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CONTINUITY ESTIMATE OF THE OPTIMAL EXERCISE BOUNDARY WITH RESPECT TO VOLATILITY FOR THE AMERICAN FOREIGN EXCHANGE PUT OPTION

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Abstract. In this paper we consider the Garman-Kohlhagen model for the American foreign exchange put option in one-dimensional diffusion model where the volatility and the domestic and foreign currency risk-free interest rates are constants. First we make preliminary estimate regarding the optimal exercise boundary of the American foreign exchange put option and then the continuity estimate with respect to volatility for the value functions of the corresponding options. Finally we establish the continuity estimate for the optimal exercise boundary of the American foreign exchange put option with respect to the volatility parameter.

Key words: foreign exchange option, optimal exercise boundary, value function, volatility.
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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t), 0 \leq t \leq T$, a one-dimensional standard Brownian motion on it. We denote by $(\mathcal{F}_t)_{0 \leq t \leq T}$ the P-completion of the natural filtration of $(W_t), 0 \leq t \leq T$. Throughout the paper we shall assume that the time horizon T is finite.
On the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P), 0 \leq t \leq T$, we consider a financial market with two currencies domestic and foreign with their corresponding non-negative constant interest rates r^d and r^f satisfying the following:

$$0 < r^d \leq \overline{r}, \quad 0 \leq r^f \leq \overline{r}. \tag{1}$$

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Consider the volatilities σ_0, σ_1 and σ_2 which satisfy:

$$0 < \sigma_0 \le \sigma_1 \le \sigma_2 \le \overline{\sigma}. \tag{2}$$

We consider American put option problem written on the foreign exchange rate $(Q_t^{(i)}, \mathcal{F}_t), 0 \le t \le T, i = 0, 1, 2$ (where $Q_t^{(i)}$ gives the units of domestic currency per unit of foreign currency at time t) with the payoff function

$$g(x) = (K - x)^+, \qquad x \ge 0, \tag{3}$$

where the exchange rate processes $Q_t^{(i)}$ satisfy the following stochastic differential equation

$$dQ_t^{(i)} = Q_t^{(i)} \cdot (r^d - r^f) dt + Q_t^{(i)} \cdot \sigma_i \cdot dW_t, \quad Q_0^{(i)} = Q_0 > 0, i = 0, 1, 2, \ 0 \le t \le T. \tag{4}$$

Denote the American put value functions by $v_0(t, x), v_1(t, x)$ and $v_2(t, x)$, $0 \le t \le T, \ x \ge 0$, corresponding to volatilities $\sigma = \sigma_0, \sigma = \sigma_1$ and $\sigma = \sigma_2$.
It is well-known (section 2.7 of Karatzas and Shreve [6] and section 8.4 of Shreve [9]) that American option valuation problem is related to the corresponding optimal stopping problem of a diffusion process in the following manner

$$v_i(t, x) = \sup_{t \le \tau \le T} E \left(e^{-r^d(\tau - t)} \left(K - Q_\tau^{(i)}(t, x) \right)^+ \right), \ \ 0 \le t \le T, x \ge 0, i = 0, 1, 2, \tag{5}$$

where the supremum is taken over all $(\mathcal{F}_u)_{0 \le u \le T}-$ stopping times τ such that $t \le \tau \le T$ and the family of stochastic processes $Q_u^{(i)}(t, x), t \le u \le T, x \ge 0$ satisfies the same stochastic differential equation

$$dQ_u^{(i)}(t, x) = Q_u^{(i)}(t, x)(r^d - r^f) du + Q_u^{(i)}(t, x) \cdot \sigma_i \cdot dW_u, Q_t^{(i)}(t, x) = x, i{=}0, 1, 2, \tag{6}$$

$t \le u \le T$.
According to this relationship the optimal exercise boundary for the holder of the option is the same as the optimal stopping curve of the latter optimal stopping problem.
Let us introduce the optimal exercise boundaries $b_{\sigma_0}(t), b_{\sigma_1}(t), b_{\sigma_2}(t), 0 \le t < T$ corresponding to American put option problem with volatilities σ_0, σ_1 and σ_2.
It is well-known (see, section 2.7, Karatzas and Shreve, [6]) that

$$\lim_{t \uparrow T} b_{\sigma_i}(t) = \min \left(\frac{r^d}{r^f} \cdot K, K \right), \qquad i = 0, 1, 2. \tag{7}$$

We know from the same section of [6] that the optimal exercise boundaries $b_{\sigma_0}(t), b_{\sigma_1}(t), b_{\sigma_2}(t), 0 \le t < T$ are nondecreasing continuous functions of time. The remarkable fact holds true (Ekstrom [2], El-Karoui et al. [3], Hobson [4]) that increasing the volatility, the value function of the American option with convex payoff also increases, therefore we have

$$v_0(t,x) \le v_1(t,x) \le v_2(t,x), \qquad 0 \le t \le T, x \ge 0. \tag{8}$$

We recall the definition of the optimal exercise boundaries

$$b_{\sigma_i}(t) = \inf\{x \ge 0 : v_i(t,x) > (K-x)^+\}, \qquad 0 \le t < T, i = 0,1,2. \tag{9}$$

From this definition and the inequality (8) it is evident that

$$b_{\sigma_0}(t) \ge b_{\sigma_1}(t) \ge b_{\sigma_2}(t), \qquad 0 \le t < T. \tag{10}$$

Our objective in this paper is to estimate the distance between the optimal exercise boundaries $b_{\sigma_2}(t)$ and $b_{\sigma_1}(t)$ in terms of the distance between the volatilities σ_2 and σ_1. For this we need some preliminary estimates regarding the optimal exercise boundary and the value function of the American foreign exchange put option. Some other properties of the optimal exercise boundary can be found in the works of Lamberton and Villeneuve [7], Rehman and Shashiashvili [8] and Villeneuve [10]. Since volatility is a major ingredient in option pricing, so these continuity estimates are important from practical viewpoint too.

2. Preliminary Results

Let us introduce the critical level

$$x_0 = \min\left(\frac{r^d}{r^f} \cdot K, K\right). \tag{11}$$

As the optimal exercise boundary $b_{\sigma_0}(t)$ is nondecreasing, we have the following bound from the limit relation (7)

$$b_{\sigma_0}(t) \le x_0, \qquad 0 \le t < T.$$

Here we need to prove the strict inequality.
The following lemma is crucial for the objective of this paper.

Lemma 2.1. *Let $\sigma_0 > 0$ be arbitrary positive constant, then for the optimal exercise boundary $b_{\sigma_0}(t), 0 \le t < T$, the following inequality is valid*

$$b_{\sigma_0}(t) < x_0, \qquad 0 \le t < T. \tag{12}$$

Proof. For $r^f \le r^d$ we have $x_0 = K$ and from lemma 2.2 of [8] we get

$$b_{\sigma_0}(t) < x_0, \qquad 0 \le t < T.$$

For $r^f > r^d > 0$, we have $x_0 = \frac{r^d}{r^f} \cdot K$.
It is easy to see that $b_{\sigma_0}(t) < x_0$ is equivalent to $v_0(t,x_0) > (K-x_0)^+$.
Fix $t, 0 \le t < T$ and consider the value function at (t,x_0)

$$v_0(t,x_0) = \sup_{t \le \tau \le T} E[e^{-r^d(\tau-t)}\big(K - Q_\tau^{(0)}(t,x_0)\big)^+], \quad 0 \le t \le T, \tag{13}$$

where the supremum is taken over all $(\mathcal{F}_u)_{0\le u\le T}-$stopping times τ such that $t\le\tau\le T$.

By time homogeneity of the stochastic process $Q_u^{(0)}(t,x_0), t\le u\le T$ we have

$$v_0(t,x_0)=\sup_{0\le\tau\le T-t}E[e^{-r^d\tau}\big(K-Q_\tau^{(0)}(x_0)\big)^+], \tag{14}$$

where $Q_u^{(0)}(x_0)$ is the solution of the following stochastic differential equation

$$dQ_u^{(0)}(x_0)=\big(r^d-r^f\big)Q_u^{(0)}(x_0)du+\sigma_0\cdot Q_u^{(0)}(x_0)\cdot dW_u, Q_0^{(0)}(x_0)=x_0, 0\le u\le T-t. \tag{15}$$

Writing the stochastic process $Q_u^{(0)}(x_0)$ in the explicit form we get

$$Q_u^{(0)}(x_0)=x_0\cdot e^{(r^d-r^f-\frac{1}{2}\sigma_0^2)u+\sigma_0W_u}, 0\le u\le T-t. \tag{16}$$

Denote the discounted payoff process $e^{-r^du}\big(K-Q_u^{(0)}(x_0)\big)^+$ by $X_u(x_0), 0\le u\le T-t$.

Then by applying the Tanaka-Meyer formula we have the following integral representation for the process $X_u(x_0)$ (see, section 4 of [8])

$$X_u(x_0)=\big(K-x_0\big)^++\widetilde{M}_u(x_0)+1/2\int_0^u e^{-r^dv}dL_v^0(K-Q^{(0)})+$$

$$+\int_0^u e^{-r^dv}\cdot I_{(Q_v^{(0)}(x_0)<K)}\big[Q_v^{(0)}(x_0)\cdot r^f-K\cdot r^d\big]dv, \quad 0\le u\le T-t, \tag{17}$$

where $\left(\widetilde{M}_u(x_0),\mathcal{F}_u\right), 0\le u\le T-t$, is a square-integrable martingale with $\widetilde{M}_0(x_0)=0$.

For arbitrary $\tau,\ 0\le\tau\le T-t$, we get

$$EX_\tau(x_0)\ge\big(K-x_0\big)^++e^{-r^dT}E\int_0^\tau I_{(Q_v^{(0)}(x_0)<K)}\big[Q_v^{(0)}(x_0)\cdot r^f-K\cdot r^d\big]dv. \tag{18}$$

Hence by the definition of the value function we have

$$v_0(t,x_0)\ge\big(K-x_0\big)^++e^{-r^dT}E\int_0^\tau I_{(Q_v^{(0)}(x_0)<K)}\big[Q_v^{(0)}(x_0)\cdot r^f-K\cdot r^d\big]dv. \tag{19}$$

Now let us introduce the following stopping time

$$\tau_U=\inf\{s\ge0: W_s\notin U\}\wedge(T-t), \qquad \text{where } U=\left(-\frac{1}{\sigma_0}\ln\frac{r^f}{r^d}\ ,\ \frac{1}{\sigma_0}\ln\frac{r^f}{r^d}\right).$$

Inserting explicit expression (16) to the latter inequality and replacing τ by $\tau\wedge\tau_U$ we get

$$v_0(t,x_0)\ge$$

$$\big(K-x_0\big)^++e^{-r^dT}E\int_0^{\tau\wedge\tau_U} I_{(Q_v^{(0)}(x_0)<K)}\big[x_0\cdot r^f\cdot e^{(r^d-r^f-\frac{1}{2}\sigma_0^2)v+\sigma_0\cdot W_v}-K\cdot r^d\big]dv. \tag{20}$$

Let us check that $Q_v^{(0)}(x_0) < K$ for all v such that $0 \le v < \tau \wedge \tau_U$.
Indeed if $0 \le v < \tau \wedge \tau_U$ then $v < \tau_U$ and as $r^f - r^d > 0$, we shall have

$$-\frac{1}{\sigma_0} \ln \frac{r^f}{r^d} < W_v < \frac{1}{\sigma_0} \ln \frac{r^f}{r^d} < \frac{1}{\sigma_0} \ln \frac{r^f}{r^d} + \frac{1}{\sigma_0}(r^f - r^d + \frac{1}{2}\sigma_0^2)v.$$

Therefore

$$\sigma_0 W_v < \ln \frac{r^f}{r^d} + (r^f - r^d + \frac{1}{2}\sigma_0^2)v,$$

from which we easily obtain

$$Q_v^{(0)}(x_0) = x_0 \cdot e^{(r^d - r^f - \frac{1}{2}\sigma_0^2)v + \sigma_0 \cdot W_v} < K.$$

So under the integral in (20) we can replace the indicator function by 1 to get

$$v_0(t, x_0) \ge \big(K - x_0\big)^+ + K \cdot r^d \cdot e^{-r^d T} E \int_0^{\tau \wedge \tau_U} \big[e^{(r^d - r^f - \frac{1}{2}\sigma_0^2)v + \sigma_0 \cdot W_v} - 1\big] dv, \quad (21)$$

where $0 \le \tau \le T - t$.

Introduce the function

$$g(v, y) = e^{(r^d - r^f - \frac{1}{2}\sigma_0^2)v + \sigma_0 \cdot y} - 1, \qquad 0 \le v \le T - t,\ y \in \left(-\frac{1}{\sigma_0} \ln \frac{r^f}{r^d}, \frac{1}{\sigma_0} \ln \frac{r^f}{r^d}\right).$$

We have

$$\frac{\partial g(v, y)}{\partial y} = \sigma_0 \cdot e^{(r^d - r^f - \frac{1}{2}\sigma_0^2)v + \sigma_0 \cdot y}.$$

Therefore $g(0, 0) = 0$ and $\frac{\partial g(0,0)}{\partial y} = \sigma_0 > 0$.
By lemma 3.1 of Villeneuve [10], there exists such a stopping time $\tau, \tau \le \tau_U$, for which

$$E \int_0^{\tau \wedge \tau_U} g(v, W_v) dv > 0.$$

Hence we have the strict inequality

$$v_0(t, x_0) > (K - x_0)^+,$$

from which we ultimately get the desired result (12). □

Now we state and prove the continuity estimate for the value functions $v_1(t, x)$ and $v_2(t, x)$ with respect to volatilities σ_1 and σ_2 respectively which we will need in our main theorem (3.1). For a more general result see section 4 of Achdou [1]. The proof is standard but it has been written for the sake of completeness.

Lemma 2.2. *For the difference of American put value functions* $v_2(t, x)$ *and* $v_1(t, x), 0 \le t \le T, x \ge 0$, *the following estimate does hold*

$$| v_2(t, x) - v_1(t, x) | \le c_1 \cdot x \mid \sigma_2 - \sigma_1 |, \quad 0 \le t \le T,\ x \ge 0, \qquad (22)$$

where the constant c_1 *depends only on* $\overline{r}, \overline{\sigma}$ *and* T.

Proof. Let us consider the difference of value functions $v_2(t,x)$ and $v_1(t,x)$, $\mid v_2(t,x) - v_1(t,x) \mid$

$$\leq \sup_{t\leq\tau\leq T} E\left(e^{-r^d(\tau-t)}\left|\left(K - Q_\tau^{(2)}(t,x)\right)^+ - \left(K - Q_\tau^{(1)}(t,x)\right)^+\right|\right)$$

$$\leq \sup_{t\leq\tau\leq T} E\left|Q_\tau^{(2)}(t,x) - Q_\tau^{(1)}(t,x)\right|$$

$$\leq E\left(\sup_{t\leq u\leq T}\left|Q_u^{(2)}(t,x) - Q_u^{(1)}(t,x)\right|\right).$$

For arbitrary random variable $X(\omega)$ we have $E|X(\omega)| \leq \left(E\big(X(\omega)\big)^2\right)^{1/2}$. Hence

$$|v_2(t,x) - v_1(t,x)| \leq \left(E\left(\sup_{t\leq u\leq T}\left|Q_u^{(2)}(t,x) - Q_u^{(1)}(t,x)\right|\right)^2\right)^{1/2}. \tag{23}$$

Now we will bound the right hand side of the latter inequality.
Denote by $\widehat{Q}_u(t,x)$ the difference of the stochastic processes $Q_u^{(2)}(t,x)$ and $Q_u^{(1)}(t,x)$

$$\widehat{Q}_u = Q_u^{(2)} - Q_u^{(1)}, \qquad t \leq u \leq T, \widehat{Q}_t = 0. \tag{24}$$

Then we have

$$\widehat{Q}_s = \int_t^s \widehat{Q}_v\cdot\big(r^d - r^f\big)dv + \int_t^s [Q_v^{(1)}\cdot(\sigma_2-\sigma_1) + \widehat{Q}_v\cdot\sigma_2]dW_v, \quad t \leq s \leq T. \tag{25}$$

From here we can write

$$\sup_{t\leq s\leq u} \widehat{Q}_s^2 \leq 2\big(r^d - r^f\big)^2\cdot T\int_t^u \widehat{Q}_v^2 dv + 2\sup_{t\leq s\leq u}\Big(\int_t^s [Q_v^{(1)}\cdot(\sigma_2-\sigma_1) + \widehat{Q}_v\cdot\sigma_2]dW_v\Big)^2.$$

Taking mathematical expectation on both sides of the latter inequality together with the use of Doob's classical maximal inequality we get

$$E\sup_{t\leq s\leq u} \widehat{Q}_s^2 \leq 2\big(r^d - r^f\big)^2\cdot T\int_t^u E\widehat{Q}_v^2 dv + 8\int_t^u E[Q_v^{(1)}\cdot(\sigma_2-\sigma_1) + \widehat{Q}_v\cdot\sigma_2]^2 dv. \tag{26}$$

Denote $\phi(u) = E\sup_{t\leq s\leq u}\widehat{Q}_s^2, \qquad t \leq u \leq T,$

then from the latter inequality and assumptions (1),(2) we obtain
$\phi(u) \leq$

$$2\cdot\overline{r}^2\cdot T\int_t^u \phi(v)dv + 16\cdot\overline{\sigma}^2\int_t^u \phi(v)dv + 16(\sigma_2-\sigma_1)^2\int_t^u E(Q_v^{(1)})^2 dv, \quad t \leq u \leq T.$$

Now we use the standard bound (see, for example theorem 2.9, chapter 5 of Karatzas, Shreve [5])

$$E(Q_v^{(1)})^2 \leq b \cdot x^2, \qquad t \leq v \leq T,$$

where constant b depends on $\overline{r}, \overline{\sigma}$ and T.
Therefore the previous inequality becomes

$$\phi(u) \leq \left(2 \cdot \overline{r}^2 \cdot T + 16 \cdot \overline{\sigma}^2\right) \int_t^u \phi(v) dv + 16(\sigma_2 - \sigma_1)^2 \cdot b \cdot x^2 \cdot T, \qquad t \leq u \leq T. \tag{27}$$

Now applying the classical Gronwall inequality we get

$$\phi(u) \leq c_2 \cdot x^2 \cdot (\sigma_2 - \sigma_1)^2, \quad t \leq u \leq T,$$

where the constant c_2 depends on $\overline{r}, \overline{\sigma}$ and T.
From here we can write

$$E\big(\sup_{t \leq u \leq T} \mid Q_u^{(2)}(t,x) - Q_u^{(1)}(t,x) \mid \big)^2 \leq c_2 \cdot x^2 \cdot (\sigma_2 - \sigma_1)^2. \tag{28}$$

Finally using the latter inequality in (23) we get the required estimate

$$\mid v_2(t,x) - v_1(t,x) \mid \leq c_1 \cdot x \mid \sigma_2 - \sigma_1 \mid, \quad 0 \leq t \leq T, x \geq 0,$$

where the constant c_1 depends only on $\overline{r}, \overline{\sigma}$ and T. □

3. The Main Result

In this section we will establish a bound for the difference of the optimal exercise boundaries of the American foreign exchange put option for two distinct values of the volatility parameter.
We remind that by x_0 we denote the following critical level

$$x_0 = \min\left(\frac{r^d}{r^f} \cdot K, K\right).$$

Theorem 3.1. *For the optimal exercise boundaries $b_{\sigma_2}(t)$ and $b_{\sigma_1}(t), 0 \leq t < T$ of the corresponding American put option problem (5), the following estimates are valid:*

If $0 \leq r^f \leq r^d$, *then*

$$\left(r^d \cdot K - r^d \cdot b_{\sigma_0}(t)\right)^{1/2} \cdot |b_{\sigma_2}(t) - b_{\sigma_1}(t)| \leq c \cdot K^{3/2} \cdot |\sigma_2 - \sigma_1|^{1/2}, \quad 0 \leq t < T. \tag{29}$$

If $r^f > r^d > 0$, *then*

$$\left(r^d \cdot K - r^f \cdot b_{\sigma_0}(t)\right)^{1/2} \cdot |b_{\sigma_2}(t) - b_{\sigma_1}(t)| \le c \cdot K^{3/2} \cdot |\sigma_2 - \sigma_1|^{1/2}, \qquad 0 \le t < T, \tag{30}$$

where the constant c *depends only on* $\overline{r}, \overline{\sigma}$ *and* T *and* $b_{\sigma_0}(t)$ *denotes the optimal exercise boundary for the same problem when volatility* $\sigma = \sigma_0$.

Proof. Consider the domain consisting of points $(t,x), 0 \le t < T, 0 < x < \infty$ for which $b_{\sigma_2}(t) < x < \min\left(\frac{r^d}{r^f} \cdot K, K\right)$.
The value function $v_2(t,x)$ of the corresponding optimal stopping problem (5) satisfies in this domain the following partial differential equation (section 2.7 of Karatzas and Shreve [6])

$$\frac{\partial v_2(t,x)}{\partial t} + \frac{\sigma_2^2 \cdot x^2}{2} \cdot \frac{\partial^2 v_2(t,x)}{\partial x^2} + (r^d - r^f) \cdot x \cdot \frac{\partial v_2(t,x)}{\partial x} - r^d \cdot v_2(t,x) = 0, \;\; 0 \le t < T. \tag{31}$$

Let us denote

$$u(t,x) = v_2(t,x) - (K - x), \qquad 0 \le t < T, b_{\sigma_2}(t) \le x < x_0. \tag{32}$$

Then $u(t,x) \ge 0$, and we have the following properties:

$$u(t, b_{\sigma_2}(t)) = 0 \qquad \text{(continuous-fit property)}, \tag{33}$$

$$\frac{\partial u}{\partial x}(t, b_{\sigma_2}(t)) = 0 \qquad \text{(smooth-fit property)}. \tag{34}$$

By writing the Taylor formula for the function $u(t,x)$ with respect to argument x at $x = b_{\sigma_2}(t)$ we get

$$u(t,x) = (x - b_{\sigma_2}(t)) \cdot \frac{\partial u}{\partial x}(t, b_{\sigma_2}(t)) + \frac{1}{2} \cdot (x - b_{\sigma_2}(t))^2 \cdot \frac{\partial^2 u}{\partial x^2}(t, \widehat{x}), \tag{35}$$

where $\widehat{x}$ is some point such that $b_{\sigma_2}(t) < \widehat{x} < x$.
Using the smooth-fit property we come to the equality

$$v_2(t,x) - (K - x) = \frac{1}{2} \cdot (x - b_{\sigma_2}(t))^2 \cdot \frac{\partial^2 v_2}{\partial x^2}(t, \widehat{x}), \text{ where } b_{\sigma_2}(t) < \widehat{x} < x. \tag{36}$$

Now let us consider two cases: either $0 \le r^f \le r^d$ or $r^f > r^d > 0$.
Consider at first the case when $0 \le r^f \le r^d$.
We know that $v_2(t,x)$ is non-increasing in t, i.e.

$$\frac{\partial v_2(t,x)}{\partial t} \le 0, \qquad 0 \le t < T, b_{\sigma_2}(t) < x < \min\left(\frac{r^d}{r^f} \cdot K, K\right). \tag{37}$$

Also it is well-known (see, section 2.7 of [6]) that

$$-1 \le \frac{\partial v_2(t,x)}{\partial x} \le 0.$$

So from the partial differential equation (31) we deduce

$$\begin{aligned} \frac{\overline{\sigma}^2 \cdot K^2}{2} \cdot \frac{\partial^2 v_2(t,x)}{\partial x^2} &\geq \frac{\sigma_2^2 \cdot x^2}{2} \cdot \frac{\partial^2 v_2(t,x)}{\partial x^2} \\ &\geq r^d \cdot v_2(t,x) \\ &\geq r^d (K-x), \end{aligned}$$

If $b_{\sigma_2}(t) < x \leq b_{\sigma_0}(t)$ then from the latter inequality we get

$$\frac{\partial^2 v_2(t,x)}{\partial x^2} \geq \frac{2 \cdot r^d}{\overline{\sigma}^2 \cdot K^2} \cdot (K - b_{\sigma_0}(t)) \,. \tag{38}$$

Using the above estimate in the equality (36) we obtain

$$v_2(t,x) - (K-x) \geq \frac{r^d \cdot (K - b_{\sigma_0}(t))}{\overline{\sigma}^2 \cdot K^2} \cdot (x - b_{\sigma_2}(t))^2 \,. \tag{39}$$

Let us take $x = b_{\sigma_1}(t)$ in the latter inequality then we shall have

$$v_2(t, b_{\sigma_1}(t)) - v_1(t, b_{\sigma_1}(t)) \geq \frac{\left(r^d \cdot K - r^d \cdot b_{\sigma_0}(t)\right)}{\overline{\sigma}^2 \cdot K^2} \cdot (b_{\sigma_1}(t) - b_{\sigma_2}(t))^2 \,. \tag{40}$$

Using lemma 2.2 we can write

$$\mid v_2(t, b_{\sigma_1}(t)) - v_1(t, b_{\sigma_1}(t)) \mid \leq c_1 \cdot K \mid \sigma_2 - \sigma_1 \mid, \tag{41}$$

and hence from the previous inequality we obtain

$$\left(r^d \cdot K - r^d \cdot b_{\sigma_0}(t)\right) (b_{\sigma_2}(t) - b_{\sigma_1}(t))^2 \leq c_1 \cdot \overline{\sigma}^2 \cdot K^3 \left|\sigma_2 - \sigma_1\right| , \tag{42}$$

otherwise

$$\left(r^d \cdot K - r^d \cdot b_{\sigma_0}(t)\right)^{1/2} \left|b_{\sigma_2}(t) - b_{\sigma_1}(t)\right| \leq c \cdot K^{3/2} \left|\sigma_2 - \sigma_1\right|^{1/2} , \tag{43}$$

where c is a constant dependent on $\overline{r}, \overline{\sigma}$ and T.
We move to consider the second case when $r^f > r^d > 0$.
From the partial differential equation (31) we have

$$\begin{aligned} \frac{\overline{\sigma}^2 \cdot K^2}{2} \cdot \frac{\partial^2 v_2(t,x)}{\partial x^2} &\geq \frac{\sigma_2^2 \cdot x^2}{2} \cdot \frac{\partial^2 v_2(t,x)}{\partial x^2} \\ &\geq r^d \cdot K - r^f \cdot x \\ &> r^d \cdot K - r^f \cdot b_{\sigma_0}(t), \end{aligned}$$

that is

$$\frac{\partial^2 v_2(t,x)}{\partial x^2} \geq \frac{2}{\overline{\sigma}^2 \cdot K^2} \left(r^d \cdot K - r^f \cdot b_{\sigma_0}(t)\right) . \tag{44}$$

Therefore we get from equality (36)

$$v_2(t,x) - (K-x) \geq \frac{r^d \cdot K - r^f \cdot b_{\sigma_0}(t)}{\overline{\sigma}^2 \cdot K^2} (x - b_{\sigma_2}(t))^2 \,. \tag{45}$$

Take $x = b_{\sigma_1}(t)$ in the latter inequality then we shall have

$$v_2(t, b_{\sigma_1}(t)) - v_1(t, b_{\sigma_1}(t)) \geq \frac{r^d \cdot K - r^f \cdot b_{\sigma_0}(t)}{\overline{\sigma}^2 \cdot K^2} \left(b_{\sigma_1}(t) - b_{\sigma_2}(t)\right)^2 . \tag{46}$$

Using bound (41) once again we get

$$\Big(r^d \cdot K - r^f \cdot b_{\sigma_0}(t)\Big) \left(b_{\sigma_2}(t) - b_{\sigma_1}(t)\right)^2 \leq c_1 \cdot \overline{\sigma}^2 \cdot K^3 \left|\sigma_2 - \sigma_1\right| , \tag{47}$$

i.e.

$$\Big(r^d \cdot K - r^f \cdot b_{\sigma_0}(t)\Big)^{1/2} \left|b_{\sigma_2}(t) - b_{\sigma_1}(t)\right| \leq c \cdot K^{3/2} \left|\sigma_2 - \sigma_1\right|^{1/2} , \tag{48}$$

where the constant c depends on $\overline{r}, \overline{\sigma}$ and T. □

References

[1] Y. Achdou, An inverse problem for a parabolic variational inequality arising in volatility calibration with American options, SIAM J. Control Optim. 43 (5) (2005) pp.1583-1615.

[2] E. Ekstrom, Properties of American option prices, Stoch. Process. Appl. 114 (2) (2004) pp.265-278.

[3] N. El-Karoui, M. Jeanblanc-Picque, S.E. Shreve, Robustness of the Black and Scholes formula, Math. Finance 8 (2) (1998) pp.93-126.

[4] D.G. Hobson, Volatility misspecification, option pricing and superreplication via coupling, Ann. Appl. Probab. 8 (1) (1998) pp.193-205.

[5] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, 2nd Edn. Springer-Verlag, New York, 1991.

[6] I. Karatzas, S.E. Shreve, Methods of Mathematical Finance, Springer, New York, 1998.

[7] D. Lamberton, S. Villeneuve, Critical Price Near Maturity for an American Option on a Dividend-Paying Stock, Ann. Appl. Probab. 13 (2) (2003) pp.800-815.

[8] N. Rehman, M. Shashiashvili, The American Foreign Exchange Option in Time-Dependent One-Dimensional Diffusion Model for Exchange Rate, Appl. Math. Optim. 59 (3) (2009) pp.329-363.

[9] S.E. Shreve, Stochastic Calculus for Finance II, Springer, New York, 2004.

[10] S. Villeneuve, Exercise Regions of American Options on Several Assets, Fin. and Stoch. 3 (3) (1999) pp.295-322.