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FORCING EDGE DETOUR NUMBER OF AN EDGE DETOUR GRAPH

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Abstract. For two vertices u and v in a graph $G = (V, E)$, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. A set $S \subseteq V$ is called an *edge detour set* if every edge in G lies on a detour joining a pair of vertices of S. The *edge detour number* $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is an *edge detour basis* of G. A connected graph G is called an *edge detour graph* if it has an edge detour set. A subset T of an edge detour basis S of an edge detour graph G is called a *forcing subset* for S if S is the unique edge detour basis containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing edge detour number* $fdn_1(S)$ of S, is the minimum cardinality of a forcing subset for S. The *forcing edge detour number* $fdn_1(G)$ of G, is $\min\{fdn_1(S)\}$, where the minimum is taken over all edge detour bases S in G. The general properties satisfied by these forcing subsets are discussed and the forcing edge detour numbers of certain classes of standard edge detour graphs are determined. The parameters $dn_1(G)$ and $fdn_1(G)$ satisfy the relation $0 \leq fdn_1(G) \leq dn_1(G)$ and it is proved that for each pair a, b of integers with $0 \leq a \leq b$ and $b \geq 2$, there is an edge detour graph G with $fdn_1(G) = a$ and $dn_1(G) = b$.

Key words: detour, edge detour set, edge detour basis, edge detour number, forcing edge detour number.
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1. Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic definitions and terminologies, we refer to [1, 8].

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For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest u–v path in G. A u–v path of length $D(u, v)$ is called a *u–v detour*. It is known that the detour distance is a metric on the vertex set V. Detour distance and detour center of a graph were studied by Chartrand et al. in [2, 7].

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. A vertex v that belongs to every detour basis of G is a *detour vertex* in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G. These concepts were studied by Chartrand et al. in [3] and have interesting applications in Channel Assignment Problem in radio technologies [4, 9].

An edge e of G is said to lie on a u–v detour P if e is an edge of the path P. In general, there are graphs G for which there exist edges which do not lie on a detour joining any pair of vertices of V. For the graph G given in Figure 1, the edge v_1v_2 does not lie on a detour joining any pair of vertices of V. This motivated us to introduce the concepts of *weak edge detour set* of a graph [10] and *edge detour graphs* [11].

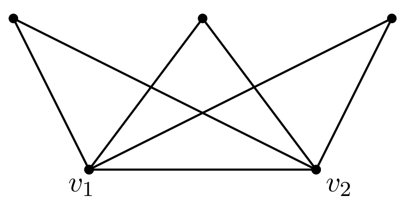


FIGURE 1. G

A set $S \subseteq V$ is called a *weak edge detour set* of G if every edge in G has both its ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is called a *weak edge detour basis* of G. Weak edge detour sets and weak edge detour number of a graph were introduced and studied by Santhakumaran and Athisayanathan in [10].

A set $S \subseteq V$ is called an *edge detour set* of G if every edge in G lies on a detour joining a pair of vertices of S. The *edge detour number* $dn_1(G)$ of G is the minimum order of its edge detour sets and any edge detour set of order $dn_1(G)$ is an *edge detour basis* of G. A graph G is called an *edge detour graph* if it has an edge detour set. A vertex v in an edge detour graph G is an *edge detour vertex* if v belongs to every edge detour basis of G. If G has a unique edge detour basis S, then every vertex in S is an edge detour vertex of G. Edge detour graphs were introduced in [11] and further studied in [12] by Santhakumaran and Athisayanathan.

For the graph G given in Figure 2 (a), the sets $S_1 = \{u, x\}$, $S_2 = \{u, w, x\}$ and $S_3 = \{u, v, x, y\}$ are detour basis, weak edge detour basis and edge detour basis of G respectively and hence $dn(G) = 2$, $dn_w(G) = 3$ and $dn_1(G) = 4$. For the graph G given in Figure 2 (b), the set $S = \{u_1, u_2\}$ is a detour basis, weak edge detour basis and an edge detour basis for G so that $dn(G) = dn_w(G) = dn_1(G) = 2$. The graphs G given in Figure 2 are edge detour graphs.

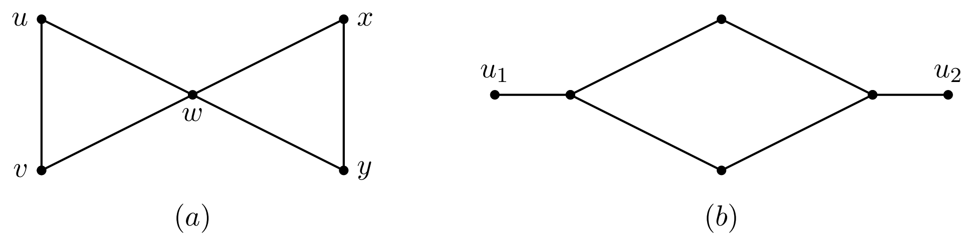


FIGURE 2. G

For the graph G given in Figure 1, the set $S = \{v_1, v_2\}$ is a detour basis and also a weak edge detour basis. However, it does not contain an edge detour set and so the graph G in Figure 1 is not an edge detour graph.

The following theorems are used in the sequel.

Theorem 1 ([11]). *For any edge detour graph G of order $p \geq 2$, $2 \leq dn_1(G) \leq p$.*

Theorem 2 ([11]). *If G is an edge detour graph of order $p \geq 3$ such that $\{u, v\}$ is an edge detour basis of G, then u and v are not adjacent.*

A vertex of degree 1 is an *end-vertex* of G.

Theorem 3 ([11]). *Every end-vertex of an edge detour graph G belongs to every edge detour set of G. Also if the set S of all end-vertices of G is an edge detour set, then S is the unique edge detour basis for G.*

Theorem 4 ([11]). *If T is a tree with $k \geq 2$ end-vertices, then $dn_1(T) = k$.*

Theorem 5 ([11]). *Let G be an edge detour graph with cut-vertices and S an edge detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

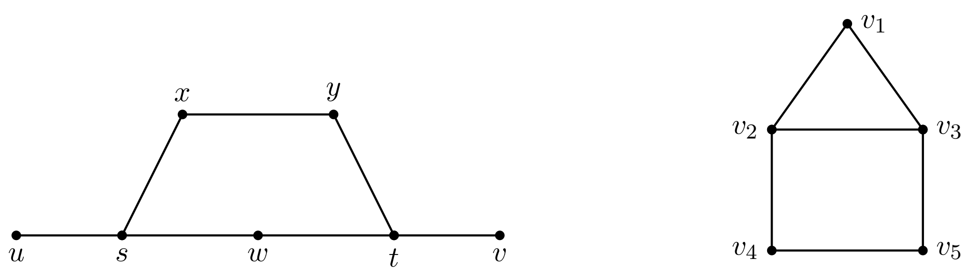
Throughout this paper G denotes a connected graph with at least two vertices.

2. Forcing Edge Detour Sets in a Graph

Forcing geodetic number and forcing dimension of a graph was introduced and studied in [5, 6].
In this section we introduce the forcing edge detour number of an edge detour graph and determine the same for some standard classes of edge detour graphs.

Definition 1. *Let G be an edge detour graph and S an edge detour basis of G. A subset $T \subseteq S$ is called a* forcing subset *for S if S is the unique edge detour basis containing T. A forcing subset for S of minimum cardinality is a* minimum forcing subset *of S. The* forcing edge detour number *of S, denoted by $fdn_1(S)$, is the cardinality of a minimum forcing subset for S. The* forcing edge detour number *of G, denoted by $fdn_1(G)$, is $fdn_1(G) = min\ \{fdn_1(S)\}$, where the minimum is taken over all edge detour bases S in G.*

Example 1. *For the graph G given in Figure 2(a), $S = \{u, v, x, y\}$ is the unique edge detour basis of G so that $fdn_1(G) = 0$. For the graph G given in Figure 3 (a), $S_1 = \{u, v, x\}$, $S_2 = \{u, v, y\}$ and $S_3 = \{u, v, w\}$ are the only edge detour bases of G and so $fdn_1(G) = 1$. For the graph G given in Figure 3 (b), $S_1 = \{v_1, v_2, v_4\}$, $S_2 = \{v_1, v_2, v_5\}$, $S_3 = \{v_1, v_3, v_4\}$, $S_4 = \{v_1, v_3, v_5\}$ and $S_5 = \{v_1, v_4, v_5\}$ are the only five edge detour bases of G so that $fdn_1(G) = 2$.*



(a) Figure 3. G (b)

The next three theorems are easy consequences of the respective definitions.

Theorem 6. *For every edge detour graph G, $0 \leq fdn_1(G) \leq dn_1(G)$.*

Remark 1. *The bounds in Theorem 6 are sharp. For the graph G given in Figure 2(a), $fdn_1(G) = 0$. For the complete graph K_4, $fdn_1(K_4) = dn_1(K_4) = 3$. Also, all the inequalities in Theorem 6 can be strict. For the graph G given in Figure 3(a), $fdn_1(G) = 1$ and $dn_1(G) = 3$ so that $0 < fdn_1(G) < dn_1(G)$.*

Theorem 7. *Let G be an edge detour graph. Then*
i) *$fdn_1(G) = 0$ if and only if G has a unique edge detour basis,*
ii) *$fdn_1(G) = 1$ if and only if G has at least two edge detour bases, one of which is a unique edge detour basis containing one of its elements, and*

iii) *$fdn_1(G) = dn_1(G)$ if and only if no edge detour basis of G is the unique edge detour basis containing any of its proper subsets.*

Theorem 8. *Let G be an edge detour graph and W the set of all edge detour vertices of G. Then $fdn_1(G) \leq dn_1(G) - |W|$.*

Remark 2. *The bound in Theorem 8 is sharp. For the graph G given in Figure 3(a), $dn_1(G) = 3$, $|W| = 2$ and $fdn_1(G) = 1$ as in Example 1. Also, the inequality in Theorem 8 can be strict. For the cycle C_4, $dn_1(C_4) = 2$, $|W| = 0$ and $fdn_1(C_4) = 1$. Thus $fdn_1(G) < dn_1(G) - |W|$.*

The following theorems give the forcing edge detour numbers of certain classes of graphs.

Theorem 9. *Let G be a complete graph K_p or an odd cycle C_p of order $p \geq 3$. Then a set $S \subseteq V$ is an edge detour of basis of G if and only if S consists of any three vertices of G.*

Proof. Let $G = K_p\,(p \geq 3)$. If $\{u, v\}$ is any set of two vertices of G, then all the edges of G other than uv lie on a u–v detour. Hence it follows that no two element subset of V is an edge detour set of G. Let $S = \{u, v, w\}$ be any set of three vertices of G. Then every edge e of G lies on a detour joining a pair of vertices of S. Hence it follows that S is an edge detour basis of G. Now, assume that S is an edge detour basis of G. By Theorem 2, $|S| \geq 3$. It follows from the first part of the proof that $|S| = 3$ and S consists of any three vertices of G.

Let G be an odd cycle $C_p(p \geq 3)$. If $\{u, v\}$ is any set of two vertices of G, then no edge on the u–v geodesic lies on the u–v detour in G and so no two element subset of V is an edge detour set of G. Let $S = \{u, v, w\}$ be any set of three vertices of G. Then every edge in G lies on any one of the u–v , v–w or u–w detours so that S is an edge detour basis of G. Now, assume that S is an edge detour basis of G. Since any set of two vertices of G is not an edge detour set of G, it follows just as above that $|S| = 3$ and S consists of any three vertices of G. □

If u and v are two vertices in a graph G, then the *distance* $d(u, v)$ is the length of a shortest u-v path in G. A u-v path of length $d(u, v)$ is a *u-v geodesic*. The *diameter* $d(G)$ of a connected graph G is the length of any longest geodesic. Two vertices u, v in G are *antipodal* if $d(u, v) = d(G)$.

Theorem 10. *Let G be an even cycle. Then a set $S \subseteq V$ is an edge detour basis of G if and only if S consists of two antipodal vertices of G.*

Proof. It is clear that every set S of two antipodal vertices of G is an edge detour set and so S is an edge detour basis of G. On the other hand, if u and v are not antipodal vertices, then the edges of u–v geodesic do not lie on the u–v detour in G and so the set $\{u, v\}$ is not an edge detour set of G. Therefore, every edge detour basis of G must consist of two antipodal vertices of G. □

Corollary 11. *For a complete graph $K_p\,(p \geq 3)$,*
i) $dn_1(K_3) = 3$ *and* $fdn_1(K_3) = 0$.
ii) $dn_1(K_p) = fdn_1(K_p) = 3$ *for* $p \geq 4$.

Proof. i) By Theorem 9, $dn_1(K_3) = 3$. Let v_1, v_2, v_3 be the vertices of K_3. Then, by Theorem 9, it is clear that the set $\{v_1, v_2, v_3\}$ is the unique edge detour basis of K_3 and hence by Theorem 7(i), $fdn_1(K_3) = 0$.
ii) Let $p \geq 4$. By Theorem 9, $dn_1(K_p) = 3$. Since $p \geq 4$, it follows from Theorem 9 that no subset of V of cardinality at most 2 is a forcing subset for any edge detour basis of K_p. Therefore, by Theorem 7(iii), $fdn_1(K_p) = dn_1(K_p) = 3$. □

Corollary 12. *If G is the cycle $C_p\,(p \geq 4)$, then*
i) $dn_1(G) = 2$ *and* $fdn_1(G) = 1$ *for p even.*
ii) $dn_1(G) = fdn_1(G) = 3$ *for p odd.*

Proof. i) Let p be even. It follows from Theorem 10 that $dn_1(G) = 2$ and that each vertex in G belongs to exactly only one edge detour basis of G. Hence every set consisting of a single vertex of G is a forcing subset for an edge detour basis of G so that $fdn_1(G) = 1$.
ii) Let p be odd. Then the result follows from Theorems 9 and 7(iii) and the proof is similar to that of Corollary 11(ii). □

A set S of vertices in a graph is *independent* if no two vertices in S are adjacent.

Theorem 13. *Let G be a complete bipartite graph $K_{m,n}\,(2 \leq m \leq n)$. Then a set $S \subseteq V$ is an edge detour basis of G if and only if S consists of any two independent vertices of G.*

Proof. Let X and Y be the partite sets of G with $|X| = m$ and $|Y| = n$. Let $S = \{u, v\}$ be any set of two independent vertices in G. Then it is clear that $D(u, v) = 2m - 2$ or $D(u, v) = 2m$ according to whether $u, v \in X$ or $u, v \in Y$. First assume that $u, v \in X$. Let xy be an edge such that $x \in X$ and $y \in Y$. If $x \neq u$, then the edge xy lies on the u–v detour $P : u, y, x, \ldots, v$ of length $2m - 2$. If $x = u$, then the edge xy lies on the u–v detour $P : u = x, y, \ldots, v$ of length $2m - 2$. Hence S is an edge detour set of G. The case when $u, v \in Y$ is similar.

Now, assume that S is an edge detour basis of G. It follows from the first part of the proof that $|S| = 2$. Let $S = \{u, v\}$. Then, by Theorem 2, u and v are not adjacent. Thus S consists of two independent vertices of G. □

Corollary 14. *For a complete bipartite graph $G = K_{m,n}\,(2 \le m \le n)$,*
i) $dn_1(G) = 2$ *and* $fdn_1(G) = 1$ *for* $m = 2$ *and* $n \ge 2$.
ii) $dn_1(G) = fdn_1(G) = 2$ *for* $m, n \ge 3$.

Proof. Let $X = \{u_1, u_2, \dots, u_m\}$ and $Y = \{v_1, v_2, \dots, v_n\}$ be the bipartite sets of $K_{m,n}\,(2 \le m \le n)$.
i) If $m = 2$ and $n = 2$, then $G = C_4$ and the result follows from Corollary 12(i). Let $m = 2$ and $n \ge 3$. By Theorem 13, $dn_1(G) = 2$. By Theorem 13, the set $X = \{u_1, u_2\}$ is the unique edge detour basis of G such that $\{u_1\}$ is a forcing subset for X. Since $n \ge 3$, there is more than one edge detour basis and it follows that $fdn_1(G) = 1$.
ii) By Theorem 13, $dn_1(G) = 2$. Also it follows from Theorem 13 that each vertex belongs to more than one edge detour basis of G and so $fdn_1(G) > 1$. Since $dn_1(G) = 2$, it follows that $fdn_1(G) = 2$. □

Theorem 15. *If T is a tree with $k \ge 2$ end-vertices, then $dn_1(G) = k$ and $fdn_1(G) = 0$.*

Proof. By Theorem 4, $dn_1(G) = k$. Since the set of all end-vertices of a tree is the unique edge detour basis, the result follows from Theorem 7(i). □

In view of Theorem 6, we have the following realization result.

Theorem 16. *For each pair a, b of integers with $0 \le a \le b$ and $b \ge 2$, there is an edge detour graph G with $fdn_1(G) = a$ and $dn_1(G) = b$.*

Proof. **Case 1:** $a = 0$. For each $b \ge 2$, let G be a tree with b end-vertices. Then $fdn_1(G) = 0$ and $dn_1(G) = b$ by Theorem 15.
Case 2: $a \ge 1$. For each $i\,(1 \le i \le a)$, let $F_i : u_i, v_i, w_i, x_i, u_i$ be the cycle of order 4 and let $H = K_{1,b-a}$ be the star at v whose set of end-vertices is $\{z_1, z_2, \dots, z_{b-a}\}$. Let G be the graph obtained by joining the central vertex v of H to the vertices u_i and w_i of each $F_i\,(1 \le i \le a)$. Clearly the graph G is connected and is shown in Figure 4.

Let $W = \{z_1, z_2, \dots, z_{b-a}\}$ be the set of all $(b-a)$ end-vertices of G. First, we show that $dn_1(G) = b$. By Theorems 3 and 5, every edge detour basis contains W and at least one vertex from each $F_i\,(1 \le i \le a)$. Thus $dn_1(G) \ge (b-a)+a = b$. On the other hand, since the set $S_1 = W \cup \{v_1, v_2, \dots, v_a\}$ is an edge detour set of G, it follows that $dn_1(G) \le |S_1| = b$. Therefore G is an edge detour graph and $dn_1(G) = b$.

Next we show that $fdn_1(G) = a$. It is clear that W is the set of all edge detour vertices of G. Hence it follows from Theorem 8 that $fdn_1(G) \le dn_1(G) - |W| = b - (b-a) = a$. Now, since $dn_1(G) = b$, it is easily seen that a set S is an edge detour basis of G if and only if S is of the form $S = W \cup \{y_1, y_2, \dots, y_a\}$, where $y_i \in \{v_i, x_i\} \subseteq V(F_i)\,(1 \le i \le a)$. Let T be a subset of S with $|T| < a$. Then there is a vertex $y_j\,(1 \le j \le a)$ such that

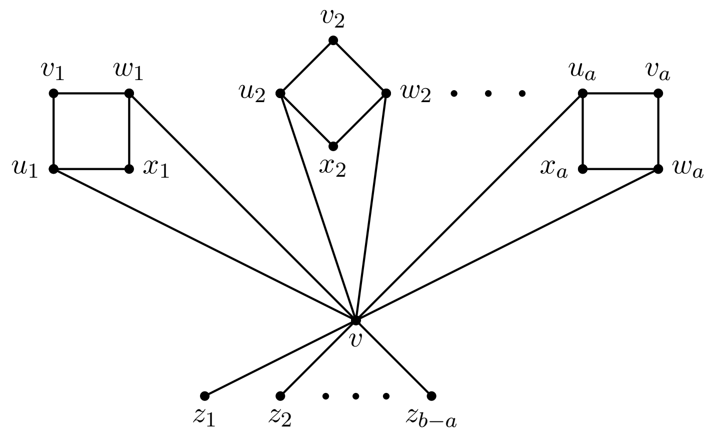


FIGURE 4. G

$y_j \notin T$. Let $s_j \in \{v_j, x_j\} \subseteq V(F_j)$ distinct from y_j. Then $S' = (S - \{y_j\}) \cup \{s_j\}$ is an edge detour basis that contains T. Thus S is not the unique edge detour basis containing T. Thus $fdn_1(S) \geq a$. Since this is true for all edge detour bases of G, it follows that $fdn_1(G) \geq a$ and so $fdn_1(G) = a$. □

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