Journal of Prime Research in Mathematics Vol. **8***(2012), 22-27*

THE BANACH-SAKS INDEX OF INTERSECTION

NOVIKOVA A. I.[1]

Abstract. In this paper we estimate Banach-Saks index of intersection of two spaces with symmetric bases from below by indices of these spaces. We also show on example of Orlicz spaces that we can't estimate Banach-Saks index of intersection in the same way from above.

Key words: Banach-Saks index, symmetric basis, Orlicz space.
AMS subject: 46E30.

1. Introduction

Banach-Saks p-property has its origin in the classical work of Banach and Saks [3]. In the sequel a lot of works were devoted to studying of Banach-Saks p-property in rearrangement invariant spaces (see, for example, [1, 4, 10]). For sequence spaces instead of rearrangement invariance we can talk about spaces with symmetric basis. Many authors investigate Banach-Saks p-property in classic Orlicz, Marcinkiewic and Lorentz sequence spaces ([2, 8, 9]). Banach-Saks p-property is closely related with other properties of Banach spaces as separability, reflexivity, p-convexity and plays an important role in geometry of Banach spaces. Intersection and sum of two spaces appear naturally in interpolation theory. This article is devoted to estimating Banach-Saks index of intersection $E \cap F$ by indices of E and F.

Definition 1. *By non-increasing rearrangement of sequence $\{a_n\}_{n=1}^{\infty}$ from c_0 we will mean non-increasing sequence $\{a_n^*\}_{n=1}^{\infty}$ obtained from the sequence $\{|a_n|\}_{n=1}^{\infty}$ by a proper permutation of natural numbers.*

Definition 2. *Basis $\{x_n\}_{n=1}^{\infty}$ of Banach space E is called symmetric if for every permutation π of natural numbers basis $\{x_{\pi(n)}\}_{n=1}^{\infty}$ is equivalent to $\{x_n\}_{n=1}^{\infty}$ (* [6, p.115-136]*).*

[1]Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan. Email: *novikova.anna18@gmail.ru*.
The research is supported by RFBR grant 11-01-00614-a.

It's well-known that for every separable sequence space with symmetric basis the following inclusion holds $l_1 \subset E \subset c_0$ and

$$\|x\|_{l_\infty} \leq \|x\|_E \leq \|x\|_{l_1}.$$

The natural basis of sequence spaces $l_p, 1 \leq p \leq \infty$, is symmetric.

The other examples of the spaces with symmetric basis are Orlicz sequence spaces. For convex, non-decreasing continuous function M on $[0, \infty)$, such that $M(0) = 0$ and $\lim\limits_{t\to\infty} M(t) = \infty$ (Orlicz function), Orlicz space l_M consists of all sequences $x = (x_1, x_2, ...)$ such that $\sum\limits_{n=1}^{\infty} M(\frac{|x_n|}{\rho}) < \infty$ for some $\rho > 0$. Conditions imposed on M make l_M be Banach space equipped with the norm

$$\|x\|_{l_M} = \inf\{\rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|x_n|}{\rho}\right) \leq 1\}.$$

We are interested in separable part h_M of l_M that consists of all sequences $x \in l_M$ such that $\sum\limits_{n=1}^{\infty} M(\frac{|x_n|}{\rho}) < \infty$ for any $\rho > 0$. Structure of sequence Orlicz spaces were studied by many authors (see [5, 7]).

Proposition 1. *(* [6, 4.a.5]*). Let M_1 and M_2 be two Orlicz functions. Then the following statements are equivalent:*

$$\begin{array}{ll} (i) & l_{M_1} = l_{M_2}; \\ (ii) & \exists\, k > 0, K > 0, t_0 > 0 : \ \forall 0 \leq t \leq t_0 \\ & K^{-1} M_2(k^{-1}t) \leq M_1(t) \leq K M_2(kt). \end{array}$$

The second statement means that functions M_1 and M_2 are equivalent at zero. So Orlicz sequence space is determined by the behavior of its normalizing function near zero.

We will consider intersection of two spaces E and F with symmetric bases with its usual norm:

$$\|x\|_{E\cap F} = \|(x_1, x_2, ...)\|_{E\cap F} = \max(\|x\|_E, \|x\|_F).$$

The natural basis of $E \cap F$ will be also symmetric. It's known that for two sequence Orlicz spaces l_{M_1} and l_{M_2} intersection $l_{M_1} \cap l_{M_2} = l_M$, where M is equivalent to $\max(M_1, M_2)$ [6].

Definition 3. *We will say that a Banach space X has Banach-Saks p-property (p-BS property), $p \geq 1$, and write $X \in pBS$ if every weakly null sequence $\{x_n\} \subset X$ contains subsequence $\{y_n\} \subset \{x_n\}$ and there is a number $0 < C < \infty$ such that*

$$\|\sum_{k=1}^{m} y_k\|_X \leq C m^{\frac{1}{p}}, \ m \in \mathbb{N}.$$

Banach-Saks index of the space X is supremum of $p \geq 1$ for which X has p-BS property:

$$\gamma(X) = \sup\{p \geq 1 : \ X \in pBS\}.$$

Clearly, every Banach space has 1-BS-property and for every non-separable space X with symmetric basis Banach-Saks index $\gamma(X)$ equals to 1 [10]. Indeed, if X is non-separable then it contains universal space l_∞ as a subspace. Further since l_∞ by turn also contains a subspaces without p-BS property (for any $p > 1$) then l_∞ and X does not have p-BS property.

We will use the famous Rakov criterium of p-BS property.

Theorem 2. [9, Proposition 2]. *Let X be a Banach space with symmetric basis. Then X has p-BS property if and only if the following is true*

$$\sup_n \phi_X(n)/n^{1/p} < \infty,$$

where $\phi_X(n)$ is a lower bound of numbers c such that any weakly null sequence $\{x_i\}_i$ from X contains n elements $\{x_{k(i)}\}_{i \leq n}$, for which holds

$$\|\sum_{i \leq n} x_{k(i)}\| \leq c \max_{i \leq n} \|x_{k(i)}\|.$$

The next theorem gives a good estimation to Banach-Saks index of Orlicz spaces in terms of delation index of normalizing Orlicz function M.

Theorem 3. [9, Theorem 4]. *Let h_M be an Orlicz sequence space with normalizing function M. Then the following statement is true*

$$\liminf_{t \to 0} \frac{M(2t)}{M(t)} = 2 \ \Leftrightarrow \ \gamma(h_M) = 1 \ \text{or} \ h_M \approx l_1.$$

2. Main Results

Theorem 4. *Let E and F be two Banach spaces with symmetric bases. Then*

$$\gamma(E \cap F) \geq \min(\gamma(E), \gamma(F)).$$

Proof. We have to show that if both E and F have p-BS property then $E \cap F$ also has p-BS property. Using theorem 2 we have

$$E \in pBS \Leftrightarrow \sup_n \frac{\phi_E(n)}{n^{1/p}} < \infty,$$

where $\phi_E(n)$ can be calculated by formula

$$\phi_E(n) = \sup_{\substack{x_n \xrightarrow[E]{w} 0 \\ \|x_n\|=1}} \inf_{\{x_{n_k}\}_{k=1}^N} \Big\| \sum_{k=1}^N x_{n_k} \Big\|_E.$$

Then observing that $\|x\|_{E\cap F}\le\|x\|_E+\|x\|_F$, for $x\in E\cap F$, we have

$$\phi_{E\cap F}(N)=\sup_{\substack{x_n\xrightarrow[E\cap F]{w}0\\ \|x_n\|_\cap=1}}\inf_{\{x_{n_k}\}_{k=1}^N}\left[\max(\|\sum_{k=1}^N x_{n_k}\|_E,\|\sum_{k=1}^N x_{n_k}\|_F)\right]\le$$

$$\le\sup_{\substack{x_n\xrightarrow[E\cap F]{w}0\\ \|x_n\|_\cap=1}}\inf_{\{x_{n_k}\}_{k=1}^N}\left[\|\sum_{k=1}^N x_{n_k}\|_E+\|\sum_{k=1}^N x_{n_k}\|_F\right]\le$$

$$\le\sup_{\substack{x_n\xrightarrow[E\cap F]{w}0\\ \|x_n\|_\cap=1}}\inf_{\{x_{n_k}\}_{k=1}^N}\|\sum_{k=1}^N x_{n_k}\|_E+\sup_{\substack{x_n\xrightarrow[E\cap F]{w}0\\ \|x_n\|_\cap=1}}\inf_{\{x_{n_k}\}_{k=1}^N}\|\sum_{k=1}^N x_{n_k}\|_F\le$$

$$\le\sup_{\substack{x_n\xrightarrow[E]{w}0\\ \|x_n\|_E\le1}}\inf_{\{x_{n_k}\}_{k=1}^N}\|\sum_{k=1}^N x_{n_k}\|_E+\sup_{\substack{x_n\xrightarrow[F]{w}0\\ \|x_n\|_F\le1}}\inf_{\{x_{n_k}\}_{k=1}^N}\|\sum_{k=1}^N x_{n_k}\|_F=$$

$$=\phi_E(N)+\phi_F(N).$$

Thus if $E\in pBS$ and $F\in pBS$ then

$$\sup_n\frac{\phi_{E\cap F}(n)}{n^{1/p}}\le\sup_n\frac{\phi_E(n)+\phi_F(n)}{n^{1/p}}\le$$
$$\le\sup_n\frac{\phi_E(n)}{n^{1/p}}+\sup_n\frac{\phi_F(n)}{n^{1/p}}<\infty$$

which means that also $E\cap F\in pBS$. So we get

$$\gamma(E\cap F)\ge\min(\gamma(E),\gamma(F)).$$

□

The following example shows that the opposite inequality can't be obtained.

Example 1. *There exist two Orlicz functions M_1 and M_2 so that $\gamma(h_{M_1})=\gamma(h_{M_2})=1$ but $h_{M_1}\cap h_{M_2}=l_2$ and thus $\gamma(h_{M_1}\cap h_{M_2})=2$.*

Proof. Let $u_k=\frac{1}{k!}$ $(u_k=\frac{1}{k}\cdot u_{k-1}), k=1,2,\dots$ Consider continuous piecewise linear function $f(t)$ so that its linear parts are tangents to $g(t)=t^2$ at points $\{u_k\}_1^\infty$, i.e.

$$f(t)=2u_nt-u_n^2 \text{ when } t\in[\frac{u_n+u_{n+1}}{2},\frac{u_{n-1}+u_n}{2}],\ n\in\mathbb{N},$$

(see figure 1). Denote $v_n=\frac{u_n+u_{n-1}}{2}$.

Take

$$M_1(t)=\begin{cases} t^2, & t\in[u_{n+1},u_n], n \text{ is odd or } t\in[1,\infty),\\ f(t), & \text{otherwise;}\end{cases}$$

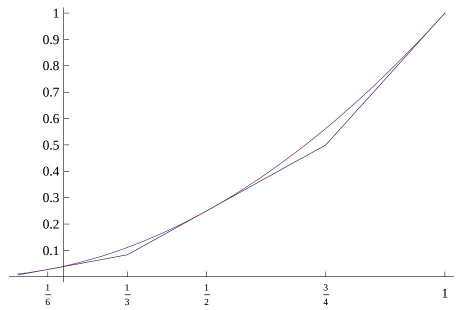

FIGURE 1. Graphs of $f(t)$ and t^2 ($u_1 = 1, u_2 = 1/2, u_3 = 1/6, v_2 = 3/4, v_3 = 1/3$.)

and

$$M_2(t) = \begin{cases} t^2, & t \in [u_{n+1}, u_n], n \text{ is even or } t \in [1, \infty), \\ f(t), & \text{otherwise;} \end{cases}$$

(see figure 2).

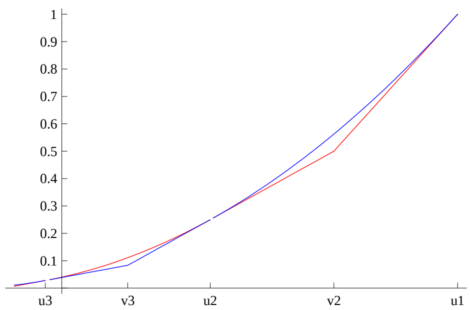

FIGURE 2. Graphs of $M_1(t)$ and $M_2(t)$.

Functions M_1 and M_2 are continuous, increasing and convex, $\lim\limits_{t\to 0} M_{1,2}(t) = 0$. We are interested in behavior of these functions when t tends to zero. Taking $t = v_n$ we have $\frac{t}{2} \geq u_n$ $(u_n + u_{n-1} \geq 4u_n,\ k \cdot u_n \geq 3u_n),\ n \geq 3$. Thus

$$2 \leq \liminf_{t\to 0} \frac{M_1(2t)}{M_1(t)} \leq \lim_{\substack{k\to\infty \\ k-\text{even}}} \frac{M_1(v_k)}{M_1(\frac{v_k}{2})} = \\ = \lim_{\substack{k\to\infty \\ k-\text{even}}} \frac{u_k \cdot u_{k-1}}{u_k \cdot v_k - u_k^2} = \lim_{\substack{k\to\infty \\ k-\text{even}}} \frac{2u_{k-1}}{u_{k-1} - u_k} = \\ = \lim_{\substack{k\to\infty \\ k-\text{even}}} \frac{2}{1 - 1/k} = 2.$$

Hence by theorem 3 we have $\gamma(h_{M_1}) = 1$ and similarly $\gamma(h_{M_2}) = 1$. But $h_{M_1} \cap h_{M_2} = h_{\max(M_1,M_2)} = h_{t^2} = l_2$ and $\gamma(h_{M_1} \cap h_{M_2}) = \gamma(l_2) = 2$.

$\square$

References

[1] Astashkin S.V., Semenov E.M., Sukochev F.A.: *The Banach-Saks p-property*, Mathematische Annalen. 332(2005), 879–900.

[2] Astashkin S.V., Sukochev F.A.: *Banach-Saks property in Marcinkiewicz spaces*, J. Math. Anal. Appl. 336(2007), 1231–1258.

[3] Banach S., Saks S.: *Sur la convergence forte dans les champs L^p*, Ibid. 2(1930), 51–57.

[4] Dodds P.G., Semenov E.M., Sukochev F.A.: *The Banach-Saks property in rearrangement invariant spaces*, Studia Math. 162, N.3(2004), 263–294.

[5] Lindberg K.J.: *On subspaces of Orlicz sequence spaces*, Studia Math. 45(1973), 119–146.

[6] Lindennstrauss J., Tzafriri L.: Classical Banach Spaces I: Sequence Spaces, Springer-Verlag, 1977.

[7] Lindennstrauss J., Tzafriri L.: *On Orlicz sequence spaces*, Israel J. Math. 10(1971), 379–390.

[8] Novikova A.I., Semenov E.M., Sukochev F.A.: *Banach-Saks index in spaces with symmetric basis*, Doklady Mathematics 77, N.3(2008), 396–397.

[9] Rakov S.A.: *On Banach-Saks index of some Banach sequence spaces*, Math. notes 32, N.5(1982), 613–626.

[10] Semenov E.M., Sukochev F.A.: *Banch-Saks index*, Math. sbornik 195, N.2(2004), 263–285.