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GRAPHS WITH SAME DIAMETER AND METRIC DIMENSION

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ABSTRACT. The cardinality of a metric basis of a connected graph G is called its metric dimension, denoted by $\dim(G)$ and the maximum value of distance between vertices of G is called its diameter. In this paper, the graphs G with diameter 2 are characterized when $\dim(G) = 2$.

Key words: distance, eccentricity, diameter, basis, metric dimension.
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1. INTRODUCTION

For a connected graph G the distance $d(u, v)$ from u to v is the length of a shortest u-v path in G. The maximum value of $d(u, v)$ for all $u, v \in G$, is its diameter denoted by $\text{diam}(G)$. An ordered subset $W = \{w_1, w_2, ..., w_k\}$ of vertices of minimum cardinality in a connected graph is called a basis of G if for each pair of vertices $u, v \in V(G)$, $(d(v, w_1), d(v, w_2), ..., d(v, w_k)) = (d(u, w_1), d(u, w_2), ..., d(u, w_k))$ holds exactly when $v = u$. In this case k is called the metric dimension $\dim(G)$ of G [2], [3], [4], [5], [6] and [8]. If graphs G and H are isomorphic we denote this by $G \cong H$.

We are interested in the classification of graphs of small diameter and lower dimension. Moon and Moser [1] were first who show that almost all graphs have diameter two, and discussion of graphs of small diameter includes most graphs. Thus the classification of graphs of diameter 2 with a given dimension n could be of help in answering many questions in graph theory. In this paper we make a humble effort of determining all graphs G of diameter 2 when

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$\dim(G) = 2$. The following general result by G. Chartrand et al. and S. Khullar et al. are well-known, see [7], [4].

Theorem 1. *Let G be a graph with metric dimension 2 and let $\{a, b\} \subset V(G)$ be a metric basis in G. Then the following are true:*
(1) There is a unique shortest path P between a and b.
(2) The degrees of a and b are at most 3.
(3) Every other node on P has degree at most 5.

Theorem 2. *Let G be a graph with metric dimension k and $|V(G)| = n$. Let d be the diameter of G. Then $|V(G)| \leq d^k + k$.*

Theorem 1 captures a few properties of graphs with metric dimension 2, whereas Theorem 2 gives an upper bound for the number of vertices of a graph in terms of its metric dimension and diameter. By this result $|V(G)| \leq 6$ for the graphs of diameter and metric dimension 2.

2. Graphs of Diameter and Metric Dimension 2

The main theorem of this paper is:

Theorem 3. *There are exactly 37 non-isomorphic connected graphs whose diameter as well as metric dimension is 2.*

We prove the above theorem in a sequence of lemmas. In the proofs of these lemmas we shall not consider some subcases yielding isomorphic graphs.

Lemma 4. *The number of non-isomorphic 2-dimensional connected graphs of diameter 2 with 4 or less vertices is exactly 4.*

Proof. It is trivial to note that no such graph exist for $|V(G)| = 2$ or 3.
Now let $V(G) = \{u, v, w, x\}$. Without loss of generality we can assume $\{v, w\}$ is a basis of G. Also, suppose that $d(v, w) = 1$ and u is not adjacent to x. In order to have a path of length 2 between the vertices u and x, it is necessary that both u and x are adjacent to one of $\{v, w\}$. If u and x are adjacent to v then $\text{diam}(G) = 2$, but u and x have same representations with respect to $\{v, w\}$ which contradicts our choice of $\{v, w\}$. To overcome this x must be adjacent to w. We are thus left with the only graph G_1 in fig. 1.
Now suppose u is adjacent to x. Then we have the following cases.
(a) There is only one edge between $\{v, w\}$ and $\{u, x\}$. But this yields $\text{diam}(G) = 3$, a contradiction.
(b) There exist two edges between $\{v, w\}$ and $\{u, x\}$. If both the edges are incident to v or to w, then there is a contradiction to our choice of $\{v, w\}$ as a resolving set. It forces that one edge is incident to v and other is incident to w. Hence there are two graphs G_2 and G_3 where $G_3 \cong G_1$ in fig. 1.
(c) There are three edges between $\{v, w\}$ and $\{u, x\}$. It is not possible that

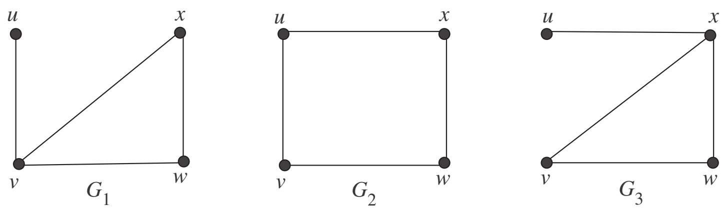

FIGURE 1

$deg(v) = 4$ or $deg(w) = 4$, since G has only four vertices. It follows that 2 edges are incident to v(say) and one edge is incident to w. Then, there is only one graph G_4 in fig. 2.
(d) There are four edges between $\{v, w\}$ and $\{u, x\}$. But this yields diam$(G) = 1$, a contradiction.
Finally, we suppose $d(v, w) = 2$ and u is a vertex on the path between v and w. Since G is connected, x must be adjacent to at least one of vertices u, v and w. If x is adjacent to both v and w, then there exist two shortest paths between v and w, which contradicts Theorem 1(1). If x is adjacent to one of $\{v, w\}$ and not to u, then diam$(G) = 3$, a contradiction. Hence x must be adjacent to u. If x is adjacent only to u, then there is only one graph G_5 in fig. 2. Finally, if x is adjacent to one of $\{v, w\}$ and also to u, then there is only one graph $G_6 \cong G_1$ in fig. 2. □

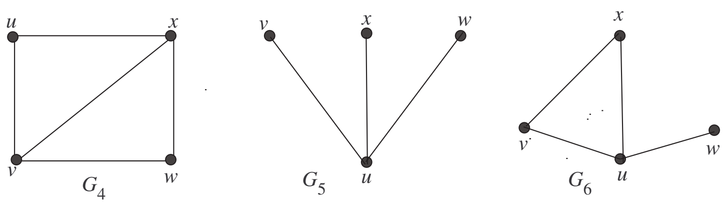

FIGURE 2

Lemma 5. *There are exactly 8 non-isomorphic connected graphs of order 5 such that their metric dimension as well as diameter is 2.*

Proof. Let $V(G) = \{u, v, w, x, y\}$. Without loss of generality we can assume $\{v, w\}$ is a basis of G. Also, suppose that $d(v, w) = 1$ and u, x and y are pairwise nonadjacent. Since G is connected each of u, x, y must be joined by at least one edge with vertices v or w. By Theorem 1(2), $deg(v), deg(w) \leq 3$. It follows that v and w are joined to at most two vertices of $\{u, x, y\}$ which implies that $\text{diam}(G) = 3$, a contradiction.

Let there be only one edge between two vertices of the subset $\{u, x, y\}$, e.g. $uy \in E(G)$. In order to have a path of length two from x to each of u, y such that $deg(v), deg(w) \leq 3$, it is necessary that x is adjacent to both v and w, v is adjacent to u and w is adjacent to y. Then there is only one graph G_1 in fig. 3.

Suppose there are two edges between the vertices of the subset $\{u, x, y\}$, e.g. $uy, yx \in E(G)$. Then we have the following possibilities.

(a) There is only one edge between $\{v, w\}$ and $\{u, x, y\}$. Then $\text{diam}(G) \geq 3$, a contradiction.

(b) There exist two edges between $\{v, w\}$ and $\{u, x, y\}$. If both the edges are incident to v or to w, then $\text{diam}(G) = 3$, a contradiction. It forces that one edge is incident to v and other is incident to w. When both v and w are adjacent to one of $\{u, x\}$, then again $\text{diam}(G) = 3$, a contradiction. If both v and w are adjacent to y, then u and x have same representations with respect to $\{v, w\}$, again a contradiction. If v is adjacent to one of $\{u, x\}$ and w is adjacent to y, then $\text{diam}(G) = 3$, a contradiction. Finally, if v is adjacent to one of $\{u, x\}$ and w is adjacent to the remaining vertex, then there is only one graph G_2 given in fig. 3.

(c) There are three edges between $\{v, w\}$ and $\{u, x, y\}$. If all three edges are incident to v or to w, then v or w have degree more than 3, a contradiction to the fact that $deg(v), deg(w) \leq 3$. It forces that two edges are incident to v(say) and one edge is incident to w. If v is adjacent to u and x, and also w is adjacent to y, then u and x have same representations relatively to $\{v, w\}$ and if $uv, yv, uw \in E(G)$, then $\text{diam}(G) = 3$, a contradiction. Since $\text{diam}(G) = 2$ and $\{v, w\}$ is a basis, it follows that there are two graphs $G_3 \cong G_1$ and G_4 in fig. 3.

(d) There are four edges between $\{v, w\}$ and $\{u, x, y\}$.

Since $deg(v), deg(w) \leq 3$, it follows that two edges are incident to v and other two are incident to w. Then there are two graphs G_5 and G_6 in fig. 3 of diameter as well as metric dimension 2.

Note that if there exist five or six edges between $\{v, w\}$ and $\{u, x, y\}$, then there exist $t \in \{v, w\}$ such that $deg(t) = 4$, a contradiction with Theorem 1(2).

We assume there exist three edges between the vertices of the subset $\{u, x, y\}$, e.g. $ux, xy, uy \in E(G)$. We consider the following cases.

(a) There is only one edge between $\{v, w\}$ and $\{u, x, y\}$. Then $\text{diam}(G) = 3$, a contradiction.

(b) There exist two edges between $\{v, w\}$ and $\{u, x, y\}$. If both the edges are incident to v or to w, then $\text{diam}(G) = 3$. It forces that one edge is incident to

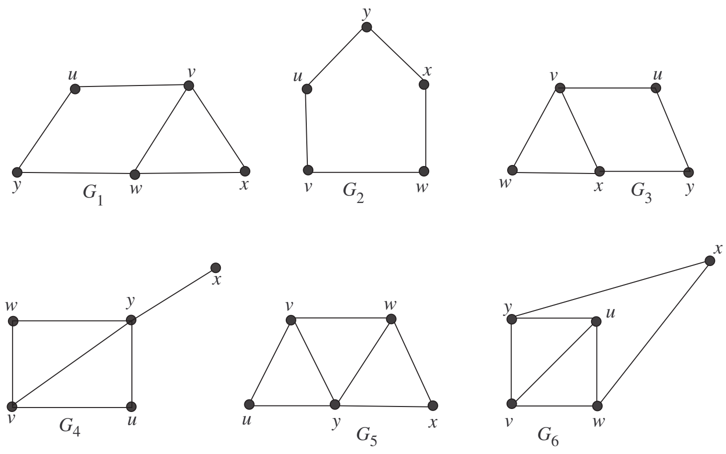

FIGURE 3

v and other is incident to w. When both v and w are adjacent to exactly one of u, x, y, there exist two vertices having same representations with respect to $\{v, w\}$, which contradicts our choice of $\{v, w\}$ as a basis. If both v and w are adjacent to any two of $\{u, x, y\}$, then these graphs are isomorphic. One of them is $G_7 \cong G_1$ given in fig. 4.
(c) There exist three edges between $\{v, w\}$ and $\{u, x, y\}$. Since $deg(v), deg(w) \leq 3$, it follows that two edges are incident to v(say) and one edge is incident to w. If v is adjacent to any two of $\{u, x, y\}$ and w is adjacent to the remaining one, there exist two vertices having same representations with respect to $\{v, w\}$. However, if v is adjacent to any two of $\{u, x, y\}$ and w is adjacent to any one of these two vertices, then we obtain the graph $G_8 \cong G_5$ in fig. 4.
(d) There exist four edges between $\{v, w\}$ and $\{u, x, y\}$.
Since $deg(v), deg(w) \leq 3$, it follows that two edges are incident to v and two edges are incident to w. If v is adjacent to any two of $\{u, x, y\}$ and w is also adjacent to both the vertices which are adjacent to v, there exist two vertices having same representations with respect to $\{v, w\}$. However, if v and w are adjacent to any two of $\{u, x, y\}$, then these graphs are isomorphic. One of them is G_9 given in fig. 4.
If there exist five or six edges between $\{v, w\}$ and $\{u, x, y\}$, then $deg(t) = 4$ for $t \in \{v, w\}$, a contradiction with Theorem 1(2).
Now suppose $d(v, w) = 2$ and u is the vertex on the shortest path between v and w. Also, suppose that x and y are nonadjacent. In order to have a path of length two from y to x, it is necessary that both y and x are adjacent to at least one vertex from $\{u, v, w\}$. If x and y are adjacent to v and w then

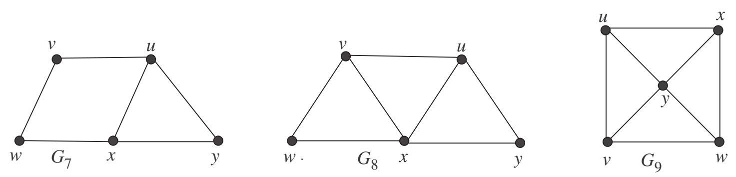

FIGURE 4

they have same representations relatively to $\{v, w\}$, a contradiction. Since $\operatorname{diam}(G) = 2$ and $\{v, w\}$ is a basis, it follows that only possibilities are G_{10} and G_{11} from fig. 5 and a graph isomorphic to G_4 from fig. 3.

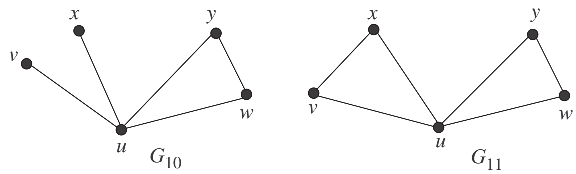

FIGURE 5

If x is adjacent to y, we shall obtain in a similar way as above graphs isomorphic to G_2, G_3, G_4 and G_5. □

Lemma 6. *There are exactly 25 non-isomorphic connected graphs of order 6 such that their diameter as well as metric dimension is 2.*

Proof. Let $V(G) = \{u, v, w, x, y, z\}$. Without loss of generality we can assume $\{v, w\}$ is a basis of G. If $W = \{v, w\}$ is a metric basis of a graph G, then the vertices in $V(G)\backslash W$ have distinct representations relative to the ordered set W of the form $(1, 1)$, $(1, 2)$, $(2, 1)$ or $(2, 2)$. The vertex x having representation $(1, 1)$, i.e., having $d(x, v) = d(x, w) = 1$ will be called major vertex and vertices with representations $(1, 2)$ or $(2, 1)$ minor vertices. We shall consider two cases: A. $d(v, w) = 1$ and B. $d(v, w) = 2$.
A. If $d(v, w) = 1$, suppose first that u, x, y and z are pairwise nonadjacent. By the connectedness each of u, x, y, z must be joined by at least one edge with vertices v or w. Since $deg(v), deg(w) \leq 3$, it follows that v and w are joined to exactly two vertices from $\{u, x, y, z\}$, which implies that $\operatorname{diam}(G) = 3$, a contradiction.
Let there be only one edge between two vertices of the subset $\{u, x, y, z\}$, e.g.

$xy \in E(G)$. In order to have a path of length two from z to each of u, x, y this path must use one of v, w and this uses two edges incident to v, w. Since $deg(v), deg(w) \leq 3$, this is not possible for all u, x, y.
By the similar argument, no graph in each of the following three cases exist.
1) There exist 2 adjacent edges in the subgraph induced by $\{u, x, y, z\}$;
2) There exist 2 nonadjacent edges in this subgraph;
3) There exist 3 edges inducing K_3 in this subgraph.
If there exist 3 edges $ux, xy, yz \in E(G)$ inducing P_4 in G, since diam$(G) = 2$ we have two distinct cases: z is major, which implies that u and y are minor and x has representation $(2, 2)$ or u and x are minor and y has representation $(2, 2)$. In the first case we get graph G_1 and in the second case G_2 (fig. 6).

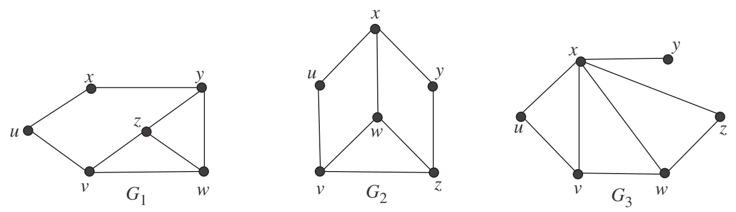

Figure 6

If three edges ux, xy, xz induce a star, then we also have 2 possibilities: x is major, u and z are minor and y has representation $(2, 2)$, thus yielding graph G_3 (fig. 6), or u is major, y and z are minor and $x(2, 2)$, then the corresponding graph is isomorphic to G_2.
If in G we have four edges ux, xy, yz, uz inducing C_4, then there are two distinct cases: z is major, u and x are minor and $y(2, 2)$ or z is major, u and y are minor and $x(2, 2)$. Corresponding graphs are G_4 and G_5 (fig. 7).
If four edges ux, xy, yu, yz induce a 3-clique $\{u, x, y\}$ with pendant edge yz, we have six cases: u is major, x and z are minor and $y(2, 2)$, the resulting graph is isomorphic to G_5; z is major, x and u are minor and $y(2, 2)$, the resulting graph is G_6; z is major, y and u are minor and $x(2, 2)$, the resulting graph is G_7; u is major, y and z are minor and $x(2, 2)$ - G_8; y is major, u and z are minor and $x(2, 2)$ - G_9; y is major, u and x are minor and $z(2, 2)$ - graph G_{10} in fig. 7.
If there exist five edges ux, xy, yz, uz, uy inducing $K_4 - e$, then we obtain graphs $G_{11} - G_{14}$ in fig. 8, when u is major, x and z are minor, $y(2, 2)$; u is major, y and z are minor, $x(2, 2)$; x is major, u and z are minor, $y(2, 2)$, x is major, u and y are minor and $z(2, 2)$, respectively.
If there exist six edges ux, xy, yz, uz, uy, xz inducing K_4, then a unique graph

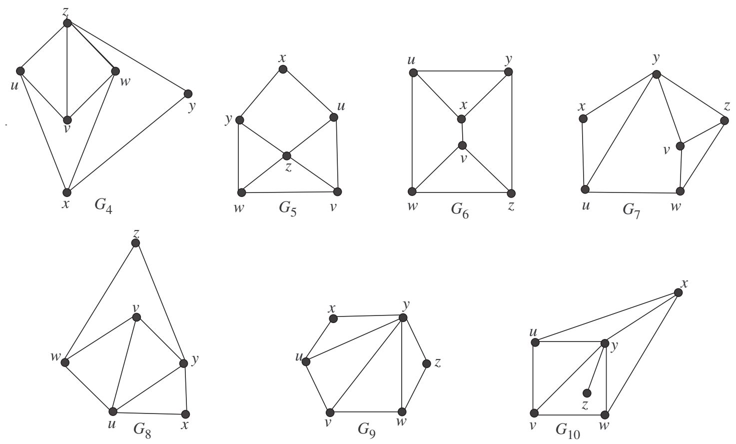

FIGURE 7

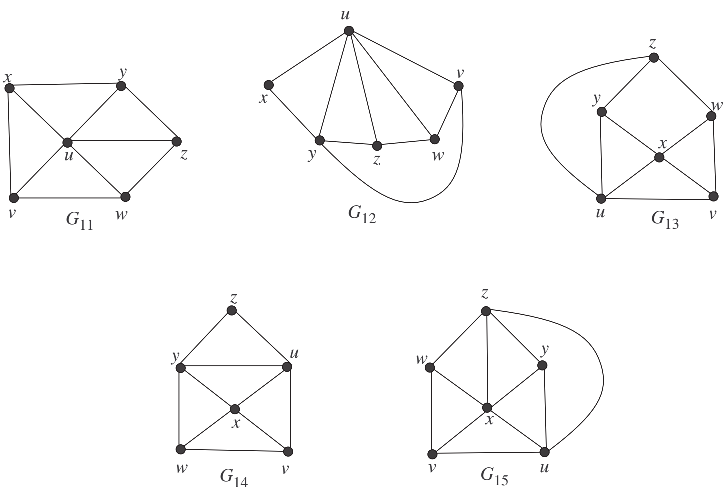

FIGURE 8

(up to isomorphism) is possible, namely G_{15}, x being a major vertex, u and z minor and $y(2,2)$ (fig. 8).
B. Suppose $d(v,w)=2$, and u is a vertex on the shortest path between v and w.This implies that u is a major vertex.
Suppose that the subgraph induced by $\{x,y,z\}$ has no edge. In this case x

and z are minor vertices and $y(2,2)$, yielding graph G_{16} (fig. 9).

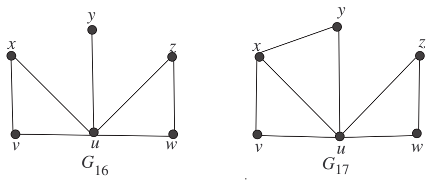

FIGURE 9

If this subgraph has one edge, $xy \in E(G)$, then we obtain G_{17} when x and z are minor vertices and $y(2,2)$ and a graph isomorphic to G_3 when x and y are minor and $z(2,2)$.
If there exist two edges $xy, yz \in E(G)$, then we have the following cases: x and y are minor vertices and $z(2,2)$, - G_{18}, G_{20}, G_{21} and G_{22}; x and z are minor and $y(2,2)$,- G_{19} and a graph isomorphic to G_9 (fig. 10).

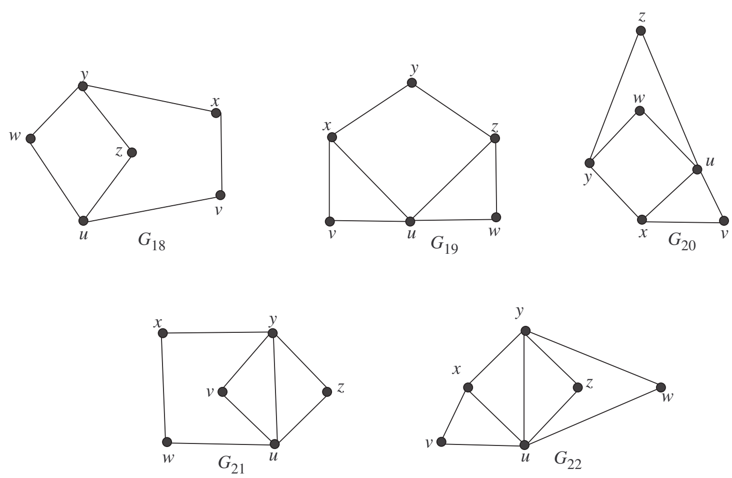

FIGURE 10

Finally, if there exist three edges $xy, yz, xz \in E(G)$ inducing K_3 we deduce that two vertices are minor and the third has representation $(2,2)$, yielding the following non-isomorphic graphs: one is isomorphic to G_{19}, another isomorphic to G_2 and new graphs G_{23}, G_{24} and G_{25} (fig. 11), which concludes the proof. □

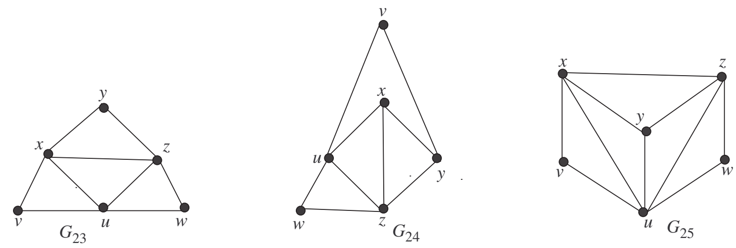

FIGURE 11

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