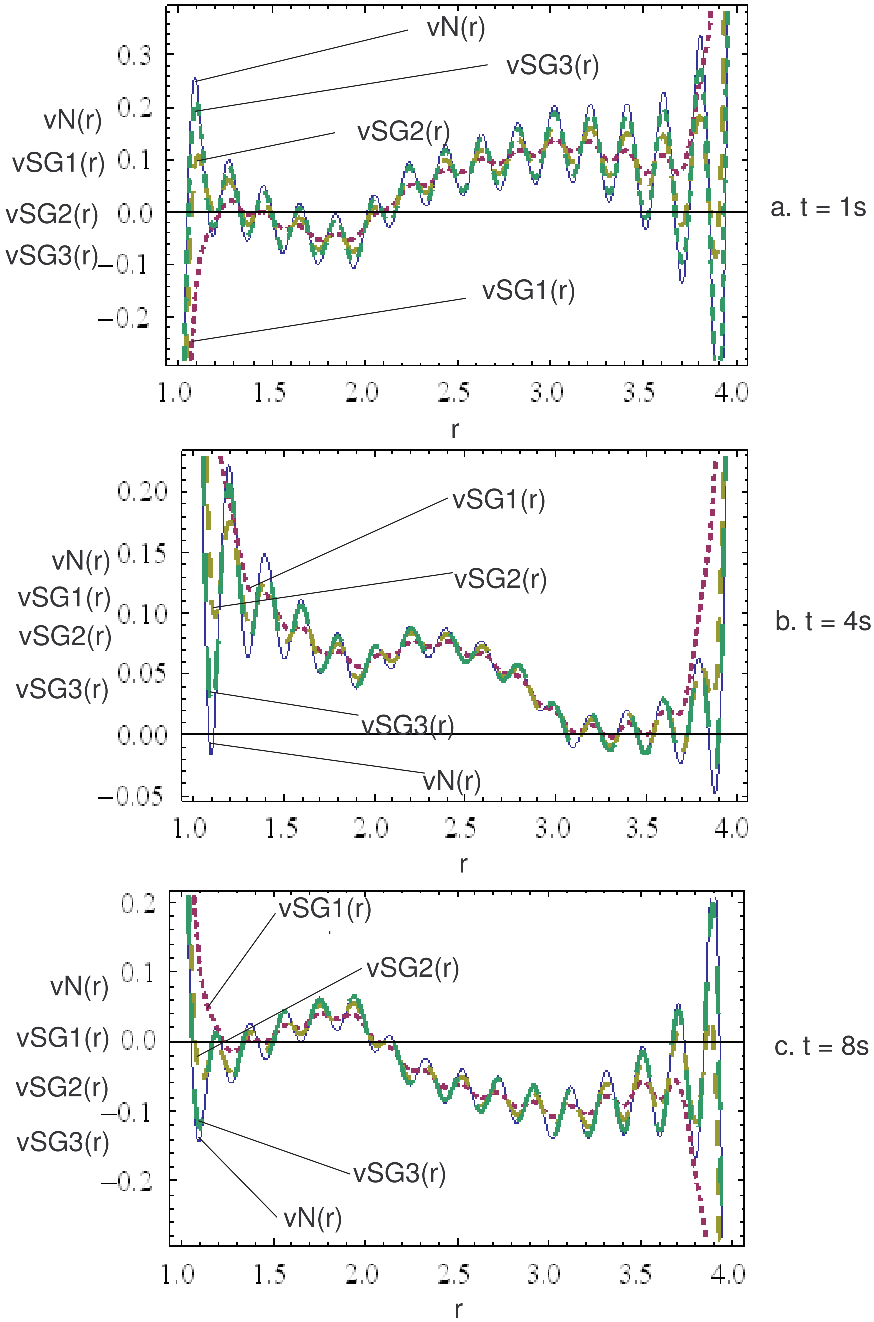

FIGURE 1. Velocity profiles for different values of time t. vSG1(r) for $\alpha = 0.005$, vSG2(r) for $\alpha = 0.001$ and vSG3(r) for $\alpha = 0.0002$.

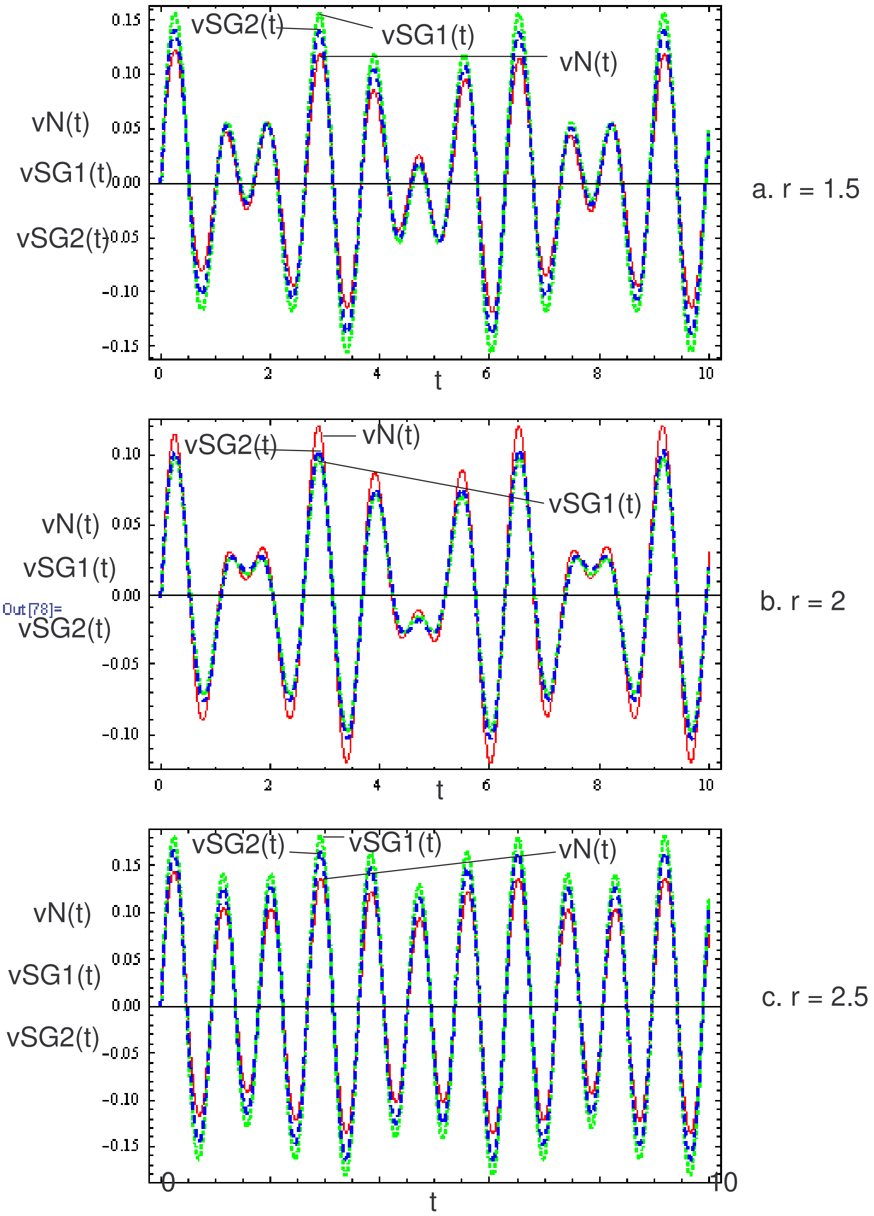

FIGURE 2. Time variation of velocity at different positions. vSG1(t) for $\alpha = 0.005$ and vSG2(t) for $\alpha = 0.001$.

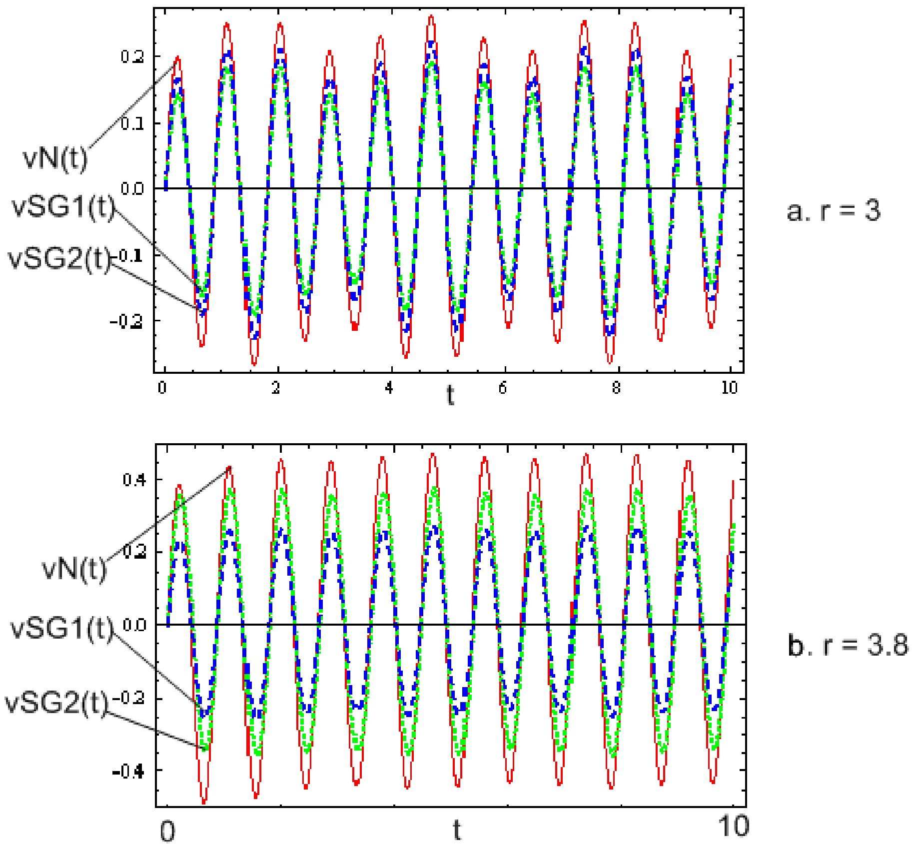

FIGURE 3. Time variation of velocity at different positions. vSG1(t) for $\alpha = 0.005$ and vSG2(t) for $\alpha = 0.001$.

Finally the graphical illustrations are given to show the comparison between the flow of second grade fluid and that of Newtonian fluid. These graphs also shows the influence of the parameter α on the velocity $v(r,t)$. In all figures we have considered $R_1 = 1$, $R_2 = 4$, $V_1 = 1$, $V_2 = 4$, $\Omega_1 = 5$, $\Omega_2 = 7$ and $\nu = 1.1746 \times 10^{-3}$, while SI units for all the parameters have been chosen. The roots r_n have been approximated by $n\pi/(R_2 - R_1)$ [19].

In Fig. 1, the profiles of the velocity $v(r,t)$, corresponding to the motion of Newtonian fluid $\big($the curves vN(r)$\big)$ and those for second grade fluid $\big($the curves vSG1(r) for $\alpha = 0.005$, vSG2(r) for $\alpha = 0.001$ and vSG3(r) for $\alpha = 0.0002\big)$ are plotted for different values of the time t. Fig. 2 and 3 depict the histories of the velocity field $v(r,t)$ at the positions $r = 1.5$, 2, 2.5, 3 and 3.8, for $t \in [0, 10]$ and different values of α. It is clear from all these figures that the curves corresponding to the flow of second grade fluid tend to that of Newtonian fluid, as α become smaller and smaller. This shows the correctness of our calculi and of the solutions that have been obtained.

Appendix

Some results used in the text:

(A1) The finite Hankel transform of the function

$$a(r) = \frac{A\ln(R_2/r) + B\ln(r/R_1)}{\ln(R_2/R_1)}$$

satisfying $a(R_1) = A$ and $a(R_2) = B$ is

$$a_n = \int_{R_1}^{R_2} ra(r)\, B_0(rr_n)dr = \frac{2B}{\pi r_n^2} - \frac{2A}{\pi r_n^2}\frac{J_0(R_2r_n)}{J_0(R_1r_n)}\,.$$

In order to prove (A1), we integrate by parts and use the next identities:

$$\int J_1(u)du = -J_0(u), \qquad J_1(R_1r_n)Y_0(R_1r_n) - J_0(R_1r_n)Y_1(R_1r_n) = \frac{2}{\pi R_1 r_n}$$

and

$$J_1(R_2r_n)Y_0(R_2r_n) - J_0(R_2r_n)Y_1(R_2r_n) = \frac{2}{\pi R_2 r_n} \quad \text{if} \quad B_0(R_1r_n) = 0\,.$$

(A2) If $u_1(t) = L^{-1}\{\overline{u}_1(q)\}$ and $u_2(t) = L^{-1}\{\overline{u}_2(q)\}$ then

$$L^{-1}\{\overline{u}_1(q)\overline{u}_2(q)\} = (u_1 * u_2)(t) = \int_0^t u_1(t-s)u_2(s)ds = \int_0^t u_1(s)u_2(t-s)ds\,.$$

(A3) $\frac{d}{du}[Y_0(u)] = -Y_1(u)$ and $\frac{d}{du}[J_0(u)] = -J_1(u)\,.$

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