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STABILITY ESTIMATE FOR THE MULTIDIMENSIONAL ELLIPTIC OBSTACLE PROBLEM WITH RESPECT TO THE OBSTACLE FUNCTION

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Abstract. The stability estimate of the energy integral established by Danelia, Dochviri and Shashiashvili [1] for the solution of the multidimensional obstacle problem in case of the Laplace operator is generalized to the case of arbitrary linear second order self-adjoint elliptic operator.

This estimate asserts that if two obstacle functions are close in the L^∞-norm, then the gradients of the solutions of the corresponding obstacle problem are close in the weighted L^2-norm.

Key words: stability estimate, unilateral elliptic obstacle problem, energy integral.
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1. Introduction

The classical obstacle problem, which is a particular example of the variational inequality, is stated as follows:
find the equilibrium position $u = u(x)$, $x \in D \subset R^2$ of an elastic membrane constrained to lie above a given obstacle $\psi(x)$ under the action of an external force function $f(x)$.

According to the famous French mathematician J. L. Lions, this problem is simple, beautiful and deep. It has been the subject of investigation for several decades and there are a few manuals dedicated to it, see e.g. Kinderlehrer and Stampacchia [3], Rodrigues [4], Troianiello [5].

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The function $u(x)$ turns out to be the unique solution of the following unilateral elliptic obstacle problem

$$\begin{cases} u(x) \geq \psi(x)\ ,\ \Delta u(x) \leq f(x), \\ \big(\Delta u(x) - f(x)\big) \cdot \big(u(x) - \psi(x)\big) = 0 \end{cases} \quad a.e.\ in\, D, \tag{1}$$

where $\Delta u(x)$ denotes the Laplace operator.
The following intuitively expected stability estimate with respect to obstacle function is the well-known classical result (see e.g; Rodrigues [4, chapter 4, Theorem 7.4])

$$\|\tilde{u}(x) - u(x)\|_{L^\infty(D)} \leq \|\tilde{\psi}(x) - \psi(x)\|_{L^\infty(D)}, \tag{2}$$

where $u(x)(respectively\, \tilde{u}(x))$ is the solution of the obstacle problem for the obstacle function $\psi(x)(respectively\, \tilde{\psi}(x))$.

It was found out by Danelia, Dochviri and Shashiashvili [1] that the following energy integral

$$\int_D \big|grad\, \tilde{u}(x) - grad\, u(x)\big|^2\, h(x)\, dx \tag{3}$$

can also be bounded through the L^∞-norm $\|\tilde{\psi}(x) - \psi(x)\|_{L^\infty(D)}$, where $h(x)$ is a particular weight function.
The objective of the present paper is the generalization of the latter estimate to the case of arbitrary linear second order self-adjoint elliptic operator $Lu(x)$ using only classical functional analytical methods.

The exact formulation of the multidimensional elliptic obstacle problem reads as follows.
Consider an n-dimensional bounded domain D with a boundary ∂D of the class $C^{2+\gamma},\ 0 < \gamma \leq 1$. Denote by $Lu(x)$ the linear second order elliptic self-adjoint differential operator acting on a function $u(x)$ from the Sobolev space $H^2(D) \cap H_0^1(D)$,

$$Lu(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\ \frac{\partial u(x)}{\partial x_i}\right) + c(x)\, u(x) \tag{4}$$

where we assume that

$$a_{ij}(x) = a_{ji}(x),\ a_{ij}(x) \in C^{1+\gamma}(\overline{D})\,,\ c(x) \in C^\gamma(\overline{D})\ \ and\ \ c(x) \leq 0\ in\ D \tag{5}$$

and the uniform ellipticity condition is satisfied:

$$\sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \geq \alpha |y|^2,\ \ y = (y_1, \ldots, y_n) \in \mathbb{R}^n,\ \ \alpha > 0. \tag{6}$$

An external force function $f(x)$, $f(x) \in L^2(D)$ will be fixed throughout the paper. Consider the obstacle function $\psi(x)$, such that

$$\psi(x) \in H^2(D), \ \max(0, \psi(x)) \in H_0^1(D). \tag{7}$$

Here $H_0^1(D)$ is the closure of $C_0^\infty(D)$ in $H^1(D)$.
Find the function $u(x)$, $u(x) \in H^2(D) \cap H_0^1(D)$, such that

$$\begin{cases} u(x) \geq \psi(x), \quad Lu(x) \leq f(x), \\ \big(f(x) - Lu(x)\big) \cdot \big(u(x) - \psi(x)\big) = 0 \end{cases} \quad \text{a.e. in } D. \tag{8}$$

By Troianiello [5, Theorem 5.2], there exists a unique solution $u(x)$ of the latter multidimensional unilateral elliptic obstacle problem.

The paper is organized as follows. In Section 2, we prove an auxiliary inequality for the functions from the Sobolev space $H^2(D) \cap H_0^1(D)$. In Section 3, we establish the basic result of this paper, which is the stability estimate of the energy integral for the solution of the multidimensional unilateral elliptic obstacle problem.

2. An Inequality for Functions from Sobolev Space

Let us define the weight function h(x) as the unique smooth solution of the following Dirichlet problem

$$\begin{cases} Lh(x) = -1 \quad in \ D \\ h(x) = 0 \quad on \ \partial D \end{cases} \tag{9}$$

By the global regularity theorem (6.14) in Gilbarg, Trudinger [2, chapter 6], we know that the latter Dirichlet problem has a unique solution $h(x)$ which is smooth up to the boundary ∂D, i.e. $h(x) \in C^{2+\gamma}(\overline{D})$.

Now by the Hopf's strong maximum principle we obtain

$$h(x) > 0 \ \ in \ D. \tag{10}$$

The following proposition is an important step in proving the basic energy inequality of this paper.

Theorem 1. *Suppose $v(x) \in H^2(D) \cap H_0^1(D)$. Then the following inequality is valid for the function $v(x)$*

$$2\alpha \int_D \big|grad\, v(x)\big|^2 h(x) dx + \int_D v^2(x) dx \leq -2 \int_D v(x) \ Lv(x)\, h(x)\, dx \tag{11}$$

Proof. We shall prove the latter inequality at first for smooth functions $v(x)$, such that $v(x) \in C^2(\overline{D})$, $v(x) = 0$ *on* ∂D and then we shall extend it to functions $v(x)$, $v(x) \in H^2(D) \cap H_0^1(D)$.

We start from classical Green's second formula for $v(x),\ v(x) \in C^2(\overline{D})$ with $v(x) = 0$ on ∂D and $h(x) \in C^2(\overline{D})$:

$$\int_D Lv(x)\, h(x)\, dx \;-\; \int_D v(x)\, Lh(x)\, dx$$

$$= \int_{\partial D} \sum_{i=1}^{n} \Bigg[\sum_{j=1}^{n} \Bigg(h(x)\, a_{ij}(x)\, \frac{\partial v(x)}{\partial x_j} \;-\; v(x)\, a_{ij}(x)\, \frac{\partial h(x)}{\partial x_j} \;-$$

$$- \, v(x)\, h(x)\, \frac{\partial a_{ij}(x)}{\partial x_j} \Bigg) \cdot n_i(x) \Bigg] d\sigma, \tag{12}$$

where the boundary integral is $(n-1)$-dimensional surface integral and $(n_i)_{i=1,\dots,n}$ is the outer normal vector.
Clearly this boundary integral vanishes as $h(x) = 0,\ v(x) = 0$ on boundary ∂D.
Hence we get the equality

$$\int_D Lv(x)\, h(x)\, dx = -\int_D v(x)\, dx \tag{13}$$

Let us put $v^2(x)$ instead of $v(x)$ in the latter formula, we have

$$\int_D Lv^2(x)\, h(x)\, dx = -\int_D v^2(x)\, dx \tag{14}$$

It is easy to check that

$$Lv^2(x) = \, 2 \sum_{i,j=1}^{n} a_{ij}(x)\, \frac{\partial v(x)}{\partial x_i}\, \frac{\partial v(x)}{\partial x_j} + \, 2v(x)\; Lv(x) - c(x)\; v^2(x) \tag{15}$$

From the latter equalities (14)-(15) we get

$$2 \int_D \sum_{i,j=1}^{n} a_{ij}(x)\, \frac{\partial v(x)}{\partial x_i}\, \frac{\partial v(x)}{\partial x_j}\, h(x)\, dx \; + \; 2 \int_D v(x)\; Lv(x)\, h(x)\, dx \, -$$

$$\int_D c(x)\; v^2(x)\, h(x)\, dx \; = -\int_D v^2(x) dx$$

from which we come to the inequality

$$2 \int_D \sum_{i,j=1}^{n} a_{ij}(x)\, \frac{\partial v(x)}{\partial x_i}\, \frac{\partial v(x)}{\partial x_j}\, h(x)\, dx + \int_D v^2(x)\, dx \le \, -2 \int_D v(x)\; Lv(x)\, h(x)\, dx \tag{16}$$

For $y = grad\, v(x)$, the uniform ellipticity condition (6) gives us that

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \frac{\partial v(x)}{\partial x_i}\, \frac{\partial v(x)}{\partial x_j} \ge \, \alpha\, \left| grad\, v(x) \right|^2, \quad x \in D\, (\alpha > 0). \tag{17}$$

So from the inequality (16) we get the following estimate

$$2\alpha \int_D \left|grad\, v(x)\right|^2 h(x)\, dx + \int_D v^2(x)\, dx \leq -2 \int_D v(x)\ Lv(x)\, h(x)\, dx \quad (18)$$

for arbitrary $v(x),\, v(x) \in C^2(\overline{D})$ *with* $v(x) = 0$ *on* ∂D.

Now we will extend the equality (13) and the inequality (18) for functions $v(x),\, v(x) \in H^2(D) \cap H_0^1(D)$.
It is known from Gilbarg, Trudinger [2, chapter 9, problem 9.6] that the subspace

$$\{v(x) \in C^2(\overline{D}) | v(x) = 0 \ on\ \partial D\}$$

is dense in $H^2(D) \cap H_0^1(D)$.
Hence there exists a sequence $v_m(x)$ such that $v_m(x) \in C^2(\overline{D})$ with $v_m(x) = 0$ *on* ∂D *and*

$$\|v_m(x) - v(x)\|_{H^2(D)} \to 0\ ,\ m \to \infty. \quad (19)$$

Let us write the equality (13) for functions $v_m(x)$, $m = 1, 2, ...$

$$\int_D Lv_m(x)\, h(x)\, dx = -\int_D v_m(x)\, dx. \quad (20)$$

Consider the difference

$$Lv_m(x) - Lv(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\ \frac{\partial(v_m(x) - v(x))}{\partial x_i}\right) +\ c(x)\ (v_m(x) - v(x))$$

$$= \sum_{i,j=1}^{n} a_{ij}(x)\ \frac{\partial^2(v_m(x) - v(x))}{\partial x_i \partial x_j} +$$

$$+ \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j}\right)\ \frac{\partial(v_m(x) - v(x))}{\partial x_i}\ +\ c(x)\, (v_m(x) - v(x))$$

From the assumption (5), it is easy to see that the functions $a_{ij}(x)$, $\frac{\partial a_{ij}(x)}{\partial x_j}$, $i, j = 1, 2, ..., n$ *and* $c(x)$ are bounded on the closure $\overline{D}$ by some constant $\tilde{C}$.

Therefore

$$\left\|Lv_m(x) - Lv(x)\right\|_{L^2(D)} \leq n\tilde{C}\ \left(\sum_{i,j=1}^{n} \left\|\frac{\partial^2(v_m(x) - v(x))}{\partial x_i \partial x_j}\right\|_{L^2(D)} + \right.$$

$$\left. + \sum_{i=1}^{n} \left\|\frac{\partial(v_m(x) - v(x))}{\partial x_i}\right\|_{L^2(D)} + \left\|v_m(x) - v(x)\right\|_{L^2(D)}\right) \quad (21)$$

From here we get

$$\left\| Lv_m(x) - Lv(x) \right\|_{L^2(D)} \underset{m\to\infty}{\longrightarrow} 0. \tag{22}$$

Passing to the limit as $m \longrightarrow \infty$ in the equality (20) we obtain for arbitrary $v(x),\ v(x) \in H^2(D) \cap H_0^1(D)$ the equality

$$\int_D Lv(x)\, h(x)\, dx = -\int_D v(x)\, dx. \tag{23}$$

Let us write the inequality (18) for the functions $v_m(x)$, $m = 1, 2, ...$

$$2\alpha \int_D \left|grad\, v_m(x)\right|^2 h(x)\, dx + \int_D v_m^2(x)\, dx \le\ -2 \int_D v_m(x)\ Lv_m(x)\, h(x)\, dx, \tag{24}$$

$m = 1, 2, ...$.

We have

$$\left| \int_D v_m(x)\ Lv_m(x)\, h(x)\, dx\ -\ \int_D v(x)\ Lv(x)\, h(x)\, dx \right|$$

$$\le \sup_{\overline{D}} h(x) \left[\int_D \left|v_m(x)\ L(v_m(x) - v(x))\right| dx + \int_D \left|(v_m(x) - v(x))\, Lv(x)\right| dx \right] \tag{25}$$

As $\|v_m(x)\|_{L^2(D)}$, $\|Lv(x)\|_{L^2(D)}$ are bounded by some constant and

$\|v_m(x) - v(x)\|_{L^2(D)} \longrightarrow 0$, $\left\|L(v_m(x) - v(x))\right\|_{L^2(D)} \underset{m\to\infty}{\longrightarrow} 0$ by (22),

applying the Cauchy-Schwarz inequality to (25) we obtain

$$\int_D v_m(x)\ Lv_m(x)\, h(x)\, dx\ \longrightarrow\ \int_D v(x)\ Lv(x)\, h(x)\, dx \tag{26}$$

as $m \longrightarrow \infty$.

Consider the difference

$$\left| \int_D |grad\, v_m(x)|^2\, h(x)\, dx - \int_D\ |grad\, v(x)|^2\, h(x)\, dx \right|$$

$$\le \sup_{\overline{D}} h(x) \int_D \left| |grad\, v_m(x)|^2 - |grad\, v(x)|^2 \right| dx$$

$$\le \sup_{\overline{D}}\ h(x) \left(\int_D |grad\, (v_m(x) - v(x))|^2 dx + \right.$$

$$\left. +2 \int_D \left|grad\, v(x)\right| \left|grad\, (v_m(x) - v(x))\right| dx \right), \tag{27}$$

where we have used the following identity for n-dimensional vectors y_1 and y_2

$$|y_2|^2 - |y_1|^2 = |y_2 - y_1|^2 + 2y_1 \cdot (y_2 - y_1). \tag{28}$$

As $\quad \|v_m(x) - v(x)\|_{H^2(D)} \underset{m\to\infty}{\longrightarrow} 0$, *we get*

$$\left\| grad(v_m(x) - v(x)) \right\|_{L^2(D)} \underset{m\to\infty}{\longrightarrow} 0$$

.
Hence

$$\int_D |grad\, v_m(x)|^2\, h(x)\, dx \longrightarrow \int_D |grad\, v(x)|^2\, h(x)\, dx \ , \ as\, m \longrightarrow \infty. \tag{29}$$

Finally we pass to limit $m \longrightarrow \infty$ in the inequality (24) and get the desired estimate (11). □

3. Main Results The Energy Inequality for the Difference of Solutions of the Obstacle Problem

Let $f(x),\ f(x) \in L^2(D)$ be a fixed external force function. Consider the obstacle functions $\psi(x),\ \tilde{\psi}(x)$ such that

$$\psi(x),\ \tilde{\psi}(x) \in H^2(D)\ and\ \max(0, \psi(x)), \max(0, \tilde{\psi}(x)) \in H_0^1(D). \tag{30}$$

We recall the unilateral obstacle problem :
the function $u(x)$ (respectively $\tilde{u}(x)$) belonging to the intersection $H^2(D) \cap H_0^1(D)$ is called the solution of the obstacle problem for the operator L if

$$\begin{cases} u(x) \geq \psi(x),\ Lu(x) \leq f(x), \\ \qquad\qquad\qquad\qquad\qquad\qquad a.e.\ in\, D, \\ \big(Lu(x) - f(x)\big) \cdot \big(u(x) - \psi(x)\big) = 0\ , \end{cases} \tag{31}$$

(respectively),

$$\begin{cases} \tilde{u}(x) \geq \tilde{\psi}(x),\ L\tilde{u}(x) \leq f(x), \\ \qquad\qquad\qquad\qquad\qquad\qquad a.e.\ in\, D, \\ \big(L\tilde{u}(x) - f(x)\big) \cdot \big(\tilde{u}(x) - \tilde{\psi}(x)\big) = 0. \end{cases} \tag{32}$$

The following proposition is the basic result of this paper.

Theorem 2. *Let the external force function* $f(x)$ *belong to* $L^2(D)$ *and the obstacle functions* $\psi(x), \tilde{\psi(x)}$ *satisfy condition* (30). *Suppose that the difference* $\tilde{\psi}(x) - \psi(x)$ *belongs to* $L^\infty(D)$, *i.e.* $\|\tilde{\psi}(x) - \psi(x)\|_{L^\infty(D)} < \infty$. *Then for the difference* $\tilde{u}(x) - u(x)$ *of solutions of the obstacle problems* (31) *and* (32) *we have the following energy estimate*

$$\alpha \int_D \left| grad\, \tilde{u}(x) - grad\, u(x) \right|^2 h(x)\, dx + \frac{1}{2} \int_D (\tilde{u}(x) - u(x))^2 dx \leq$$

$$\|\tilde{\psi}(x) - \psi(x)\|_{L^\infty(D)} \left[2 \int_D |f(x)|\, h(x) dx + \big(meas(D)\big)^{\frac{1}{2}} \big(\|\tilde{u}(x)\|_{L^2(D)} + \|u(x)\|_{L^2(D)} \big) \right] \tag{33}$$

Proof. Denote $v(x) = \tilde{u}(x) - u(x)$. Since $v(x) \in H^2(D) \cap H_0^1(D)$, from the inequality (11) of section-2 we get

$$\alpha \int_D \left| grad\,(\tilde{u}(x) - u(x)) \right|^2 h(x) dx + \frac{1}{2} \int_D (\tilde{u}(x) - u(x))^2\, dx$$
$$\leq - \int_D (\tilde{u}(x) - u(x))\; L(\tilde{u}(x) - u(x))\, h(x)\, dx \qquad (34)$$

Consider the right hand side of the inequality (34), we have

$$- \int_D (\tilde{u}(x) - u(x))\, L(\tilde{u}(x) - u(x))\, h(x)\, dx$$
$$= - \int_D [(\tilde{u}(x) - \tilde{\psi}(x)) + (\tilde{\psi}(x) - \psi(x)) + (\psi(x) - u(x))]\; L(\tilde{u}(x) - u(x))\, h(x)\, dx.$$

Let us rewrite this equality in the following manner

$$- \int_D (\tilde{u}(x) - u(x))\, L(\tilde{u}(x) - u(x))\, h(x)\, dx$$
$$= - \int_D (\tilde{u}(x) - \tilde{\psi}(x)) \left[(L\tilde{u}(x) - f(x)) + (f(x) - Lu(x)) \right] h(x)\, dx -$$
$$- \int_D (\tilde{\psi}(x) - \psi(x))\; (L\tilde{u}(x) - Lu(x))\, h(x)\, dx -$$
$$- \int_D (\psi(x) - u(x)) \left[(L\tilde{u}(x) - f(x)) + (f(x) - Lu(x)) \right] h(x)\, dx \qquad (35)$$

Now using the formulation of obstacle problems (31) and (32), the latter equality takes the following form

$$- \int_D (\tilde{u}(x) - u(x))\; L(\tilde{u}(x) - u(x))\, h(x)\, dx$$
$$= - \int_D (\tilde{u}(x) - \tilde{\psi}(x))\, (f(x) - Lu(x))\, h(x)\, dx -$$
$$- \int_D (\tilde{\psi}(x) - \psi(x))\; (L\tilde{u}(x) - Lu(x))\, h(x)\, dx -$$
$$- \int_D (u(x) - \psi(x))(f(x) - L\tilde{u}(x))\, h(x)\, dx$$
$$\leq - \int_D (\tilde{\psi}(x) - \psi(x))\, \big(L\tilde{u}(x) - Lu(x) \big)\, h(x)\, dx.$$

Thus we arrive to the following inequality

$$- \int_D (\tilde{u}(x) - u(x))\; L(\tilde{u}(x) - u(x))\, h(x)\, dx$$
$$\leq - \int_D (\tilde{\psi}(x) - \psi(x))\; \left[(L\tilde{u}(x) - f(x)) + (f(x) - Lu(x)) \right] h(x)\, dx \qquad (36)$$

From the latter inequality we can write

$$
\begin{aligned}
&-\int_D (\tilde{u}(x)-u(x))\; L(\tilde{u}(x)-u(x))\, h(x)\, dx \\
&\quad\le \int_D (\tilde{\psi}(x)-\psi(x))\; \big[|L\tilde{u}(x)-f(x)|+|f(x)-Lu(x)|\big]\, h(x)\, dx \\
&\le \|\tilde{\psi}(x)-\psi(x)\|_{L^\infty(D)} \big[2\int_D f(x)\, h(x)\, dx - \int_D L(\tilde{u}(x)+u(x))\, h(x)\, dx\big]
\end{aligned} \tag{37}
$$

Applying the equality (23) for the function $v(x)=\tilde{u}(x)+u(x)$ we have

$$
\begin{aligned}
-\int_D L(\tilde{u}(x)+u(x))\, h(x)\, dx &= \int_D (\tilde{u}(x)+u(x))\, dx \\
&\le \big(meas(D)\big)^{\frac{1}{2}}\; \big(\; \|\tilde{u}(x)\|_{L^2(D)} + \|u(x)\|_{L^2(D)}\big)
\end{aligned} \tag{38}
$$

Using the bounds (37) and (38) we get

$$
\begin{aligned}
&-\int_D (\tilde{u}(x)-u(x))\, L(\tilde{u}(x)-u(x))\, h(x)\, dx \\
&\quad\le \|\tilde{\psi}(x)-\psi(x)\|_{L^\infty(D)} \Big[2\int_D |f(x)|\, h(x)\, dx + \\
&\qquad\qquad + \big(meas(D)\big)^{\frac{1}{2}}\; \big(\; \|\tilde{u}(x)\|_{L^2(D)} + \|u(x)\|_{L^2(D)}\big)\Big]
\end{aligned} \tag{39}
$$

Now from the inequalities (34) and (39) we come to the desired estimate (33). □

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