Journal of Prime Research in Mathematics Vol. **8***(2012), 36-44*

BOYD INDICES FOR QUASI-NORMED REARRANGEMENT INVARIANT SPACES

G. E. KARADZHOV[1]

Abstract. We calculate the Boyd indices for the sum and intersection of two quasi-normed rearrangement invariant spaces. An application to Lorentz type spaces is also given.

Key words: rearrangement invariant function spaces, Boyd indices, interpolation spaces.
AMS subject: Primary 46E30, 47B38.

1. Introduction

Let L_{loc} be the space of all locally integrable functions f on $\mathbf{R}^n$ with the Lebesgue measure and let L be the cone of all locally integrable functions $g \geq 0$ on $(0, \infty)$ with the Lebesgue measure.

Let f^* be the decreasing rearrangement of f, given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},\ t > 0,$$

where μ_f is the distribution function of f, defined by

$$\mu_f(\lambda) = |\{x \in \mathbf{R}^n : |f(x)| > \lambda\}|_n\,,$$

and $|\cdot|_n$ denotes Lebesgue's $n-$measure. If $g \in L$ we define g^* analogously. We use the notations $a_1 \preceq a_2$ or $a_2 \succeq a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \preceq a_2$ and $a_1 \succeq a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

We consider rearrangement invariant quasi-normed spaces E, consisting of all $f \in L_{loc}$, such that $\|f\|_E := \rho_E(f^*) < \infty$, where ρ_E is a quasi-norm, defined on L with values in $[0, \infty]$. In this way equivalent quasi-norms ρ_E give

[1]Insitute of Mathematics and Informatics, Bulgarian Academy of Sciences and Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan. Email: *geremika@yahoo.com*.
Research partially supported by the Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan and by a grant from HEC, Pakistan.

the same space E. We suppose that $L^1 \cap L^\infty \hookrightarrow E \hookrightarrow L^1 + L^\infty$, i.e. E is an intermediate space for the couple (L^1, L^∞). There is an equivalent quasi-norm $\rho_p \approx \rho_E$ that satisfies the triangle inequality $\rho_p^p(g_1 + g_2) \leq \rho_p^p(g_1) + \rho_p^p(g_2)$, $g_1, g_2 \in L$, for some $p \in (0, 1]$ that depends only on the space E (see [9]). We say that the quasi-norm ρ_E is $K-$monotone (cf. [4], Definition 1.16, p. 305) if

$$g_1^{**} \leq g_2^{**} \text{ implies } \rho_E(g_1^*) \preceq \rho_E(g_2^*),\ g_1 \in L,\ g_2 \in L, \tag{1}$$

where $g^{**}(t) = \frac{1}{t}\int_0^t g^*(s)ds$, and we say that ρ_E is monotone if $g_1 \leq g_2$ implies $\rho_E(g_1^*) \leq \rho_E(g_2^*)$.

We say that the quasi-norm ρ_E satisfies Minkowski's inequality if for the equivalent quasi-norm ρ_p,

$$\rho_p^p\left(\sum g_j\right) \preceq \sum \rho_p^p(g_j),\ g_j \in L. \tag{2}$$

For example, if E is a rearrangement invariant Banach function space as in [4], then by the Luxemburg representation theorem $\|f\|_E = \rho_E(f^*)$ for some norm ρ_E satisfying (1), (2). More general example is given by the Riesz-Fischer monotone spaces as in [4], p. 304, 305.

We recall some basic definitions of the real interpolation for quasi-normed spaces [5]. Let (A_1, A_2) be a couple of two quasi-normed spaces, such that both are continuously embedded in some quasi-normed space (see [5]) and let

$$K(t, f) = K(t, f; A_1, A_2) = \inf_{f = f_1 + f_2}\{\|f_0\|_{A_1} + t\,\|f_2\|_{A_2}\},\ f \in A_1 + A_2,$$

be the $K-$functional of Peetre (see [5]). By definition, the $K-$interpolation space $A_\Phi = (A_1, A_2)_\Phi$ has a quasi-norm

$$\|f\|_{A_\Phi} = \|K(., f)\|_\Phi\,,$$

where Φ is a quasi-normed function space with a monotone quasi-norm on $(0, \infty)$ with the Lebesgue measure and such that $\min\{1, t\} \in \Phi$. Then (see [5])

$$A_1 \cap A_2 \hookrightarrow A_\Phi \hookrightarrow A_1 + A_2.$$

where by $X \hookrightarrow Y$ we mean that X is continuously embedded in Y. If

$$\|g\|_\Phi = \left(\int_0^\infty [w(t)t^{-\theta}g(t)]^q dt/t\right)^{1/q},\ 0 \leq \theta \leq 1,\ 0 < q \leq \infty,\ w \in L,$$

we write $(A_1, A_2)_{wt^{-\theta},q}$ instead of $(A_1, A_2)_\Phi$ (see [5]).

By definition,

$$\|f\|_{A_1 \cap A_2} = \|f\|_{A_1} + \|f\|_{A_2},\ \|f\|_{A_1 + A_2} = K(1, f; A_1, A_2).$$

Denote by $Int(L^1, L^\infty)$ the class of all quasi-normed interpolation spaces E for the couple (L^1, L^∞). This means that E is an intermediate space for the couple (L^1, L^∞) and if T is a bounded linear operator in both L^1 and L^∞, then T is also bounded in E. Note that if E is an intermediate space, ρ_E is $K-$monotone and $f_1^{**} \leq f_2^{**}$, $f_2 \in E$ implies $f_1 \in E$, then $E \in Int(L^1, L^\infty)$.

For example, consider the Gamma spaces $\Gamma^q(w)$, $0 < q \leq \infty$, w - positive weight, i.e. a positive function from L, with a quasi-norm $\|f\|_{\Gamma^q(w)} := \rho_{w,q}(f^*)$, where

$$\rho_{w,q}(g) := \left(\int_0^\infty [g^{**}(t)w(t)]^q dt/t\right)^{1/q}, \ g \in L, \ 0 < q < \infty;$$

$$\rho_{w,\infty}(g) := vraisup_{t>0} g^{**}(t)w(t)$$

and

$$\int_0^\infty \min(1, t^{-q})w^q(t)dt/t < \infty, \ 0 < q < \infty;$$

$$vraisup_{t>0} \min(1, t^{-1})w(t) < \infty, \ q = \infty.$$

Then $\Gamma^q(w) = (L^1, L^\infty)_{w(t)/t,q}$. The space $E = \Gamma^q(w)$ with $\rho_E = \rho_{w,q}$ satisfies the conditions (1), (2).

The Lorentz spaces $\Lambda^q(w)$, $0 < q \leq \infty$, w - positive weight, $w(2t) \approx w(t)$, are defined with a quasi-norm

$$\|f\|_{\Lambda^q(w)} := \left(\int_0^\infty [w(t)f^*(t)]^q dt/t\right)^{1/q}, \ 0 < q < \infty$$

and

$$\|f\|_{\Lambda^\infty(w)} := vraisup_{t>0} w(t)f^*(t).$$

We suppose that they are not trivial.

Recall the definition of the lower and upper Boyd indices α_E and β_E. Let

$$h_E(u) = \sup\left\{\frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in L\right\}, \ g_u(t) := g(t/u)$$

be the dilation function generated by ρ_E. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \text{ and } \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

The function h_E is submultiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(1/u) \geq 1$, hence $0 \leq \alpha_E \leq \beta_E$ and if $E \in Int(L^1, L^\infty)$ then by interpolation, (analogously to [4], p. 148) we see that $h_E(s) \leq \max(1, s)$. Hence $\beta_E \leq 1$.

Using Minkowski's inequality for the equivalent quasi-norm ρ_p and monotonicity of f^*, we see that

$$\rho_E(f^*) \approx \rho_E(f^{**}) \text{ if } \beta_E < 1. \tag{3}$$

In particular, $\Lambda^q(w) = \Gamma^q(w)$ if $\beta_E < 1$ for $E = \Gamma^q(w)$.

We need also the modified Boyd indices $\tilde{\alpha}_E$ and $\tilde{\beta}_E$, defined as follows. Let

$$\tilde{h}_E(u) = \sup\left\{\frac{\rho_E(((\chi g)^*_u)}{\rho_E((\chi g)^*)} : g \in L\right\}, \ (\chi g)_u(t) := (\chi g)(t/u)$$

be the modified dilation function, generated by ρ_E. Here χ is the characteristic function of the interval $(0,1)$. Then

$$\tilde{\alpha}_E := \sup_{0<t<1} \frac{\log \tilde{h}_E(t)}{\log t} \text{ and } \tilde{\beta}_E := \inf_{1<t<\infty} \frac{\log \tilde{h}_E(t)}{\log t}.$$

Since $\tilde{h}_E \leq h_E$, it follows $0 \leq \alpha_E \leq \tilde{\alpha}_E \leq \tilde{\beta}_E \leq \beta_E$. For example, if $E = L^1 + L^\infty$, then $\alpha_E = 0$, $\beta_E = 1$, $\tilde{\alpha}_E = \tilde{\beta}_E = 1$.

The Boyd indices are useful in various problems concerning continuity of operators acting in rearrangement invariant spaces [4] or in optimal couples of rearrangement invariant spaces [7], [2], [8], and in the problems of optimal embeddings [1], [3], [10]. The main goal of this paper is to provide formulas for the Boyd indices for intersection or sum of two quasi-normed spaces and to apply these results to the case of Lorentz type spaces.

2. Boyd Indices for the Sum of Two Quasi-normed Spaces

First we characterize the sum $E_1 + E_2$ via the quasi-norm $\rho_{E_1+E_2}$.

Theorem 1. *Let E_1 and E_2 be intermediate spaces for the couple (L^1, L^∞) and let ρ_{E_1}, ρ_{E_2} be $K-$monotone. Then*

$$\|f\|_{E_1+E_2} \approx \rho_{E_1+E_2}(f^*), \tag{4}$$

where for $g \in L$,

$$\rho_{E_1+E_2}(g) := \inf\{\rho_{E_1}(g_1^*) + \rho_{E_2}(g_2^*) : \ g = g_1 + g_2, \ g_1, g_2 \in L\}, \tag{5}$$

where g^* for $g \in L$ is taken with respect to the Lebesgue measure on $(0,\infty)$.

Proof. If $f = f_1 + f_2$, $f \in L_{loc}$, then $f^*(t) \leq f_1^*(t/2) + f_2^*(t/2)$, whence

$$\rho_{E_1+E_2}(f^*) \preceq \rho_{E_1}(f_1^*) + \rho_{E_2}(f_2^*),$$

therefore the right-hand side in (4) is majorized by the left-hand side. For the reverse, suppose that $f \in L_{loc}$ and $f^* = g_1 + g_2$, $g_1, g_2 \in L$. Then by the Hardy-Littlwood inequality,

$$f^{**}(t) \leq \frac{1}{t}\int_0^t g_1^*(u)du + \frac{1}{t}\int_0^t g_2^*(u)du,$$

hence by the divisibility theorem (see [6]), there exist $f_1, f_2 \in L_{loc}$ such that $f = f_1 + f_2$ and

$$f_j^{**}(t) \preceq \frac{1}{t}\int_0^t g_j^*(u)du, \ j = 1,2.$$

Using $K-$monotonicity of ρ_{E_1} and ρ_{E_2}, we get $\rho_{E_j}(f_j^*) \preceq \rho_{E_j}(g_j^*)$, $j = 1,2$. Hence

$$\|f\|_{E_1+E_2} \preceq \rho_{E_1}(g_1^*) + \rho_{E_2}(g_2^*).$$

Taking the infimum, we obtain $\|f\|_{E_1+E_2} \preceq \rho_{E_1+E_2}(f^*)$.

□

Now we calculate the Boyd indices of the sum of two quasi-normed spaces.

Theorem 2. *Let E_1 and E_2 be intermediate spaces for the couple (L^1, L^∞) and let ρ_{E_1}, ρ_{E_2} be $K-$monotone. Then*

$$\alpha_{E_1+E_2} \geq \min(\alpha_{E_1}, \alpha_{E_2}),\ \beta_{E_1+E_2} \leq \max(\beta_{E_1}, \beta_{E_2}). \tag{6}$$

Proof. Since

$$\rho_{E_1+E_2}(g) := \inf\{\rho_{E_1}(g_1^*) + \rho_{E_2}(g_2^*) :\ g = g_1 + g_2,\ g_1, g_2 \in L\},\ g \in L$$

and $g_u^*(t) \leq g_{1u}^*(t/2) + g_u^*(t/2)$, it follows

$$\rho_{E_1+E_2}(g_u^*) \leq h_{E_1}(2u)\rho_{E_1}(g_1^*) + h_{E_2}(2u)\rho_{E_2}(g_2^*).$$

Therefore,

$$h_{E_1+E_2}(u) \preceq h_{E_1}(u) + h_{E_2}(u),\ u > 0. \tag{7}$$

Then for $u > 1$ and any $\varepsilon > 0$,

$$h_{E_1+E_2}(u) \preceq u^{\beta_{E_1}+\varepsilon} + u^{\beta_{E_2}+\varepsilon} \preceq u^{\max(\beta_{E_1},\beta_{E_2})+\varepsilon},$$

whence the second inequality in (6) follows. The proof of the first inequality is analogous.

□

Theorem 3. *Let ρ_{E_1}, ρ_{E_2} satisfy*

$$\rho_{E_1}(\chi_{(0,1)}g^*) \preceq \rho_{E_2}(g^*),\ \rho_{E_2}(\chi_{(1,\infty)}g^*) \preceq \rho_{E_1}(g^*),\ g \in L. \tag{8}$$

Then

$$\|f\|_{E_1+E_2} \approx \rho_{E_1}(\chi_{(0,1)}f^*) + \rho_{E_2}(\chi_{(1,\infty)}f^*). \tag{9}$$

Moreover, the left-hand side in (9) is always dominated by the right-hand side, even without the condition (8).

Proof. If $f = f_1 + f_2$, then $f^*(t) \leq f_1^*(t/2) + f_2^*(t/2)$ and

$$\rho_{E_1}(\chi_{(0,1)}f^*) \preceq \rho_{E_1}(\chi_{(0,1)}f_1^*) + \rho_{E_1}(\chi_{(0,1)}f_2^*),$$

whence by (8)

$$\rho_{E_1}(\chi_{(0,1)}f^*) \preceq \rho_{E_1}(f_1^*) + \rho_{E_2}(f_2^*),$$

and taking the infimum, we get

$$\rho_{E_1}(\chi_{(0,1)}f^*) \preceq \|f\|_{E_1+E_2}.$$

We have

$$\rho_{E_2}(\chi_{(1,\infty)}f^*) \preceq \rho_{E_2}(\chi_{(1/2,\infty)}f_1^*) + \rho_{E_2}(\chi_{(1/2,\infty)}f_2^*),$$

whence by (8)

$$\rho_{E_2}(\chi_{(1,\infty)}f^*) \preceq \rho_{E_2}(\chi_{(1/2,1)})f_1^*(1/2) + \rho_{E_1}(f_1^*) + \rho_{E_2}(f_2^*),$$

hence, using also $\rho_{E_1}(f_1^*) \geq \rho_{E_1}(\chi_{(0,1/2)} f_1^*) \succeq f_1^*(1/2)$, we obtain

$$\rho_{E_2}(\chi_{(1,\infty)} f^*) \preceq \|f\|_{E_1+E_2}.$$

Thus one inequality in the equivalence (9) is proved. For the reverse, let $f \in L^1 + L^\infty$. Define $f_1(x) = sign f(x)|f(x)|\chi_e(x)$, $e = \{x : |f(x)| > f^*(1)\}$ and $f_2 = f - f_1$. Then

$$f_1^*(u) \leq \chi_{(0,1)}(u) f^*(u),\ f_2^*(u) \leq \min(f^*(u), f^*(1)).$$

Therefore, $\rho_{E_1}(f_1^*) \leq \rho_{E_1}(\chi_{(0,1)} f^*)$ and $\rho_{E_2}(f_2^*) \leq \rho_{E_2}(\chi_{(0,1)}) f^*(1) + \rho_{E_2}(\chi_{(1,\infty)} f^*)$. Since $f^*(1) \preceq \rho_{E_1}(\chi_{(0,1)} f^*)$, it follows

$$\|f_1\|_{E_1} + \|f_2\|_{E_2} \preceq \rho_{E_1}(\chi_{(0,1)} f^*) + \rho_{E_2}(\chi_{(1,\infty)} f^*).$$

Thus the second inequality in (9) is proved without the condition (8).

□

Theorem 4. *Let E_1 and E_2 be intermediate spaces for the couple (L^1, L^∞) and let ρ_{E_1}, ρ_{E_2} be $K-$monotone, satisfying (8). If $\alpha_{E_1} \geq \alpha_{E_2}$, $\beta_{E_1} \geq \beta_{E_2}$ and*

$$\rho_{E_1}(\chi_{(0,1)}(t) t^{\varepsilon - \beta_{E_1}}) < \infty,\ \ \rho_{E_2}(\chi_{(1,\infty)}(t) t^{-\varepsilon - \alpha_{E_2}}) < \infty, \tag{10}$$

for some small $\varepsilon \in (0, \beta_{E_1})$, then

$$\alpha_{E_1+E_2} = \alpha_{E_2},\ \beta_{E_1+E_2} = \beta_{E_1}. \tag{11}$$

Proof. We have for $g(t) = \chi_{(1,\infty)}(t) t^{-\varepsilon - \alpha_{E_2}}$,

$$h_{E_1+E_2}(u) \succeq \rho_{E_2}(\chi_{(1,\infty)}(t) g_u^*(t)) \succeq u^{\alpha_{E_2}+\varepsilon},$$

whence $\alpha_{E_1+E_2} \leq \alpha_{E_2}$. Analogously $\beta_{E_1+E_2} \geq \beta_{E_1}$. It remains to use (6).

□

Theorem 5. *Let E_1 and E_2 be intermediate spaces for the couple (L^1, L^∞) and let ρ_{E_1}, ρ_{E_2} be $K-$monotone, satisfying (8). Then*

$$\tilde{\alpha}_{E_1+E_2} = \tilde{\alpha}_{E_1},\ \tilde{\beta}_{E_1+E_2} = \tilde{\beta}_{E_1}. \tag{12}$$

Proof. We have $\rho_{E_1+E_2}(f^*) \approx \rho_{E_1}(\chi_{(0,1)} f^*) + \rho_{E_2}(\chi_{(1,\infty)} f^*)$, whence

$$\rho_{E_1+E_2}(g_u^*) \approx \rho_{E_1}(\chi_{(0,1)} g_u^*) + \rho_{E_2}(\chi_{(1,\infty)} g_u^*),\ g \in L.$$

Since $(\chi_{(0,1)} g)^* \leq \chi_{(0,1)} g^*$, $g \in L$, we have

$$\rho_{E_1+E_2}((\chi_{(0,1)} g)_u^*) \approx \rho_{E_1}(\chi_{(0,1)}(\chi_{(0,1)} g)_u^*) \approx \rho_{E_1}((\chi_{(0,1)} g)_u^*),$$

whence $\tilde{h}_{E_1+E_2} \approx \tilde{h}_{E_1}$. Therefore $\tilde{\alpha}_{E_1+E_2} = \tilde{\alpha}_{E_1}$, $\tilde{\beta}_{E_1+E_2} = \tilde{\beta}_{E_1}$.

□

Recall that the positive weight w is slowly varying on $(1,\infty)$ (in the sense of Karamata [11]), if for all $\varepsilon > 0$ the function $t^{\varepsilon}w(t)$ is equivalent to a non-decreasing function, and the function $t^{-\varepsilon}w(t)$ is equivalent to a non-increasing function. By symmetry, we say that w is slowly varying on $(0,1)$ if the function $t \mapsto w(1/t)$ is slowly varying on $(1,\infty)$. Finally, w is slowly varying if it is slowly varying on $(0,1)$ and $(1,\infty)$.

Now we give examples.

Example 1. *If* $E = \Lambda^q(t^a w)$ *or* $E = \Gamma^q(t^a w)$, $0 \le a \le 1$, $0 < q \le \infty$, *where* w *is slowly varying, then* $\alpha_E = \beta_E = \tilde{\alpha}_E = \tilde{\beta}_E = a$.

Proof. We give a proof for $E = \Lambda^q(t^a w)$ and $0 < q < \infty$, the other cases being analogous. We have

$$\rho_E(g_u^*) = \left(\int_0^\infty [g^*(t/u)t^a w(t)]^q dt/t\right)^{1/q}$$

and by a change of variables,

$$\rho_E(g_u^*) = \left(\int_0^\infty [g^*(t)(tu)^a w(tu)]^q dt/t\right)^{1/q}. \tag{13}$$

It follows from the definition of a slowly varying function that for every $\varepsilon > 0$, we have $t^{-\varepsilon}w(t) \approx d(t)$, where d is a decreasing function. If $u > 1$, then $d(tu) \le d(t)$, thus

$$\begin{aligned} t^{-\varepsilon}w(t) &\succeq d(tu) \\ &\approx u^{-\varepsilon}t^{-\varepsilon}w(tu), \end{aligned}$$

which gives

$$w(tu) \preceq u^{\varepsilon}w(t). \tag{14}$$

Inserting this estimate in (13), we arrive at

$$\rho_E(g_u^*) \preceq u^{a+\varepsilon}\rho_E(g^*),\ u > 1,$$

which yields $h_E(u) \preceq u^{a+\varepsilon},\ u > 1$. Then it follows $\beta_E \le a + \varepsilon$. Analogously, $\alpha_E \ge a - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $\alpha_E = \beta_E = a$.

Further,

$$\rho_E((\chi g)_u^*) = \left(\int_0^\infty [(\chi g)^*(t/u)t^a w(t)]^q dt/t\right)^{1/q}$$

and by a change of variables,

$$\rho_E((\chi g)_u^*) = \left(\int_0^\infty [(\chi g)^*(t)(tu)^a w(tu)]^q dt/t\right)^{1/q}, \tag{15}$$

or

$$\rho_E((\chi g)_u^*) \preceq u^{a+\varepsilon}\rho_E((\chi g)^*),\ u > 1,$$

which yields $\tilde{h}_E(u) \preceq u^{a+\varepsilon}$, $u > 1$. Then it follows $\tilde{\beta}_E \leq a+\varepsilon$ and analogously $\tilde{\alpha}_E \geq a-\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain $\tilde{\alpha}_E = \tilde{\beta}_E = a$.

□

Example 2. *If $E = E_1 + E_2$, $E_j = \Gamma^{q_j}(t^{a_j} w_j)$, $1 \geq a_1 > a_2 \geq 0$, $0 < q_j \leq \infty$, $j = 1, 2$, where w_1 and w_2 are slowly varying, then applying Theorem 4 and Theorem 5, and the results of the previous example, we obtain $\alpha_E = a_2$, $\beta_E = \tilde{\beta}_E = \tilde{\alpha}_E = a_1$.*

3. Boyd Indices for Intersection of Two Quasi-normed Spaces

Theorem 6. *Let $E_j \in Int(L^1, L^\infty)$, $j = 1, 2$. Then*

$$\alpha_{E_1 \cap E_2} \geq \min(\alpha_{E_1}, \alpha_{E_2}),\ \beta_{E_1 \cap E_2} \leq \max(\beta_{E_1}, \beta_{E_2}). \tag{16}$$

Also,

$$h_{E_1 \cap E_2}(u) \succeq \rho_{E_1}(\chi_{(0,u)}) + \rho_{E_2}(\chi_{(0,u)}). \tag{17}$$

Proof. We have $\|f\|_{E_1 \cap E_2} = \rho_{E_1 \cap E_2}(f^*)$, where by definition,

$$\rho_{E_1 \cap E_2}(g) = \rho_{E_1}(g) + \rho_{E_2}(g),\ g \in L.$$

Since $\rho_{E_1 \cap E_2}(g_u^*) \leq h_{E_1}(u)\rho_{E_1}(g^*) + h_{E_2}(u)\rho_{E_2}(g^*)$, it follows

$$h_{E_1 \cap E_2}(u) \leq h_{E_1}(u) + h_{E_2}(u),\ u > 0. \tag{18}$$

Then for $u > 1$ and any $\varepsilon > 0$,

$$h_{E_1 \cap E_2}(u) \preceq u^{\beta_{E_1}+\varepsilon} + u^{\beta_{E_2}+\varepsilon} \preceq u^{\max(\beta_{E_1}, \beta_{E_2})+\varepsilon},$$

whence the second inequality in (16) follows. The proof of the first inequality is analogous.

For (17) we use the test function $g = \chi_{(0,1)}$. Then $h_{E_1 \cap E_2}(u) \succeq \rho_{E_1}(g_u^*) + \rho_{E_2}(g_u^*)$ and (17) follows.

□

Theorem 7. *Let $E_j \in Int(L^1, L^\infty)$, $j = 1, 2$ satisfy*

$$\rho_{E_1}(\chi_{(0,1)} g^*) \preceq \rho_{E_2}(\chi_{(0,1)} g^*),\ \rho_{E_2}(\chi_{(1,\infty)} g^*) \preceq \rho_{E_1}(\chi_{(1/2,\infty)} g^*). \tag{19}$$

Then

$$\|f\|_{E_1 \cap E_2} \approx \rho_{E_1}(\chi_{(1,\infty)} f^*) + \rho_{E_2}(\chi_{(0,1)} f^*). \tag{20}$$

The proof follows immediately from the definitions.

Theorem 8. *If $E_j \in Int(L^1, L^\infty)$, $j = 1, 2$ satisfy (19), then*

$$\tilde{\alpha}_{E_1 \cap E_2} = \tilde{\alpha}_{E_2},\ \tilde{\beta}_{E_1 \cap E_2} = \tilde{\beta}_{E_2}. \tag{21}$$

Proof. Since the condition (19) is satisfied, it follows

$$\rho_{E_1\cap E_2}(g^*) \approx \rho_{E_2}(\chi_{(0,1)}g^*) + \rho_{E_1}(\chi_{(1,\infty)}g^*)$$

and since $(\chi_{(0,1)}g)^* \leq \chi_{(0,1)}g^*$, $g \in L$, we have

$$\rho_{E_1\cap E_2}((\chi_{(0,1)}g)^*_u) \approx \rho_{E_2}(\chi_{(0,1)}(\chi_{(0,1)}g)^*_u) \approx \rho_{E_2}((\chi_{(0,1)}g)^*_u),$$

whence $\tilde{h}_{E_1\cap E_2} \approx \tilde{h}_{E_2}$. Therefore $\tilde{\alpha}_{E_1\cap E_2} = \tilde{\alpha}_{E_2}$, $\tilde{\beta}_{E_1\cap E_2} = \tilde{\beta}_{E_2}$.

□

Example 3. *If* $E = E_1 \cap E_2$, $E_j = \Gamma^{q_j}(t^{a_j}w_j)$, $1 \geq a_1 > a_2 \geq 0$, $0 < q_j \leq \infty$, $j = 1, 2$, *where* w_1 *and* w_2 *are slowly varying, then applying Theorem 6 and Theorem 8 we obtain* $\beta_E = a_1$, $\alpha_E = \tilde{\beta}_E = \tilde{\alpha}_E = a_2$.

References

[1] Ahmed, I., Karadzhov, G. E., *Optimal embeddings of generalized homogeneous Sobolev spaces*, Colloq. Math., 123(2011), 1-20.

[2] Ahmed, I., Karadzhov, G. E., Nazeer, W., *Optimal couples of rearrangement invariant spaces for the fractional maximal operator*, C.r. de l'Acad. bulgare des Sci., 64(2011),

[3] Bashir, Z., Karadzhov, G. E., *Optimal embeddings of generalized Besov spaces*, Eurasian Math. J., 2(2011), 5-31.

[4] Bennett, C., Sharpley, R., *Interpolation of operators*, Academic Press, New York, 1988.

[5] Berg, J., Löfström, J., *Interpolation spaces. An introduction,* Springer-Verlag, New York, 1976.

[6] Brudnyi, Ju. A., Krugliak, N. Ja., *Interpolation spaces and interpolation functors*, North-Holland, Amsterdam, 1991.

[7] Capone, C., Fiorenza, A., Karadhov, G. E., Nazeer, W., *Optimal couples of rearrangement invariant spaces for the Riesz potential operator*, Z. Anal. Anwendungen, 30(2011), 219-236.

[8] Karadzhov, G. E., Nazeer, W., *Optimal couples of rearrangement invariant spaces for the Bessel potential*, C.r. de l'Acad. bulgare des Sci., 64(2011), 767-774.

[9] Köthe, G., *Topol. lineare Räume*, Springer, Berlin, 1966.

[10] Vybiral, J., *Optimality of function spaces for boundedness of integral operators and Sobolev embeddings*, Faculty of Math. and Physics, Charles University, Prague, 2002.

[11] A. Zygmund, *Trigonometric Series*, second edition, Cambridge, 1958.