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λ-FRACTIONAL SCHWARZIAN DERIVATIVE AND λ-FRACTIONAL MÖBIUS TRANSFORMATION

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Abstract. We denote by $\mathcal{A}$ the class of all analytic functions in the open unit disk $\mathbb{D} = \{z \mid |z| < 1\}$ which satisfy the conditions $f(0) = 0$, $f'(0) = 1$. In this paper we define a new concept of $\lambda-$ fractional Schwarzian derivative and $\lambda-$ fractional Möbius transformation for the class $\mathcal{A}$. We also formulate the criterion for a function to be univalent using the fractional Schwarzian.

Key words: λ-fractional integral, λ-fractional derivative.
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1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions of the power-series expansion of the form $f(z) = z + a_2 z^2 + \ldots, |z| < 1$ in the open unit disk $\mathbb{D}$. Let λ be a positive real number, then the λ-order *fractional integral* of $f(z) \in \mathcal{A}$ is given by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta, \tag{1}$$

where the function $f(z)$ is analytic in a simply connected region of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$ (see [3] and [4]).

The λ-order *fractional derivative* of $f(z) \in \mathcal{A}$ can be written as

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \qquad (0 \leq \lambda < 1), \tag{2}$$

where the function $f(z)$ is analytic in a simply connected region of the complex plane containing the origin and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in the definition of the fractional integral (see [3] and [4]).

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Using the definition of the fractional derivative of order λ, the fractional derivative of order $(m+\lambda)$ is defined for an analytic function $f(z) \in \mathcal{A}$ by

$$D_z^{m+\lambda} f(z) = \frac{d^m}{dz^m} D_z^{\lambda} f(z) \qquad \left(0 \le \lambda < 1,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right). \tag{3}$$

We employ the definitions (1) and (2) for the fractional integrals and fractional derivatives of univalent and analytic functions to obtain

$$D_z^{-\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\lambda)} z^{k+\lambda} \qquad (\lambda > 0,\ k > 0), \tag{4}$$

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda} \qquad (k > 0,\ k-\lambda \ne -1, -2, ..., 0 \le \lambda < 1), \tag{5}$$

and

$$D_z^{n+\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-n-\lambda)} z^{k-n-\lambda} \qquad (0 \le \lambda < 1, n \in \mathbb{N}_0). \tag{6}$$

Therefore, we end up with the following result for any real λ

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\lambda)} z^{k-\lambda}. \tag{7}$$

2. Fractional Schwarzian

Let us consider the following fractional differential equation

$$D_z^{\lambda} g(z) = h(z) \qquad (0 < \lambda < 1), \tag{8}$$

where $g(0) = 0$, $D_z^{\lambda} g(0) = 1$ and

$$h(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} z^n \qquad \text{for } |z| < 1. \tag{9}$$

Next, we propose

$$g(z) = \sum_{n=0}^{\infty} a_n z^{n+\lambda} \qquad \text{with } a_0 = \frac{1}{\Gamma(\lambda+1)} \tag{10}$$

to obtain from (8) the following equality between coefficients:

$$a_n = \frac{h^{(n)}(0)}{\Gamma(\lambda+n+1)}$$

whence the solution to (8) can be found as

$$g(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{\Gamma(\lambda+n+1)} z^{n+\lambda} \qquad \text{with } h(0) = 1.$$

Using (4) and (9), one may see that

$$g(z) = D_z^{-\lambda} h(z)$$

which leads to

$$D_z^{\lambda}g(z) = h(z) \Leftrightarrow g(z) = D_z^{-\lambda}h(z). \tag{11}$$

We can write

$$\ln D_z^{\lambda}g(z) = \ln h(z)$$

and thus

$$\frac{D_z^{\lambda+1}g(z)}{D_z^{\lambda}g(z)} = \frac{h'(z)}{h(z)}.$$

Similarly,

$$\ln\left[\frac{D_z^{\lambda+1}g(z)}{D_z^{\lambda}g(z)}\right] = \ln\left[\frac{h'(z)}{h(z)}\right]$$

can be employed to obtain

$$\frac{D_z^{\lambda+2}g(z)}{D_z^{\lambda+1}g(z)} = \frac{h''(z)}{h'(z)}$$

which in turn allows one to deduce simultaneously the Schwarzian of h and the fractional Schwarzian of g, that is,

$$Sh(z) = S_{\lambda}g(z), \tag{12}$$

where

$$Sh(z) = \left(\frac{h''(z)}{h'(z)}\right)' - \frac{1}{2}\left(\frac{h''(z)}{h'(z)}\right)^2 \tag{13}$$

is the Schwarzian derivative of the function $h(z)$ [1]. We will call

$$S_{\lambda}g(z) = \left(\frac{D_z^{\lambda+2}g(z)}{D_z^{\lambda+1}g(z)}\right)' - \frac{1}{2}\left(\frac{D_z^{\lambda+2}g(z)}{D_z^{\lambda+1}g(z)}\right)^2 \tag{14}$$

the fractional Schwarzian derivative of the function $g(z)$. Therefore we have the following theorem.

Theorem 1. *Let $g(z)$ and $h(z)$ be analytic functions in the open unit disk $\mathbb{D}$ with the conditions $g(0) = 0$, $D_z^{\lambda}g(0) = 1$ and*

$$h(z) = \sum_{n=0}^{\infty}\frac{h^n(0)}{n!}z^n, \qquad g(z) = \sum_{n=0}^{\infty}a_n z^{n+\lambda} \quad \text{with} \quad \left\{a_0 = \frac{1}{\Gamma(\lambda+1)}\right\}$$

If $D_z^{\lambda}g(z) = h(z)$ is satisfied then the λ-fractional Schwarzian derivative of $g(z)$, given by (14), exists.

3. Univalency Criterion and Fractional Schwarzian

There is a well known sharp inequality in [2] for any univalent function $h(z)$ to satisfy and it can be expressed as

$$|Sh(z)| \leq \frac{6}{\left(1-|z|^2\right)^2} \qquad \{|z|<1\}. \tag{15}$$

Using (12) and (15) the univalency of the function $g(z)$ proposed as in (10) is given by

$$|S_\lambda g(z)| \leq \frac{6}{\left(1-|z|^2\right)^2} \qquad \{|z|<1\}, \tag{16}$$

where $S_\lambda g(z)$ is the fractional Schwarzian derivative of the function $g(z)$. Thus, the univalency criterion can be formulated also for functions, expanded into λ-fractional series as in (10), in the terms of λ-fractional Schwarzian.

4. Fractional Möbius Transformation

It is known that

$$Sh(z) = 0 \tag{17}$$

if h is a linear-fractional transformation or the so-called Möbius transformation, that is,

$$h(z) = M(z) = \frac{az+\alpha}{bz+\beta} \qquad (a\beta - b\alpha \neq 0), \tag{18}$$

where a, b, α, and β are constants such that $\alpha/\beta = 1$ as a result of the normalization $h(0) = M(0) = 1$ without loss of generality. According to (12), we have

$$S_\lambda g(z) = 0$$

provided that

$$g(z) = M_\lambda(z), \tag{19}$$

where $M_\lambda(z)$ will be called the *fractional Möbius transformation* and can be obtained through (11) as

$$M_\lambda(z) = D_z^{-\lambda} M(z) = \frac{1}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} \left(\frac{a\zeta+\alpha}{b\zeta+\beta}\right) d\zeta$$

which yields

$$M_\lambda(z) = \frac{1}{\Gamma(\lambda)} \left[\frac{z^\lambda}{\lambda} M(z) - (a\beta - b\alpha) \left(\frac{z^{\lambda+1}}{\lambda+1}\right) \frac{{}_2F_1\left(1, \lambda+1; \lambda+2; \dfrac{bz}{\beta+bz}\right)}{(bz+\beta)^2} \right],$$

where

$$ {}_2F_1\left(1,\lambda+1;\lambda+2;\frac{bz}{\beta+bz}\right)= $$

$$ =\frac{\Gamma(\lambda+2)}{\Gamma(\lambda+1)}\sum_{n=0}^{\infty}\frac{\Gamma(1+n)\,\Gamma(\lambda+1+n)}{\Gamma(\lambda+2+n)\,n!}\left(\frac{bz}{\beta+bz}\right)^{n} \tag{20} $$

is the hypergeometric function expressed in terms of the so-called hypergeometric series.

The composition of two Möbius transformations $M^{(1)}(z)$ and $M^{(2)}(z)$ yields

$$ M^{(1)}(z)\circ M^{(2)}(z)=M(z), \tag{21} $$

where $M(z)$ is still the Möbius transformation. Using $M^{(1)}(z)=D_z^{\lambda}M_{\lambda}^{(1)}(z)$ and $M^{(2)}(z)=D_z^{\lambda}M_{\lambda}^{(2)}(z)$, it follows from (21) that

$$ D_z^{\lambda}M_{\lambda}^{(1)}(z)\circ D_z^{\lambda}M_{\lambda}^{(2)}(z)=D_z^{\lambda}M_{\lambda}(z) $$

which in turn leads to

$$ M_{\lambda}(z)=D_z^{-\lambda}\left(D_z^{\lambda}M_{\lambda}^{(1)}(z)\circ D_z^{\lambda}M_{\lambda}^{(2)}(z)\right). \tag{22} $$

The simple result described by (22) can be summarized within the following theorem.

Theorem 2. *The fractional Möbius transformation can be expressed by the λ-order fractional integral of the composition of the λ-order fractional derivatives of two fractional Möbius transformations.*

5. Discussion: Alternative Definitions

Equation (8) is not the unique choice to define the fractional Schwarzian. Replacing (8) by

$$ D_z^{-\lambda}g(z)=h(z) $$

and expanding $h(z)$ and $g(z)$ into series, we get

$$ a_n=\frac{h^{(n)}(0)}{\Gamma(n+1-\lambda)}. $$

Applying logarithm to both sides of

$$ \frac{D_z^{-\lambda+1}g(z)}{D_z^{-\lambda}g(z)}=\frac{h'(z)}{h(z)}, $$

we again reproduce (12), where now in the place of (14) we end up with

$$ S_{\lambda}g(z)=\left(\frac{D_z^{-\lambda+2}g(z)}{D_z^{-\lambda+1}g(z)}\right)'-\frac{1}{2}\left(\frac{D_z^{-\lambda+2}g(z)}{D_z^{-\lambda+1}g(z)}\right)^{2}. \tag{23} $$

The corresponding fractional Möbius transformation is given by

$$M_\lambda(z) = D_z^\lambda M(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z (z-\zeta)^{-\lambda} \left(\frac{a\zeta + \alpha}{b\zeta + \beta} \right) d\zeta.$$

In principle, the power of the fractional derivative in (8) may contain an integer number as well. For instance, considering

$$D_z^{\lambda+1} g(z) = h(z)$$

with

$$a_{n+1} = \frac{h^{(n)}(0)}{(n+1)\Gamma(n+2+\lambda)},$$

we will get another definition of the fractional Schwarzian.

6. Conclusions

Fractional Schwarzian, defined by (14), which exists under the conditions of Theorem 1, possesses some important properties of the standard Schwarzian (13): it allows to formulate the univalency criterion for functions expanded into λ-fractional series and to find the fraction analogue of the Möbius transformation with the composition property given by Theorem 2.

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References

[1] A. W. Goodman: *Univalent functions*, Volume I and Volume II, Mariner Publishing Comp. Inc. Tampa Florida (1983).
[2] Z. Nehari: *Conformal Mapping*, Dover Publications, Inc. New York (1952).
[3] S. Owa: *On the distorsion theorems I*, Kyungpook Math. J. 18 (1978) 53-59.
[4] H. M. Srivastava, S. Owa: (Editors), *Univalent Functions, Fractional Calculus and their applications*, John Wiley and Sons (1989).