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THE DOMINATION COVER PEBBLING NUMBER OF THE SQUARE OF A PATH

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ABSTRACT. Given a configuration of pebbles on the vertices of a connected graph G, a pebbling move (or pebbling step) is defined as the removal of two pebbles from a vertex and placing one pebble on an adjacent vertex. The domination cover pebbling number, $\psi(G)$, of a graph G is the minimum number of pebbles that have to be placed on $V(G)$ such that after a sequence of pebbling moves, the set of vertices with pebbles forms a dominating set of G, regardless of the initial configuration. In this paper, we determine the domination cover pebbling number for the square of a path.

Key words: pebbling, square of a path, cover pebbling, domination.
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1. INTRODUCTION

One recent development in graph theory suggested by, Lagarias and Saks and called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of graph pebbling [5]. There have been many developments since Hulbert's survey appeared in graph pebbling.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from $V(G)$ to $N \cup \{0\}$ representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected. A pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. The *pebbling number* [1], $f(G)$, to be the minimum number of pebbles such that regardless of their initial

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configuration, it is possible to move a pebble to any arbitrarily selected vertex v in G, using a sequence of pebbling moves. In the worst case this pebble is the very last pebble on the graph.

A set $D \subseteq V(G)$ in G is a dominating set [4] of G, if every vertex in G is either in D or adjacent to some element in D. The *cover pebbling number* [2],$\gamma(G)$, is defined as the minimum number of pebbles required such that given any initial configuration of at least $\gamma(G)$ pebbles, it is possible to make a series of pebbling moves to place at least one pebble on every vertex of G. The *domination cover pebbling number*, $\psi(G)$, is the minimum number of pebbles required such that any initial configuration of at least $\psi(G)$ pebbles can be transformed so that the set of vertices that contains pebbles form a dominating set D of G. In [3], Gardner et al. determine the domination cover pebbling number for paths, cycles and complete binary trees. In the next section, we determine $\psi(G)$ for squares of a paths.

2. The Domination Cover Pebbling Number for the Square of a Path

Definition 1. [6] *Let $G = (V(G), E(G))$ be a connected graph. The nth power of G, denoted by G^p is the graph obtained from G by adding the edge uv to G whenever $2 \leq d(u, v) \leq p$ in G, that is,*

$$G^p = (V(G), \{uv : 1 \leq d(u, v) \leq p \text{ in } G\}).$$

If $p = 1$, we define $G^1 = G$. We know that if p is large enough, that is, $p \geq n - 1$, then $G^p = K_n$ where n is the number of vertices of the graph.

Notation 1. *Let $P_n : v_1v_2...v_{n-1}v_n$ be the path of length $n - 1$. We play on P_n^2. Let $p(v_i)$ denote the number of pebbles on the vertex v_i. Let $p(P_n^2)$ denote the number of pebbles on the square of the path P_n.*

It is easy to see that, $\psi(P_3^2) = 1$, since $P_3^2 \cong K_3$, see [3].

Theorem 1. *The domination cover pebbling number for P_4^2 is $\psi(P_4^2) = 2$.*

Proof. If we place one pebble on v_1, then we cannot cover dominate the vertex v_4. Thus, $\psi(P_4^2) \geq 2$.

Now consider the distribution of two pebbles on the vertices of P_4^2.If either $p(v_2) \geq 1$ or $p(v_3) \geq 1$ or ($p(v_1) = 1$ and $p(v_4) = 1$), then we are done. Otherwise, $p(v_1) = 2$ or $p(v_4) = 2$. So, we can move one pebble to v_2 or v_3 and we are done. Thus, $\psi(P_4^2) \leq 2$. □

Theorem 2. *The domination cover pebbling number for P_5^2 is $\psi(P_5^2) = 3$.*

Proof. Consider the following configuration such that $p(v_1) = 1$, $p(v_2) = 1$, and $p(v_i) = 0$ where $i = 3, 4, 5$. Clearly we cannot cover dominate the vertex v_5. Thus, $\psi(P_5^2) \geq 3$.

Now consider the distribution of three pebbles on the vertices of P_5^2. If $p(v_3) \geq 1$ or $p(v_i) \geq 2$ where $i \neq 3$ then we are done. Otherwise, three vertices receive exactly one pebble each and so we are done. Thus, $\psi(P_5^2) \leq 3$. □

Theorem 3. *The domination cover pebbling number for P_6^2 is $\psi(P_6^2) = 5$.*

Proof. Consider the configuration such that $p(v_1) = 4$, $p(v_i) = 0$ for all $v_i \in V(P_6^2) - \{v_1\}$. Then we cannot cover dominate at least one of the vertices of P_6^2. Thus, $\psi(P_6^2) \geq 5$.

Now consider the distribution of 5 pebbles on the vertices of P_6^2. Consider the paths $P_A : v_1v_2v_3$ and $P_B : v_4v_5v_6$. Note that $\psi(P_A^2) = 1 = \psi(P_B^2)$. If $p(P_A^2) \geq 1$ and $p(P_B^2) \geq 1$ then we are done. Otherwise, $p(P_A^2) = 0$ or $p(P_B^2) = 0$. Without loss of generality let us assume that $p(P_B^2) = 0$. So, $p(P_A^2) = 5$. Using at most 4 pebbles from P_A^2 we can cover dominate P_B^2. Then, $p(P_A^2) \geq 1$ and hence we are done. Thus, $\psi(P_6^2) \leq 5$. □

Theorem 4. *The domination cover pebbling number for P_7^2 is $\psi(P_7^2) = 6$.*

Proof. Consider the configuration such that $p(v_1) = 4$, $p(v_4) = 1$, $p(v_i) = 0$ for all $v_i \in V(P_7^2) - \{v_1, v_4\}$. Then we cannot cover dominate at least one of the vertices of P_7^2. Thus, $\psi(P_7^2) \geq 6$.

Now consider the distribution of 6 pebbles on the vertices of P_7^2. Consider the paths $P_A : v_1v_2v_3v_4$ and $P_B : v_5v_6v_7$. Note that $\psi(P_A^2) = 2$ and $\psi(P_B^2) = 1$. If $p(P_A^2) \geq 2$ and $p(P_B^2) \geq 1$, then we are done. Otherwise, $p(P_A^2) \leq 1$ or $p(P_B^2) = 0$. If $p(P_B^2) = 0$, then $p(P_A^2) = 6$. Using at most 4 pebbles from P_A^2 we can cover dominate P_B^2. Then the remaining number of pebbles in P_A^2 is at least two and hence we are done. Next, if $p(P_A^2) \leq 1$, then $p(P_B^2) \geq 5$. Using at most 4 pebbles from P_B^2 we can put one pebble on v_3, and so we cover dominate P_A^2. Then the remaining number of pebbles in P_B^2 is at least one and we are done. Thus $\psi(P_7^2) \leq 6$. □

Theorem 5. *The domination cover pebbling number for P_8^2 is $\psi(P_8^2) = 9$.*

Proof. Consider the configuration such that $p(v_1) = 8$, $p(v_i) = 0$ for all $v_i \in V(P_8^2) - \{v_1\}$. Then we cannot cover dominate at least one of the vertices of P_8^2. Thus, $\psi(P_8^2) \geq 9$.

Now consider the distribution of 9 pebbles on the vertices of P_8^2. Consider the paths $P_A : v_1v_2v_3v_4$ and $P_B : v_5v_6v_7v_8$. Note that $\psi(P_A^2) = 2 = \psi(P_B^2)$. If $p(P_A^2) \geq 2$ and $p(P_B^2) \geq 2$, then we are done. Otherwise, $p(P_A^2) \leq 1$ or $p(P_B^2) \leq 1$. Without loss of generality let us assume that $p(P_B^2) \leq 1$. Then $p(P_A^2) \geq 8$. Using at most 8 pebbles from P_A^2, we can put a pebble on v_6 so that we cover dominate P_B^2. If we use exactly 7 or 8 pebbles to put a pebble on v_6, then $\{v_4, v_5\}$ contains zero pebbles. Then the remaining number of pebbles in $P_A^2 - \{v_4\}$ is at least one and hence we are done, since v_4 is already cover dominated by v_6. Otherwise, we use at most 6 pebbles to cover P_B^2. So, the remaining number of pebbles in P_A^2 is at least two and hence we are done. Thus, $\psi(P_8^2) \leq 9$. □

Theorem 6. *The domination cover pebbling number for P_9^2 is $\psi(P_9^2) = 10$.*

Proof. Consider the configuration such that $p(v_1) = 9$, and $p(v_i) = 0$ for all $v_i \in V(P_9^2) - \{v_1\}$. Then we cannot cover dominate at least one of the vertices of P_9^2. Thus, $\psi(P_9^2) \geq 10$.

Now consider the distribution of 10 pebbles on the vertices of P_9^2. Consider the paths $P_A : v_1v_2v_3v_4v_5$ and $P_B : v_6v_7v_8v_9$. Note that $\psi(P_A^2) = 3$ and $\psi(P_B^2) = 2$. If $p(P_A^2) \geq 3$ and $p(P_B^2) \geq 2$, then clearly we are done. Otherwise, $p(P_A^2) \leq 2$ or $p(P_B^2) \leq 1$. If $p(P_B^2) \leq 1$, then $p(P_A^2) \geq 9$. If $p(v_i) = 0$ for $i = 7, 8, 9$, then using at most 9 pebbles from P_A^2 we can put a pebble on v_7 and hence we cover dominate P_B^2. If we use exactly 8 (or 9) pebbles then we cover dominate the vertex v_5 (or v_4 and v_5). Then the remaining number of pebbles in $P_A^2 - \{v_5\}$ is at least two (or in $P_A^2 - \{v_4, v_5\}$ is at least one) and hence we are done. Otherwise, we can use at most 7 pebbles to put a pebble on v_7 and then the remaining number of pebbles in $P_A^2 - \{v_5\}$ is at least two and so we are done. If $1 \leq p(v_7) + p(v_8) + p(v_9) \leq p(P_B^2) \leq 1$ then the number of pebbles in $P_A^2 \cup \{v_6\}$ is at least 9 and so we are done since $\psi(P_6^2) = 5$.

Next, if $p(P_A^2) \leq 2$, then $p(P_B^2) \geq 8$. If $p(v_i) = 0$ for all $i = 1, 2, 3$ then the number of pebbles in $P_B^2 \cup \{v_5, v_4\}$ is 10, and using at most 9 pebbles we can cover dominate P_A^2. If we use exactly 9 pebbles in P_B^2 then the vertex v_6 is cover dominated and we are done since $P_B^2 - \{v_6\}$ contains at least one pebble. Otherwise, we can use at most 8 pebbles in P_B^2. Then the remaining number of pebbles in P_B^2 is at least two and hence we are done. If $1 \leq p(v_1) + p(v_2) + p(v_3) \leq p(P_A^2) \leq 2$, then we are done since the number of pebbles in $P_B^2 \cup \{v_5, v_4\}$ is at least 8 and $\psi(P_6^2) = 5$. Thus, $\psi(P_9^2) \leq 10$. □

Theorem 7. *The domination cover pebbling number for P_{10}^2 is $\psi(P_{10}^2) = 18$.*

Proof. Consider the configuration such that $p(v_1) = 17$, and $p(v_i) = 0$ for all $v_i \in V(P_{10}^2) - \{v_1\}$. Then we cannot cover dominate at least one of the vertices of P_{10}^2. Thus, $\psi(P_{10}^2) \geq 18$.

Now consider the distribution of 18 pebbles on the vertices of P_{10}^2. Consider the paths $P_A : v_1v_2v_3v_4v_5$ and $P_B : v_6v_7v_8v_9v_{10}$. Note that $\psi(P_A^2) = 3$ and $\psi(P_B^2) = 3$. If $p(P_A^2) \geq 3$ and $p(P_B^2) \geq 3$, then we are done. Otherwise, $p(P_A^2) \leq 2$ or $p(P_B^2) \leq 2$. Without loss of generality, let us assume that $p(P_B^2) \leq 2$. Then $p(P_A^2) \geq 16$. If $p(v_i) = 0$, $i = 8, 9, 10$ then $P_A^2 \cup \{v_6, v_7\}$ contains 18 pebbles. Using at most 16 pebbles from these pebbles, we can put a pebble on v_8, to cover dominate the vertex v_{10}. By putting a pebble on v_8, we can cover dominate the vertices v_6, v_7, and v_9 and hence P_B^2 is cover dominated. If we use 15 or 16 pebbles to put a pebble on v_8 then the remaining number of pebbles in P_A^2 is at least two, that is, v_1 has at least two pebbles on it. So, we can move one pebble to v_3 from v_1 and hence we are done. Otherwise we can use at most 14 pebbles to put a pebble on v_8. Then the remaining number of pebbles in P_A^2 is at least 4 if v_6 and v_7 have zero pebbles on them or the remaining number of pebbles in P_A^2 is at least 8 if either $p(v_6) = 1$ or $p(v_7) = 1$ and hence we are done.

If $1 \leq p(v_8) + p(v_9) + p(v_{10}) \leq p(P_B^2 \leq 2$ then we are done since the number of pebbles in $P_A^2 \cup \{v_6, v_7\}$ is at least sixteen and $\psi(P_7^2) = 6$. Thus $\psi(P_{10}^2) \leq 18$. □

Theorem 8. *The domination cover pebbling number for P_{11}^2 is $\psi(P_{11}^2) = 21$.*

Proof. Consider the configuration such that $p(v_1) = 20$, and $p(v_i) = 0$ for all $v_i \in V(P_{11}^2) - \{v_1\}$. Then we cannot cover dominate at least one of the vertices of P_{11}^2. Thus, $\psi(P_{11}^2) \geq 21$.

Now consider the distribution of 21 pebbles on the vertices of P_{11}^2. Consider the paths $P_A : v_1v_2v_3v_4v_5v_6$ and $P_B : v_7v_8v_9v_{10}v_{11}$. Note that $\psi(P_A^2) = 5$ and $\psi(P_B^2) = 3$. If $p(P_A^2) \geq 5$ and $p(P_B^2) \geq 3$, then we are done. Otherwise, $p(P_A^2) \leq 4$ or $p(P_B^2) \leq 2$. If $p(P_B^2) \leq 2$, then $p(P_A^2) \geq 19$. If $p(v_i) = 0$, $i = 9, 10, 11$, then $P_A^2 \cup \{v_7, v_8\}$ contains 21 pebbles. Using at most 17 pebbles from these pebbles, we can put one pebble on v_9. So we can cover dominate P_B^2. If we use exactly 17 pebbles then v_8 has a pebble on it. This implies that, the vertex v_6 is cover dominated. Then the remaining number of pebbles in $P_A^2 - \{v_6\}$ is at least 4 and we are done. Otherwise, we can use at most 16 pebbles to cover dominate P_B^2. Then the remaining number of pebbles in P_A^2 is at least 5 and hence we are done. If $1 \leq p(v_9)+p(v_{10})+p(v_{11}) \leq p(P_B^2) \leq 2$, then we are done since the number of pebbles in $P_A^2 \cup \{v_7, v_8\}$ is at least 19 and $\psi(P_8^2) = 9$.

If $p(P_A^2) \leq 4$, then $p(P_B^2) \geq 17$. If $p(v_i) = 0$, $i = 1, 2, 3$, then $P_B^2 \cup \{v_6, v_5, v_4\}$ contains 21 pebbles. Using at most 17 pebbles from these pebbles, we can put a pebble on v_3, to cover dominate the vertex v_1. In this process we cover dominate the vertices v_2, v_4, and v_5. If we use at most 14 pebbles to cover dominate P_A^2 from P_B^2, then the remaining number of pebbles in P_B^2 is at least 3 and we are done. Otherwise, P_B^2 contains at least 20 pebbles. This implies that after using at most 17 pebbles, P_B^2 contains at least 3 pebbles and hence we are done. If we use 17 pebbles then P_A^2 is cover dominated. Otherwise, P_B^2 contains all the 21 pebbles. After using at most 16 pebbles to put a pebble on v_3, $P_B^2 \cup \{v_6\}$ contains at least 5 pebbles and we are done. If $1 \leq p(v_1) + p(v_2) + p(v_3) \leq p(P_A^2) \leq 4$, then we are done since the number of pebbles in $P_B^2 \cup \{v_6, v_5, v_4\}$ is at least 17 and $\psi(P_8^2) = 9$. Thus, $\psi(P_{11}^2) \leq 21$. □

Theorem 9. *The domination cover pebbling number for P_{12}^2 is $\psi(P_{12}^2) = 37$.*

Proof. Consider the configuration such that $p(v_1) = 36$, and $p(v_i) = 0$ for all $v_i \in V(P_{12}^2) - \{v_1\}$. Then we cannot cover dominate at least one of the vertices of P_{12}^2. Thus, $\psi(P_{12}^2) \geq 37$.

Now consider the distribution of 37 pebbles on the vertices of P_{12}^2. Consider the paths $P_A : v_1v_2v_3v_4v_5v_6$ and $P_B : v_7v_8v_9v_{10}v_{11}v_{12}$. Note that $\psi(P_A^2) = 5$ and $\psi(P_B^2) = 5$. If $p(P_A^2) \geq 5$ and $p(P_B^2) \geq 5$, then we are done. Otherwise, $p(P_A^2) \leq 4$ or $p(P_B^2) \leq 4$. Without loss of generality let us assume that $p(P_B^2) \leq 4$. Then $p(P_A^2) \geq 33$. If $p(v_i) = 0$, $i = 10, 11, 12$ then $P_A^2 \cup \{v_7, v_8, v_9\}$ contains 37 pebbles. Using at most 32 pebbles we can put a pebble on v_{10} to cover dominate the vertex v_{12}. In this process we cover dominate the vertices v_8, v_9, and v_{11}. If we use at most 28 pebbles then the remaining number of pebbles in P_A^2 is at least 5 and we are done. Otherwise, $P_A^2 \cup \{v_7\}$ contains at least 6 pebbles or v_1 has 5 pebbles on it and so we are done. If $1 \leq p(v_{10}) + p(v_{11}) + p(v_{12}) \leq p(P_B^2) \leq 4$, then we are done since the number of pebbles in $P_A^2 \cup \{v_7, v_8, v_9\}$ is at least 33 and $\psi(P_9^2) = 10$. Thus, $\psi(P_{12}^2) \leq 37$. □

Theorem 10. *For $n \geq 8$, let $n \equiv \alpha(\mod 5)$ and let*

$$T(P_n^2) = \sum_{i \in I} 2^{\lfloor \frac{i}{2} \rfloor} + \begin{cases} \alpha & \text{if } \alpha = 0 \text{ or } 1; \\ \lfloor \frac{\alpha}{2} \rfloor & \text{if } 2 \leq \alpha \leq 4, \end{cases}$$

where $I = \{\alpha + 3, \alpha + 8, \alpha + 13, ..., n - 7, n - 2\}$. Then the domination cover pebbling number for the square of a path P_n^2 is $\psi(P_n^2) = T(P_n^2)$.

Proof. The result is true for n = 8 to 12, by Theorem 5 to Theorem 9. So assume that the result is true for $m < n$. Consider $p_n^2 : v_1v_2v_3 \dots v_{n-1}v_n$ $(n \geq 8)$. To cover dominate the vertex v_n we need at least $2^{\lfloor \frac{n-2}{2} \rfloor}$ pebbles from v_1. If we put one pebble at v_{n-2}, we cover dominate the square of path $v_{n-4}v_{n-3}v_{n-2}v_{n-1}v_n$. Similarly, we need $2^{\lfloor \frac{n-7}{2} \rfloor} + 2^{\lfloor \frac{n-12}{2} \rfloor} + \dots + 2^{\lfloor \frac{\alpha+8}{2} \rfloor} + 2^{\lfloor \frac{\alpha+3}{2} \rfloor}$ pebbles at v_1 to cover dominate the square of path $v_{\alpha+1} - v_{n-5}$. Thus we need $2^{\lfloor \frac{n-2}{2} \rfloor} + 2^{\lfloor \frac{n-7}{2} \rfloor} + 2^{\lfloor \frac{n-12}{2} \rfloor} + \dots + 2^{\lfloor \frac{\alpha+8}{2} \rfloor} + 2^{\lfloor \frac{\alpha+3}{2} \rfloor}$ pebbles at v_1 to cover dominate the square of path $v_{\alpha+1}v_{\alpha+2} \dots v_{n-1}v_n$. That is, we need $\sum_{i \in I} 2^{\lfloor \frac{i}{2} \rfloor}$ pebbles at v_1, where $I = \{\alpha + 3, \alpha + 8, \alpha + 13, ..., n - 7, n - 2\}$. Clearly we are done if $\alpha = 0$. If $\alpha = 1$, then we need one more pebble to cover dominate the vertex v_1 and for $2 \leq \alpha \leq 4$, we need $\lfloor \frac{\alpha}{2} \rfloor$ pebbles to cover dominate the remaining vertices of P_n^2 from v_1. Thus we need at least $T(P_n^2)$ pebbles on v_1 to cover dominate the vertices of P_n^2, that is, $\psi(P_n^2) \geq T(P_n^2)$. Let us use the induction on n to prove the upper bound for the domination cover pebbling number of P_n^2. We have to show that $p(P_n^2) = T(P_n^2)$ pebbles suffice.

Case 1: n is even.

Consider the paths $P_A : v_1v_2...v_{\frac{n}{2}}$ and $P_B : v_{\frac{n}{2}+1}v_{\frac{n}{2}+2}...v_n$. Also note that $\psi(P_A^2) = \psi(P_{\frac{n}{2}}^2)$ and $\psi(P_B^2) = \psi(P_{\frac{n}{2}}^2)$. If $p(P_A^2) \geq \psi(P_{\frac{n}{2}}^2)$ and $p(P_B^2) \geq \psi(P_{\frac{n}{2}}^2)$ then we are done. Without loss of generality, let us assume that $p(P_B^2) \leq \psi(P_{\frac{n}{2}}^2) - 1$. This implies that, $p(P_A^2) \geq T(P_n^2) - \psi(P_{\frac{n}{2}}^2) + 1$. Since, $T(P_n^2) \geq T(P_{n-5}^2) + 2^{\lfloor \frac{n-2}{2} \rfloor} \geq 2T(P_{n-3}^2) \geq 2\psi(P_{n-3}^2)$ and $\psi(P_{\frac{n}{2}}^2) < \psi(P_{n-3}^2)$ we get $p(P_A^2) \geq \psi(P_{n-3}^2)$.

If $p(v_i) = 0$, for $i = n-2, n-1, n$, then using at most $2^{\lfloor \frac{n-2}{2} \rfloor}$ pebbles, we can move a pebble to v_{n-2} and we cover dominate the vertices v_{n-4}, v_{n-3}, v_{n-1} and v_n. Then we have at least $T(P_n^2) - 2^{\lfloor \frac{n-2}{2} \rfloor} \geq 2T(P_{n-3}^2)$ pebbles in $P_A^2 \cup [P_B^2 - \{v_{n-2}v_{n-1}v_n\}]$. If $p(v_{n-3}) \leq 1$ and $p(v_{n-4}) \leq 1$ then $P_A^2 \cup [P_B^2 - \{v_{n-2}v_{n-1}v_n\}]$ contains $T(P_n^2) - 2 \geq T(P_{n-5}^2) \geq \psi(P_{n-5}^2)$ and hence we are done. Suppose $p(v_{n-3}) \geq 2$ or $p(v_{n-4}) \geq 2$ or both. Then using two pebbles from these vertices, we can move a pebble to v_{n-2} and we are done since $T(P_n^2) - 2 \geq \psi(P_{n-3}^2)$ pebbles in $P_A^2 \cup [P_B^2 - \{v_{n-2}v_{n-1}v_n\}]$. If $1 \leq p(v_{n-2}) + p(v_{n-1}) + p(v_n) \leq p(P_B^2 \leq \psi(P_{\frac{n}{2}}^2) - 1$ then we are done as $P_{n-3}^2 = P_n^2 - \{v_{n-2}v_{n-1}v_n\}$ contains $p(P_A^2) \geq \psi(P_{n-3}^2)$ pebbles.

Case 2: n is odd.

Consider the paths $P_A : v_1v_2...v_{\frac{n+1}{2}}$ and $P_B : v_{\frac{n+1}{2}+1}v_{\frac{n+1}{2}+2}...v_n$. Also note that $\psi(P_A^2) = \psi(P_{\frac{n+1}{2}}^2)$ and $\psi(P_B^2) = \psi(P_{\frac{n-1}{2}}^2)$. If $p(P_A^2) \geq \psi(P_{\frac{n+1}{2}}^2)$ and $p(P_B^2) \geq \psi(P_{\frac{n-1}{2}}^2)$ then we are done. Let us assume that $p(P_B^2) \leq \psi(P_{\frac{n-1}{2}}^2) - 1$. This implies that, $p(P_A^2) \geq T(P_n^2) - \psi(P_{\frac{n-1}{2}}^2) + 1 \geq \psi(P_{n-3}^2)$ since $T(P_n^2) \geq 2\psi(P_{n-3}^2)$ and $\psi(P_{\frac{n-1}{2}}^2) < \psi(P_{n-3}^2)$.

If $p(v_i) = 0$, for $i = n-2, n-1, n$, then using at most $2^{\lfloor \frac{n-2}{2} \rfloor} + 1$ pebbles [–For the case $p(v_1 = T(P_n^2 - 1$ and $p(v_{n-3}) = 1$, using $2^{\lfloor \frac{n-2}{2} \rfloor} - 1$ pebbles from v_1 and one pebble from v_{n-3}, we cannot move a pebble to v_{n-2}. So we need $2^{\lfloor \frac{n-2}{2} \rfloor} + 1$ pebbles for this case, to put a pebble at v_{n-2}. For the other cases, we need at most $2^{\lfloor \frac{n-2}{2} \rfloor}$ pebbles so that we can move a pebble to v_{n-2}–], we can move a pebble to v_{n-2} and we cover dominate the vertices v_{n-4}, v_{n-3}, v_{n-1} and v_n. Then we have at least $T(P_n^2) - 2^{\lfloor \frac{n-2}{2} \rfloor} - 1 \geq \psi(P_{n-5}^2) - 1$ pebbles in $P_A^2 \cup [P_B^2 - \{v_{n-2}, v_{n-1}, v_n\}]$. If we use exactly $2^{\lfloor \frac{n-2}{2} \rfloor} + 1$ pebbles then $p(v_{n-3}) = 1$ and so the vertex v_{n-5} is cover dominated. Thus, $p(P_A^2) \geq T(P_n^2) - \psi(P_{\frac{n-1}{2}}^2) - 2^{\lfloor \frac{n-2}{2} \rfloor} \geq \psi(P_{n-6}^2)$. If $p(v_{n-3}) \leq 1$ and $p(v_{n-4}) \leq 1$ then we are done. Suppose $p(v_{n-3}) \geq 2$. Then we can move a pebble to v_{n-2} and we are done since $P_A^2 \cup [P_B^2 - \{v_{n-2}v_{n-1}v_n\}]$ contains at least $T(P_n^2) - 2 \geq \psi(P_{n-3}^2)$ pebbles. If $1 \leq p(v_{n-2}) + p(v_{n-1}) + p(v_n) \leq p(P_B^2) \leq \psi(P_{\frac{n-1}{2}}^2) - 1$ then we are done as $P_{n-3}^2 = P_n^2 - \{v_{n-2}v_{n-1}v_n\}$ contains $p(P_A^2) \geq \psi(P_{n-3}^2)$ pebbles.

A similar argument is true for the case $p(P_A^2) \geq \psi(P_{\frac{n+1}{2}}^2) - 1$, by using the conditions $T(P_n^2) \geq 2\psi(P_{n-3}^2)$ and $\psi(P_{\frac{n+1}{2}}^2) < \psi(P_{n-3}^2)$. Thus we can always cover dominate the vertices of P_n^2. That is, $\psi(P_n^2) \leq T(P_n^2)$. □

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