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DIVISOR PATH DECOMPOSITION NUMBER OF A GRAPH

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ABSTRACT. A *decomposition* of a graph G is a collection Ψ of edge-disjoint subgraphs $H_1, H_2, \ldots, H_n$ of G such that every edge of G belongs to exactly one H_i. If each H_i is a path in G, then Ψ is called a *path partition* or *path cover* or *path decomposition* of G. A *divisor path decomposition* of a (p, q)-graph G is a path cover Ψ of G such that the length of all the paths in Ψ divides q. The minimum cardinality of a divisor path decomposition of G is called the *divisor path decomposition number* of G and is denoted by $\pi_D(G)$. In this paper, we initiate a study of the parameter π_D and determine the value of π_D for some standard graphs. Further, we obtain some bounds for π_D and characterize graphs attaining the bounds.

Key words: divisor path, greatest divisor path, divisor path decomposition, divisor path decomposition number.
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1. INTRODUCTION

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by p and q respectively. For terms not defined here we refer to Harary [4].

Let $P = (v_1, v_2, \ldots, v_n)$ be a path in a graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$. The vertices $v_2, v_3, \ldots, v_{n-1}$ are called *internal vertices* of P and v_1 and v_n are called *external vertices* of P. The length of a path is denoted by $l(P)$. A *spider tree* is a tree in which it has a unique vertex of degree 3.

A *decomposition* of a graph G is a collection of edge-disjoint subgraphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. If each $H_i \cong H$, then we say that G has a *H-decomposition* and we denote it

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by $H \mid G$. In this paper we extend this definition to non-isomorphic decomposition. If each H_i is a path, then it is called a *path partition* or *path cover* or *path decomposition* of G. The minimum cardinality of a path partition of G is called the path partition number of G and is denoted by $\pi(G)$ and any path partition Ψ of G for which $|\Psi| = \pi(G)$ is called a *minimum path partition* or *π-cover* of G. The parameter π was studied by Harary and Schwenk [5], Peroche [9], Stanton *et.al.,* [10] and Arumugam and Suresh Suseela [2].

Various types of path decompositions and corresponding parameters have been studied by several authors by imposing conditions on the paths in the decomposition. Some such path decomposition parameters are acyclic graphoidal covering number [2], simple path covering number [1], 2-graphoidal path covering number [6] and m-graphoidal path covering number [7]. Another such decomposition is equiparity path decomposition(EQPPD) which was defined by K.Nagarajan, A.Nagarajan and I.Sahul hamid [8].

Definition 1.1. [8] *An equiparity path decomposition(EQPPD) of a graph G is a path cover Ψ of G such that the lengths of all the paths in Ψ have the same parity.*

Since for any graph G, the edge set $E(G)$ is an equiparity path decomposition, the collection $\mathcal{P}_P$ of all equiparity path decompositions Ψ of G is non-empty. Let $\pi_P(G) = min\,|\Psi|$. Then $\pi_P(G)$ is called the *equiparity path decomposition number* of G and any equiparity path decomposition Ψ of G for which $|\Psi| = \pi_P(G)$ is called a *minimum equiparity path decomposition* of G or *π_P-cover* of G. The parameter π_P was studied in [8].

If the lengths of all the paths in Ψ are even(odd) then we say that Ψ is an *even (odd) parity path decomposition*, shortly *EPPD (OPPD).*

Remark 1.2. [8] *If G is a graph of odd size, then any EQPPD Ψ of a graph G is an OPPD and consequently $\pi_P(G)$ is odd.*

In this paper we define a new path called divisor path of a graph as follows.

Definition 1.3. *Let G be a (p, q)-graph with p vertices and q edges and let P be a path in G. If the length of the path P divides q, then P is called a divisor path in G.*

Note that the edges of a graph are divisor paths. The divisor path of length l where $1 < l < q$ is called *proper divisor path*, otherwise it is called *improper divisor path.* So, the edges of a graph are improper divisor paths. Also for a path, the path itself is a improper divisor path.

Example 1.4. *Consider the following graph G.*

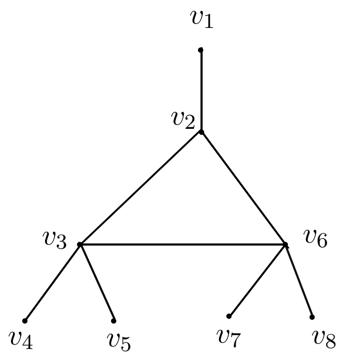

Fig 1

Here $q = 8$. *The path* $(v_4, v_3, v_2, v_6, v_8)$ *is a divisor path, but the path* (v_5, v_3, v_6, v_7) *is not a divisor path. Also the path* (v_1, v_2) *is an improper divisor path.*

Definition 1.5. *If* $q = d_1 d_2 \dots d_k$, *where* d_i*'s are proper divisors of* q, *then* $d = \max_{1 \le i \le k}\{d_i\}$ *is called the greatest divisor of* q *and is denoted by* $gd(q)$.

Definition 1.6. *Let* $\{P_i : 1 \le i \le k\}$ *be the family of all the divisor paths of a graph* G. *The path of length* $d = \max_{1 \le i \le k} l(P_i)$ *is called the greatest divisor path of* G *and the length* d *is denoted by gdpl(G).*

Note that the $gdpl(G)$ need not be $gd(q)$.

Example 1.7. *Consider the following star graph* G.

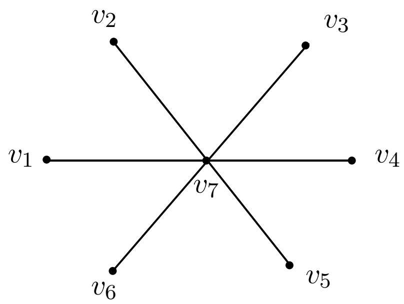

Fig 2

Here $q = 6$. *Clearly we see that* $gd(q) = 3$, *but* $gdpl(G) = 2$.

Consider the following path decomposition theorems.

Theorem 1.8. [3] *For any connected* (p, q)*-graph* G, *if* q *is even, then* G *has a* P_3*-decomposition.*

Theorem 1.9. [10] *If G is a 3-regular (p, q)-graph, then G is P_4 decomposable and $\pi(G) = \frac{q}{3} = \frac{p}{2}$.*

Theorem 1.10. [10] *A complete graph K_{2n} is hamilton path decomposable of length $2n - 1$.*

Theorem 1.8 and Theorem 1.9 give path decomposition in which all the paths are of length 2 and 3 respectively which divide q. Theorem 1.10 gives path decomposition in which all the paths are of length $2n - 1$ which divides $q = n(2n - 1)$. Thus, we observe that the lengths of all the paths in the above path decompositions divide q. This observation motivates the following definition.

Definition 1.11. *A divisor path decomposition (DPD) of a graph G is a path cover Ψ of G such that the lengths of all the paths in Ψ divide q.*

Since the edge set $E(G)$ of a graph G is a divisor path decomposition, the collection $\mathcal{P}_D$ of all divisor path decompositions of G is non-empty. Let $\pi_D(G) = min\{|\Psi| : \Psi \in \mathcal{P}_D\}$. Then $\pi_D(G)$ is called the *divisor path decomposition number* of G and any divisor path decomposition Ψ of G for which $|\Psi| = \pi_D(G)$ is called a *minimum divisor path decomposition* of G or π_D*-cover* of G.

Example 1.12. *Consider the following spider tree G.*

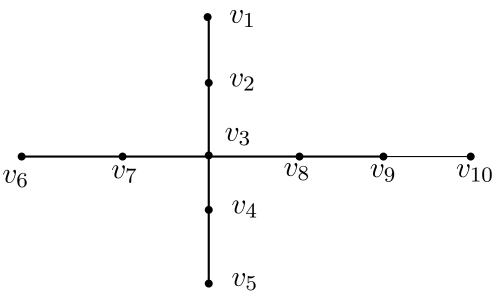

Fig 3

Here $q = 9$ and $\{(v_1, v_2, v_3, v_4), (v_4, v_5), (v_6, v_7), (v_7, v_3), (v_3, v_8, v_9, v_{10})\}$ forms a π_D-cover so that $\pi_D(G) = 5$. Note that $\{(v_1, v_2, v_3, v_4, v_5), (v_6, v_7, v_3, v_8, v_9, v_{10})\}$ forms a π-cover so that $\pi(G) = 2$.

Remark 1.13. *Let $\Psi = \{P_1, P_2, \ldots, P_n\}$ be an DPD of a (p, q)-graph G such that $l(P_1) \leq l(P_2) \leq \ \ldots \leq l(P_n)$. Since every edge of G is in exactly one path P_i, we have $\sum_{i=1}^{n} l(P_i) = q$ and hence every DPD of G gives rise to a partition of a positive integer q into the divisors(not necessarily distinct) of q.*

In this paper we initiate a study of the parameter π_D and determine the value of π_D for some standard graphs. Further, we obtain bounds for π_D and characterize graphs attaining the bounds.

2. Main Results

Hereafter, we consider G as a graph, which is not a path. We first present a general result which is useful in determining the value of π_D.

Theorem 2.1. *For any DPD Ψ of a graph G, let $t_\Psi = \sum_{P\in\Psi} t(P)$, where $t(P)$ denotes the number of internal vertices of P and let $t = \max t_\Psi$, where the maximum is taken over all divisor path decompositions Ψ of G. Then $\pi_D(G) = q - t$.*

Proof. Let Ψ be any DPD of G.

$$\begin{aligned} \text{Then } q &= \sum_{P\in\Psi} |E(P)| \\ &= \sum_{P\in\Psi} (t(P)+1) \\ &= \sum_{P\in\Psi} t(P) + |\Psi| \\ &= t_\Psi + |\Psi|. \end{aligned}$$

Hence $|\Psi| = q - t_\Psi$ so that $\pi_D = q - t$. □

Next we will find some bounds for π_D. First, we find a bound for π_D in terms of the size of G.

Theorem 2.2. *For any graph G of even size, $\pi_D(G) \leq \frac{q}{2}$.*

Proof. It follows from Theorem 1.8 that G has a P_3- decomposition, which is a DPD of G and hence $\pi_D(G) \leq \frac{q}{2}$. □

Remark 2.3. *The bound given in Theorem 2.2 is sharp. For the cycle C_4 and the star $K_{1,n}$, where n is even, $\pi_D = \frac{q}{2}$.*

The following problem naturally arises.

Problem 2.4. *Characterize graphs of an even size for which $\pi_D = \frac{q}{2}$.*

Observation 2.5. *If G is a graph with odd size q and $q \geq 3$, then $\pi_D(G) \geq 3$.*

Now, we characterize graphs attaining the extreme bounds.

Theorem 2.6. *For a (p,q)-graph G, $1 \leq \pi_D(G) \leq q$. Also $\pi_D(G) = 1$ if and only if G is a path and $\pi_D(G) = q$ if and only if G has no proper divisor paths.*

Proof. The inequalities are trivial. Further, it is obvious that $\pi_D(G) = 1$ if and only if G is a path.

Now, suppose $\pi_D(G) = q > 1$. Then it follows from Theorem 2.2 that q is odd. Then G has no proper divisor path of length 2. Suppose G has a proper divisor path P of length ≥ 3. Then the path P together with the remaining edges form a DPD(Ψ) of G so that $\pi_D(G) \leq |\Psi| = q - l(P) + 1 < q$, which is a contradiction. Thus, G has no proper divisor paths. Converse is obvious. □

Corollary 2.7. *If q is an odd prime, then $\pi_D(G) = q$.*

Proof. If q is an odd prime, then there is no proper divisor path in G. Hence the result follows from Theorem 2.6. □

Remark 2.8. *The converse of the Corollary 2.7 need not be true. For example, consider the star graph $K_{1,q}$, where q is an odd composite number. Note that $\pi_D(K_{1,q}) = q$.*

Theorem 2.9. *For any graph G, $\pi_D(G) = q - 1$ if and only if $G \cong P_3$.*

Proof. Suppose $\pi_D(G) = q - 1$. If G has a divisor path P with $l(P) \geq 3$, then the path P together with the remaining edges form a DPD (Ψ) of G so that $\pi_D(G) \leq |\Psi| = 1 + (q - l(P)) < q - 1$, which is a contradiction. Thus, every divisor path in G is of length 1 or 2. If G has divisor paths of length 1 only, then $\pi_D(G) = q$, which is a contradiction. So G has at least one divisor path of length 2. Then q is even and $\pi_D(G) \geq \frac{q}{2}$. From Theorem 2.2 , it follows that $\pi_D(G) = \frac{q}{2}$. By hypothesis, we have $q = 2$ and hence $G \cong P_3$. Converse is obvious. □

The following theorems give the lower bound for π_D in terms of π and π_P.

Theorem 2.10. *For any graph G, $\pi(G) \leq \pi_D(G)$.*

Proof. Since every divisor path decomposition is a path cover, we have $\pi(G) \leq \pi_D(G)$. □

Remark 2.11. *Equality holds in Theorem 2.10 for the star graph $K_{1,4}$ in which $\pi = 2 = \pi_D$. However, the inequality is strict. For, consider the Example 1.12 in which $\pi = 2 < 5 = \pi_D$.*

Theorem 2.12. *For any (p, q)-graph G, $\pi_P(G) \leq \pi_D(G)$ if q is odd.*

Proof. Since q is odd, the divisors of q are odd. If Ψ is a DPD of G, then the lengths of all the paths in Ψ are odd. Hence Ψ is an OPPD of G. Thus, every DPD is an OPPD so that $\pi_P(G) \leq \pi_D(G)$. □

Remark 2.13. *Equality holds in Theorem 2.12 for the cycle C_p with p odd and $p \cong 0(mod 3)$ in which $\pi_P = 3 = \pi_D$. Also the strict inequality holds for the cycle C_p with p odd prime ≥ 5 in which $\pi_P = 3$. From Corollary 2.7, it follows that $\pi_D = q \geq 5 > 3 = \pi_P$.*

The following theorem gives the lower bound for π_D in terms of the length of the greatest divisor path.

Theorem 2.14. *For any graph G, $\pi_D(G) \geq \frac{q}{d}$ where $d = gdpl(G)$.*

Proof. Let Ψ be a minimum π_D-cover of G. Since every edge of G is in exactly one path in Ψ we have $q = \sum\limits_{P\in\Psi} |E(P)|$. Also $|E(P)| \leq d$ for each P in Ψ. Hence $q \leq \pi_D d$ so that $\pi_D(G) \geq \frac{q}{d}$. □

Corollary 2.15. *If a hamilton path is a divisor path of G, then $\pi_D(G) \geq \frac{q}{p-1}$.*

Proof. If a hamilton path is a divisor path, then it is the greatest divisor path of length $p-1$ in G. Then the result follows from Theorem 2.14. □

From the above bounds the following problems will naturally arise.

Problem 2.16. *Characterize the class of graphs for which (i) $\pi_D(G) = \pi(G)$ (ii) $\pi_D(G) = \frac{q}{d}$ (iii) $\pi_D(G) = \frac{q}{p-1}$ and (iv) $\pi_P(G) = \pi_D(G)$ if q is odd.*

Theorem 2.17. *Let G be a (p, q)-graph with $q = n^2$ where n is a prime. Then $\pi_D(G) = n^2 - kn + k$ where k is the number of paths of length n.*

Proof. Since n is prime, n is the only proper divisor of q. Then we observe that any DPD Ψ of G contains either divisor paths of length n or edges of G. Since there are k divisor paths of length n in G, $|\Psi| \geq n^2 - kn + k$ and so $\pi_D(G) \geq n^2 - kn + k$. Again, since there are k divisor paths of length n in G, these k paths and the remaining edges of G form a DPD of G so that $\pi_D(G) \leq n^2 - kn + k$ and hence $\pi_D(G) = n^2 - kn + k$. □

Corollary 2.18. *If there are n divisor paths of length n, then $\pi_D(G) = n$.*

In the following theorems, we determine the divisor path decomposition number of several classes of graphs such as cycles, wheels, stars, cubic graphs and complete graphs.

Theorem 2.19. *For a cycle C_p, $\pi_D(C_p) = \frac{q}{d}$ where $d = gdpl(C_p)$.*

Proof. Let $C_p = (v_1, v_2, \ldots, v_p, v_1)$. Since d divides q, there are $\frac{q}{d}$ divisor paths of length d in C_p and they form a DPD of C_p. Hence $\pi_D(C_p) \leq \frac{q}{d}$. From Theorem 2.14, it follows that $\pi_D(C_p) = \frac{q}{d}$. □

Theorem 2.20. *For the wheel W_p on p vertices, we have*

$$\pi_D(W_p) = \begin{cases} \frac{p-1}{2} & \text{if } p \text{ is odd,} \\ \frac{p}{2} & \text{if } p \text{ is even and } p \cong 1 (mod 3), \\ \frac{p}{2} + 1 & \text{if } p \text{ is even and } p \ncong 1 (mod 3). \end{cases}$$

Proof. Let $V(W_p) = \{v_1, v_2, \ldots, v_{p-1}, v_p\}$ and let $E(W_p) = \{v_iv_{i+1} : 1 \leq i \leq p-2\} \bigcup \{v_1v_{p-1}\} \bigcup \{v_pv_i : 1 \leq i \leq p-1\}$. Note that $q = 2(p-1)$.

Case (i): p is odd. Then 4 divides q.

Let $\Psi = \bigcup_{i=1}^{i=\frac{p-3}{2}} \{(v_{i+1}, v_i, v_p, v_{\frac{p-1}{2}+i}, v_{\frac{p+1}{2}+i})\} \bigcup \{(v_{\frac{p+1}{2}}, v_{\frac{p-1}{2}}, v_p, v_{p-1}, v_1)\}$. Then Ψ is a DPD with $|\Psi| = \frac{p-1}{2}$ and hence $\pi_D(W_p) \leq \frac{p-1}{2}$. Since every odd degree vertex of W_p is an end vertex of a path in any path cover of W_p, we have $\pi_D(W_p) \geq \frac{p-1}{2}$. Thus, $\pi_D(W_p) = \frac{p-1}{2}$.

Case (ii): p is even and $p \cong 1(mod3)$.

Let $\Psi = \{(v_1, v_2, \ldots, v_{p-1}, v_p)\} \bigcup \{(v_{p-1}, v_1, v_p, v_2)\} \bigcup_{i=1}^{i=\frac{p-2}{2}} \{(v_{2i-1}, v_p, v_{2i})\}$. Then Ψ is a DPD with $|\Psi| = \frac{p}{2}$ and hence $\pi_D(W_p) \leq \frac{p}{2}$. Since every odd degree vertex of W_p is an end vertex of a path in any path cover of W_p, we have $\pi_D(W_p) \geq \frac{p}{2}$. Thus, $\pi_D(W_p) = \frac{p}{2}$.

Case (iii): p is even and $p \ncong 1(mod3)$.

Let $\Psi = \{(v_1, v_2, \ldots, v_{p-1}, v_p)\} \bigcup \{(v_{p-1}, v_1)\} \bigcup_{i=1}^{i=\frac{p-2}{2}} \{(v_{2i-1}, v_p, v_{2i})\}$. Then Ψ is a DPD with $|\Psi| = \frac{p}{2} + 1$ and hence $\pi_D(W_p) \leq \frac{p}{2} + 1$. Since p is even, 4 does not divide q and also since $p \ncong 1(mod3)$, 3 does not divide q. So, any DPD of W_p does not contain the paths of length 3 and 4. It is observed that any π-cover of W_p must contain at least one path of length either 3 or 4. Thus, $\pi_D(W_p) > \pi(W_p) = \frac{p}{2}$ and so $\pi_D(W_p) \geq \frac{p}{2} + 1$. Hence $\pi_D(W_p) = \frac{p}{2} + 1$. □

Theorem 2.21. *For a 3-regular graph G, $\pi_D(G) = \frac{p}{2}$.*

Proof. We have $q = \frac{3p}{2}$. It follows from Theorem 1.9 that every 3-regular graph is P_4 decomposable. Also P_4's are divisor paths of G and hence $\pi_D(G) \leq \frac{q}{3} = \frac{p}{2}$. Further, since every vertex of G is of odd degree, they are the end vertices of paths in any path cover of G. So, we have $\pi_D(G) \geq \frac{p}{2}$. Thus, $\pi_D(G) = \frac{p}{2}$. □

Theorem 2.22. *For a star $K_{1,n}$,*

$$\pi_D(K_{1,n}) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If n is even, then from Theorem 2.2, it follows that $\pi_D(K_{1,n}) \leq \frac{n}{2}$. Since the path of length 2 is the greatest divisor path of $K_{1,n}$, from Theorem 2.14, it follows that $\pi_D(K_{1,n}) \geq \frac{n}{2}$. Hence $\pi_D(K_{1,n}) = \frac{n}{2}$. If n is odd, then there is no proper divisor path in $K_{1,n}$, and from Theorem 2.6, it follows that $\pi_D(K_{1,n}) = n$. □

Theorem 2.23. *For any $n \geq 1$, $\pi_D(K_{2n}) = n$.*

Proof. Since K_{2n} is decomposable into n hamilton paths of length $2n-1$ and $q = n(2n-1)$, these hamilton paths are greatest divisor paths and they form a DPD of K_{2n}. It follows that $\pi_D(K_{2n}) \leq n$. Further, since every vertex of K_{2n} is of odd degree, they are the end vertices of paths in any path cover of K_{2n}. So, we have $\pi_D(K_{2n}) \geq n$ and hence $\pi_D(K_{2n}) = n$. $\square$

Theorem 2.24. *For a complete graph K_{2n+1}, $\pi_D(K_{2n+1}) = 2n+1$, if n or $2n+1$ is prime.*

Proof. Let $V(K_{2n+1}) = \{v_1, v_2, \ldots, v_{2n+1}\}$.
Case (i): n is even. Consider the paths
$P_j = (v_{2n+j}, v_{j+2}, v_{2n+j-1}, v_{j+3}, v_{2n+j-2}, v_{j+4} \ldots, v_{\frac{3n}{2}+j})$
$P'_j = (v_{j+1}, v_1, v_{n+j+1}, v_{n+j}, v_{n+j+2}, v_{n+j-1}, v_{n+j+3}, \ldots, v_{\frac{3n}{2}+j})$ for $j = 1, 2, \ldots, n$ and

$$2n+i = \begin{cases} i & i \text{ for } i \geq 2, \\ 2n+i & \text{otherwise.} \end{cases}$$

and $P_{n+1} = (v_{2n+1}, v_2, v_3, v_4, v_5, \ldots, v_{n-1}, v_n, v_{n+1})$.
Case (ii): n is odd. Consider the paths
$P_j = (v_{2n+j}, v_{j+2}, v_{2n+j-1}, v_{j+3}, v_{2n+j-2}, v_{j+4} \ldots, v_{\frac{n+2j+3}{2}})$
$P'_j = (v_{j+1}, v_1, v_{n+j+1}, v_{n+j}, v_{n+j+2}, v_{n+j-1}, v_{n+j+3}, \ldots, v_{\frac{n+2j+3}{2}})$ for $j = 1, 2, \ldots, n$

$$2n+i = \begin{cases} i & i \text{ for } i \geq 2, \\ 2n+i & \text{otherwise.} \end{cases}$$

and $P_{n+1} = (v_{2n+1}, v_2, v_3, v_4, v_5, \ldots, v_{n-1}, v_n, v_{n+1})$.
In both the cases, for the first $2n$ paths, we select two paths of length n from each hamilton cycle of length $2n+1$. The last path P_{n+1} is obtained by properly arranging the remaining one edge of each hamilton cycle. Since $q = n(2n+1)$, the above paths of length n are divisor paths of K_{2n+1}. Thus, $\pi_D(K_{2n+1}) \leq 2n+1$.
Claim: $gdpl(K_{2n+1}) = n$.

We have $q = n(2n+1)$. If $2n+1$ is prime, then clearly the result follows. Suppose n is prime and $2n+1$ is not a prime. Since $2n+1$ is odd, any divisor of $2n+1$ is less than n. Thus, the claim follows.

Now, from the Theorem 2.14, it follows that $\pi_D(K_{2n+1}) \geq \frac{q}{n} = 2n+1$ and hence $\pi_D(K_{2n+1}) = 2n+1$. $\square$

The following examples illustrate the cases considered in the proof of the Theorem 2.24.

Example 2.25. *Consider K_5. Note that $n = 2$ (even), $2n+1 = 5$ and $q = 10$. Let $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$.*

Consider the following hamilton cycles of K_5.
$C_1 = (v_5, v_3, v_4, v_1, v_2, v_5)$
$C_2 = (v_2, v_4, v_5, v_1, v_3, v_2)$
Now, we select two paths of length 2 from each hamilton cycle as follows.

$P_1 = (v_5, v_3, v_4)$, $P_1' = (v_2, v_1, v_4)$ and
$P_2 = (v_2, v_4, v_5)$, $P_2' = (v_3, v_1, v_5)$.

Now, consider the path $P_3 = (v_5, v_2, v_3)$ of length 2, which is obtained by properly arranging the remaining one edge of each hamilton cycle. Thus, we see that $\{P_1, P_1', P_2, P_2', P_3\}$ forms a DPD of K_5 so that $\pi_D(K_5) = 5 = 2n+1$.

Example 2.26. *Consider K_7. Note that $n = 3$ (odd), $2n+1 = 7$ and $q = 21$. Let $V(K_7) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.*

Consider the following hamilton cycles of K_7.
$C_1 = (v_7, v_3, v_6, v_4, v_5, v_1, v_2, v_7)$
$C_2 = (v_2, v_4, v_7, v_5, v_6, v_1, v_3, v_2)$
$C_3 = (v_3, v_5, v_2, v_6, v_7, v_1, v_4, v_3)$

Now, we select two paths of length 3 from each hamilton cycle as follows.
$P_1 = (v_7, v_3, v_6, v_4)$, $P_1' = (v_2, v_1, v_5, v_4)$,
$P_2 = (v_2, v_4, v_7, v_5)$, $P_2' = (v_3, v_1, v_6, v_5)$ and
$P_3 = (v_3, v_5, v_2, v_6)$, $P_3' = (v_4, v_1, v_7, v_6)$.

Now, consider the path $P_4 = (v_7, v_2, v_3, v_4)$ of length 3, which is obtained by properly arranging the remaining one edge of each hamilton cycle. Thus, we see that $\{P_1, P_1', P_2, P_2', P_3, P_3', P_4\}$ forms a DPD of K_7 so that $\pi_D(K_7) = 7 = 2n+1$.

Next, we will find $\pi_D(K_{2n+1})$, if both n and $2n+1$ are composite.

Theorem 2.27. *Let $d_1, d_2, \ldots, d_k$ and $d_1', d_2', \ldots, d_l'$ be the divisors of n and $2n+1$ respectively and let $d = max\{d_i d_j' : n \leq d_i d_j' < 2n+1, 1 \leq i \leq k, 1 \leq j \leq l\}$. Then $\pi_D(K_{2n+1}) = \frac{n(2n+1)}{d}$.*

Proof. We have $q = n(2n+1)$. Clearly d divides $n(2n+1)$ and by definition of d, $gdpl(K_{2n+1}) = d$. Then by Theorem 2.14, it follows that $\pi_D(K_{2n+1}) \geq \frac{q}{d}$.

Consider the following hamilton cycle decomposition of K_{2n+1}.
$C_j = (v_1, v_{j+1}, v_{2n+j}, v_{j+2}, v_{2n+j-1}, v_{j+3}, v_{2n+j-2}, \ldots, v_{n+j-1}, v_{n+j+2}, v_{n+j}, v_{n+j+1}, v_1)$ for $j = 1, 2, \ldots, n$ and

$$2n+i = \begin{cases} i & i \text{ for } i \geq 2, \\ 2n+i & \text{otherwise.} \end{cases}$$

Let P_1 be a path such that $l(P_1) = d$ in C_1 starting from the vertex v_1. Let P_2 be a path such that $l(P_2) = d$ starting from the end vertex of P_1 in C_1 and select the appropriate section of the cycle C_2. Continuing this process, we get the paths $P_1, P_2, \ldots, P_{\frac{q}{d}}$ such that $l(P_i) = d$, $1 \leq i \leq \frac{q}{d}$ and they form a DPD of K_{2n+1}. Thus, $\pi_D(K_{2n+1}) \leq \frac{q}{d}$ so that $\pi_D(K_{2n+1}) = \frac{q}{d}$. Hence the theorem. □

The Theorem 2.27 is illustrated in the following example.

Example 2.28. *Consider the complete graph K_9. Here $q = 4 \times 9 = 36$. The divisors of 4 and 9 are 1, 2, 4 and 1, 3, 9 respectively. Then $d_i = 2$, $d'_j = 3$ and so $d = 6$. Now consider the hamilton cycle decomposition of K_9.*
$C_1 = (v_1, v_2, v_9, v_3, v_8, v_4, v_7, v_5, v_6, v_1)$
$C_2 = (v_1, v_3, v_2, v_4, v_9, v_5, v_8, v_6, v_7, v_1)$
$C_3 = (v_1, v_4, v_3, v_5, v_2, v_6, v_9, v_7, v_8, v_1)$
$C_4 = (v_1, v_5, v_4, v_6, v_3, v_7, v_2, v_8, v_9, v_1)$

From these cycles, we construct the divisors paths of length 6 as follows.
$P_1 = (v_1, v_2, v_9, v_3, v_8, v_4, v_7)$
$P_2 = (v_7, v_5, v_6, v_1, v_3, v_2, v_4)$
$P_3 = (v_4, v_9, v_5, v_8, v_6, v_7, v_1)$
$P_4 = (v_1, v_4, v_3, v_5, v_2, v_6, v_9)$
$P_5 = (v_9, v_7, v_8, v_1, v_5, v_4, v_6)$
$P_6 = (v_6, v_3, v_7, v_2, v_8, v_9, v_1)$

These paths form a DPD of K_9 and hence $\pi_D(K_9) = 6$.

Remark 2.29. *The following table gives the value of $\pi_D(K_{2n+1})$ for some composite numbers n and $2n + 1$.*

n	$2n+1$	d_i	d'_j	d	q	π_D
4	9	2	3	6	36	6
10	21	5	3	15	210	14
16	33	8	3	24	528	22
22	45	11	3	33	990	30
25	51	25	1	25	1275	51
27	55	9	5	45	1485	33
28	57	14	3	42	1596	28
32	65	4	13	52	2080	40
34	69	17	3	51	2346	46
38	77	38	1	38	2926	77

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