Journal of Prime Research in Mathematics Vol. **8** *(2012), 45-50*

ON THE POWER MEAN INEQUALITY OF THE HYPERBOLIC METRIC OF UNIT BALL

BARKAT ALI BHAYO

ABSTRACT. The hyperbolic distances from the origin are changed under the radial selfmapping $x \mapsto |x|^{1/K-1}x$, $K > 1$ of the unit ball. Here author gives the power mean inequality of the hyperbolic metric under the radial mapping.

Key words: Power Mean, inequalities, hyperbolic metric, distortion function.
AMS subject: 30F45.

1. INTRODUCTION

For the statement of the main results we introduce some notations and terminologies.

For $p \in \mathbb{R}$, the Power Mean M_p of order p of two positive numbers x and y is define by

$$M_p(x,y) = \begin{cases} \left(\dfrac{x^p + y^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{x\,y} & p = 0\,. \end{cases}$$

For $a, b > 0$ and $x \in \mathbb{R}^n$, we define

$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & if\ |x| \leq 1 \\ |x|^{b-1}x & if\ |x| \geq 1, \end{cases}$$

see [4, (1.5)]. For brevity we write $\mathcal{A} = \mathcal{A}_{1/K,K}$.

Department of Mathematics & Statistics, Quaid-e-Awam University of Engineering Science & Technology, Nawabshah, Pakistan. Email: *bhayo.barkat@gmail.com*.

The hyperbolic metric $\rho(x,y)$ of the unit ball is given by

$$\begin{aligned}\rho(x,y) &= 2\,\mathrm{artanh}\left(\frac{|x-y|}{\sqrt{|x-y|^2+(1-|x|^2)(1-|y|^2)}}\right)\\ &= 2\,\mathrm{arsinh}\left(\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\right),\end{aligned}$$

for all $x,y\in\mathbb{B}^n$, $n\geq 2$ and $n\in\mathbb{Z}$ (see [6, 8]).

A decreasing homeomorphism $\mu:(0,1)\to(0,\infty)$ is defined by

$$\mu(r)=\frac{\pi}{2}\,\frac{\mathcal{K}(r')}{\mathcal{K}(r)},\quad \mathcal{K}(r)=\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}},\quad 0<r<1\,,$$

where $\mathcal{K}(r)$ is Legendre's complete elliptic integral of the first kind and $r'=\sqrt{1-r^2}$.

The Hersch-Pfluger distortion function is an increasing homeomorphism $\varphi_K:(0,1)\to(0,1)$ defined by setting

$$\varphi_K(r)=\mu^{-1}(\mu(r)/K)\,,\; r\in(0,1),\;\; K>0.$$

The main results of the paper are:

Theorem 1. *For $K\geq 1$, $p\in[-2,0]$ and $x,y\in\mathbb{B}^n$, we have*

$$M_p(\rho(0,\mathcal{A}(|x|)),\rho(0,\mathcal{A}(|y|)))\geq\rho(0,\mathcal{A}(M_p(|x|,|y|))),$$

equality holds iff $x=y$.

Theorem 2. *For $p\geq 1$, $K>1$ and $x,y\in(0,\infty)$, the following relation holds*

$$g_K(M_p(x,y))\geq M_p(g_K(x),g_K(y)),$$

where $g_K(r)=\mathrm{artanh}(\varphi_K(\tanh(r)))$, equality holds iff $x=y$.

2. Proofs

Let $f:I\to(0,\infty)$ be continuous, where I is a subinterval of $(0,\infty)$. Let M and N be any two mean values. We say that f is MN-convex (concave) if

$$f(M(x,y))\leq(\geq)N(f(x),f(y))\quad\text{for}\quad \text{all } x,y\in I\,.$$

Lemma 3. [2, Theorem 2.4(5)] *Let $I=(0,b)$, $0<b<\infty$, and let $f:I\to(0,\infty)$ be continuous. Then f is GG-convex (concave) on I if and only if $\log(f(be^{-t}))$ is convex (concave) on $(0,\infty)$, where G is the Geometric Mean.*

Lemma 4. (1) *For $m\in(0,2)$, the function*

$$h_1(y)=1-\frac{1}{3}\log(1-y)-\frac{1+m}{1+y+m(1-y)}$$

is increasing from $(0,1)$ onto $(0,\infty)$,

(2) *for* $K \geq 1$ *the function*

$$h_2(x) = \left(\frac{f_K(x)}{x}\right)^{-(1+m)} \frac{x^{1/K-1}}{K(1-x^{2/K})}$$

in increasing in $x \in (0,1)$, *where* $f_K(x) = \log((1+x^{1/K})/(1-x^{1/K}))$.

(3) *the function* $h_3(t) = \log(f_K(e^{-t}))$ *is convex in* $(0,\infty)$.

Proof. Differentiating w.r.t y we get

$$h_1'(y) = \frac{(1+m+y-my)^2 + 3(1-m^2)(1-y)}{3(1+m+y-my)^2(1-y)} > 0\,.$$

For (2), we get

$$\begin{aligned} h_2'(x) &= \xi[(1+x^{2/K} + Km(1-x^{2/K}))f_K(x) - 2(1+m)x^{1/K}] \\ &= 2\xi[(1+x^{2/K} + Km(1-x^{2/K}))\mathrm{artanh}(x^{1/K}) - (1+m)x^{1/K}] \\ &> 2x^{1/K}\xi[(1+x^{2/K} + Km(1-x^{2/K}))(1 - \frac{1}{3}\log(1-x^{2/K})) - (1+m)] \\ &> 2x^{1/K}\xi[(1+x^{2/K} + m(1-x^{2/K}))(1 - \frac{1}{3}\log(1-x^{2/K})) - (1+m)], \end{aligned}$$

by using $\mathrm{artanh}(x) > 1 - \frac{1}{3}\log(1-x^{1/K})$ (see [5, Thm 1.2(2)]). Clearly $h_2'(x)$ is positive by part (1), where

$$\xi = \frac{x^{1/K+(m-1)} f_K(x)^{1-m}}{K(1-x^{2/K}) f_K(x)^3}\,.$$

Finally we get,

$$h_3''(t) = \frac{2\,e^{-t/K}(1+e^{-2t/K})}{K^2(e^{-2t/K}-1)^2} > 0,$$

this completes the proof. ☐

Lemma 5. *For* $K \geq 1$, $p \in [-2,0]$, *and* $r, s \in (0,1)$, *we have*

$$M_p(f_K(r), f_K(s)) \geq f_K(M_p(r,s))\,,$$

where f_K *is same as in Lemma 4, equality holds iff* $r = s$.

Proof. The case $p = 0$ follows from Lemmas 4(3) and 3. For the case $p \in [-2,0)$, let $0 < x < y < 1$, and $u = \left(\frac{x^p+y^p}{2}\right)^{1/p} < x$. We define

$$g(x) = f_K(u)^p - \frac{f_K(x) + f_K(y)^p}{2}\,.$$

Differentiating w.r.t x we get $du/dx = (1/2)(x/u)^{p-1}$, and

$$\begin{aligned} g'(x) &= \frac{1}{2}p\, f_K(u)^{p-1}\frac{d}{dx}\left(f_K(u)\right)\left(\frac{x}{u}\right) - \frac{1}{2}p\, f_K(x)^{p-1}\frac{d}{dx}\left(f_K(x)\right) \\ &= p\, x^{p-1}\left[\left(\frac{f_K(u)}{u}\right)^{p-1}\frac{u^{1/K-1}}{K(1-u^{2/K})} - \left(\frac{f_K(x)}{x}\right)^{p-1}\frac{x^{1/K-1}}{K(1-x^{2/K})}\right]. \end{aligned}$$

which is positive by Lemma 4(2), hence g is increasing. This implies that $g(x) < g(y) = 0$, and this completes the proof. □

1. Proof of Theorem 1.

The proof follows from the formula $\rho(0,r) = \log((1+r)/(1-r))$ and Lemma 5. □

The proof of the following lemma follows from the definition of $\rho(x,y)$ and Theorem 1.

Corollary 6. *The following inequalities hold for $p \in [-2,0]$ and $r,s \in (0,1)$,*

$$\begin{aligned} &arsinh\left(\frac{r}{\sqrt{1-r^2}}\right)^p + arsinh\left(\frac{s}{\sqrt{1-s^2}}\right)^p \\ &\geq 2\, arsinh\left(\frac{(r^p+s^p)^{1/p}}{\sqrt{2^{2/p}-(r^p+s^p)^{2/p}}}\right)^p, \\ &artanh\left(\frac{r}{\sqrt{1-r^2}}\right)^p + artanh\left(\frac{s}{\sqrt{1-s^2}}\right)^p \\ &\geq 2\, artanh\left(\frac{(r^p+s^p)^{1/p}}{2}\right)^p, \end{aligned}$$

in both equality holds with $r = s$.

Corollary 7. *For $K \geq 1$, $p < 0$, we have*

$$\begin{aligned} M_p(\mathcal{A}(|x|),\mathcal{A}(|y|)) &\geq \mathcal{A}(M_p(|x|,|y|)), \quad x,y \in \mathbb{B}^n, \\ M_p(\mathcal{A}(|x|),\mathcal{A}(|y|)) &\leq \mathcal{A}(M_p(|x|,|y|)), \quad x,y \in \mathbb{R}^n \setminus \mathbb{B}^n. \end{aligned}$$

Both inequalities reverse for $p > 0$, and equality holds iff $x = y$.

Proof. Let $0 < r < s < 1$ or $1 < r < s$, and $u = \left(\frac{r^p+s^p}{2}\right)^{1/p} < r$. We define

$$g_2(r) = \mathcal{A}(u)^p - \frac{\mathcal{A}(r)^p + \mathcal{A}(s)^p}{2}.$$

By differentiating with respect to r we get $du/dr = (r/u)^{p-1}$, and

$$\begin{aligned} g_2'(r) &= \frac{1}{2}p\,\mathcal{A}(u)^{p-1}\frac{d}{dr}(\mathcal{A}(u))\left(\frac{r}{u}\right)^{p-1} - \frac{1}{2}p\,\mathcal{A}(r)^{p-1}\frac{d}{dr}(\mathcal{A}(r)) \\ &= \frac{p}{2}r^{p-1}\left(g_3(u) - g_3(r)\right), \end{aligned}$$

where

$$g_3(z) = \left(\frac{\mathcal{A}(z)}{z}\right)^{p-1} \frac{d}{dr}(\mathcal{A}(z))\,.$$

Case 1. When $z \in (0,1)$, then $g_3(z) = (1/K)(z^{1/K-1})^p$, which is increasing (decreasing) for $p < 0\,(p > 0)$, respectively. This implies that $g_2(r) < (>)g_2(s) = 0$, and the first inequality is obvious. Case 2. When $z > 1$, then $g_3(z) = K(z^{K-1})^p$, which is decreasing (increasing) for $p > 0\,(p < 0)$, respectively. This implies that $g_2(r) > (<)g_2(s) = 0$, and second inequality follows. This completes the proof. □

Lemma 8. [1, Theorem 10.12] *For $K > 1$, the function $g_K(r) =$ artanh$(\varphi_K(\tanh(x)))$ is strictly increasing and concave from $(0,\infty)$ onto $(0,\infty)$.*

2. Proof of Theorem 2.

Let $0 < x < y < 1$ or $1 < x < y$, and $w = \left(\frac{x^p+y^p}{2}\right)^{1/p} < x$. We define

$$g_4(x) = g_K(w)^p - \frac{g_K(x)^p + g_K(y)^p}{2}\,.$$

By differentiating w.r.t x we get $dw/dx = (x/w)^{p-1}$, and

$$\begin{aligned} g_4'(x) &= \frac{1}{2}p\,g_K(w)^{p-1}\frac{d}{dx}(g_K(w))\left(\frac{x}{w}\right)^{p-1} - \frac{1}{2}p\,g_K(x)^{p-1}\frac{d}{dx}(g_K(x)) \\ &= \frac{p}{2}x^{p-1}\left(g_5(w) - g_5(x)\right), \end{aligned}$$

where

$$g_5(z) = \left(\frac{g_K(z)}{z}\right)^{p-1} \frac{d}{dx}(g_K(z))\,.$$

The function g_5 is decreasing by Lemma 8 and [1, Theorem 1.25]. This implies that $g_4(x) \geq g_4(y) = 0$. This completes the proof. □

Acknowledgments. Author wishes to acknowledge the referee for his/her valuable comments.

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