Journal of Prime Research in Mathematics Vol. **8***(2012), 95-101*

ON TWO FAMILIES OF GRAPHS WITH CONSTANT METRIC DIMENSION

M. ALI[1], M. T. RAHIM[1], G. ALI[1]

Abstract. If G is a connected graph, the distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The representation $r(v|W)$ of v with respect to W is the k-tuple $(d(v, w_1), d(v, w_2), ..., d(v, w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a resolving set or locating set for G. A resolving set of minimum cardinality is called a basis for G and this cardinality is the metric dimension of G, denoted by $dim(G)$.
A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) does not depend upon the choice of G in $\mathcal{G}$. In this paper, we show that the graphs D_n^* and D_n^p, obtained from prism graph have constant metric dimension.

Key words: Metric dimension, basis, resolving set, prism.
2000 AMS subject: Primary 05C12.

1. Introduction

If G is a connected graph, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* of v with respect to W denoted by $r(v|W)$ is the k-tuple $(d(v, w_1), d(v, w_2), ..., d(v, w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* or *locating set* for G [2]. A resolving set of minimum cardinality is called a *metric basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$. For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the

[1]Department of Mathematics, National University of Computer & Emerging Sciences, FAST, Peshawar, Pakistan. Email: *murtaza_psh@yahoo.com*, $\{$*tariq.rahim*, *gohar.ali*$\}$*@nu.edu.pk*.

ith component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.
By denoting $G + H$ the join of G and H, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* $J_{2n}, (n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Caceres *et al.* [1] found the metric dimension of fan f_n and Tomescu *et al.*[13] found the metric dimension of *Jahangir graph* J_{2n}. Also Tomescu *et al.* [14] evaluated the partition and connected partition dimension of wheels.
In [2] Chartrand et al. proved that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2. The *prisms* D_n are the trivalent plane graphs obtained by the cross product of the path P_2 with a cycle C_n; they also constitute a family of 3-*regular graphs* with constant metric dimension. Also Javaid *et al.* proved in [5] that the set of *antiprisms* A_n constitutes a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$.
In this paper, we extend this study by considering some prism related graphs. We define the graph D_n^* as an extension of the prism graph defined as follows. For each vertex b_i, of the outer cycle we introduce a new vertex a_i, and join a_i to b_i and b_{i-1}, $i = 1, 2, ..., n$, where $b_0 = b_n$. Thus $V(D_n^*) = \bigcup_{i=1}^n \{a_i, b_i, c_i\}$. Here $\{c_i\}$, are inner cycle vertices and $\{b_i\}$, are outer cycle vertices and $\{a_i\}$, $i = 1, 2, ..., n$ are adjacent vertices to outer cycle. We define the graph D_n^p as an extension of the prism graph defined as follows. For each vertex b_i, for $i = 1, 2, ..., n$, of the outer cycle we introduce a new vertex a_i and join a_i to b_i, $i = 1, 2, ..., n$. Thus $V(D_n^p) = \bigcup_{i=1}^n \{a_i, b_i, c_i\}$. Here $\{c_i\}$ are inner cycle vertices, $\{b_i\}$ are outer cycle vertices and $\{a_i\}$, are vertices pendant to outer cycle for $i = 1, 2, ..., n$. We show that these graphs constitute families of graphs with constant metric dimension.

2. Prism Related Graphs with Constant Metric Dimension

In this section we show that D_n^*, D_n^p have constant metric dimension.

Theorem 1. *For $n \geq 6$ we have $dim(D_n^*) = 3$*

Proof. We consider two cases.
Case(1). Suppose $n = 2k$, $k \geq 3$, $k \in I\!N$. We consider $W = \{c_1, c_2, c_{k+1}\} \subset V(D_n^*)$. We show that W is a resolving set for $V(D_n^*)$. The representations of the vertices of $V(D_n^*) \setminus W$ with respect to W are:

$$r(c_i|W) = \begin{cases} (i-1, i-2, 1+k-i), & \text{for } 3 \leq i \leq k; \\ (2k-i+1, 2k+2-i, i-1-k), & k+2 \leq i \leq n. \end{cases}$$

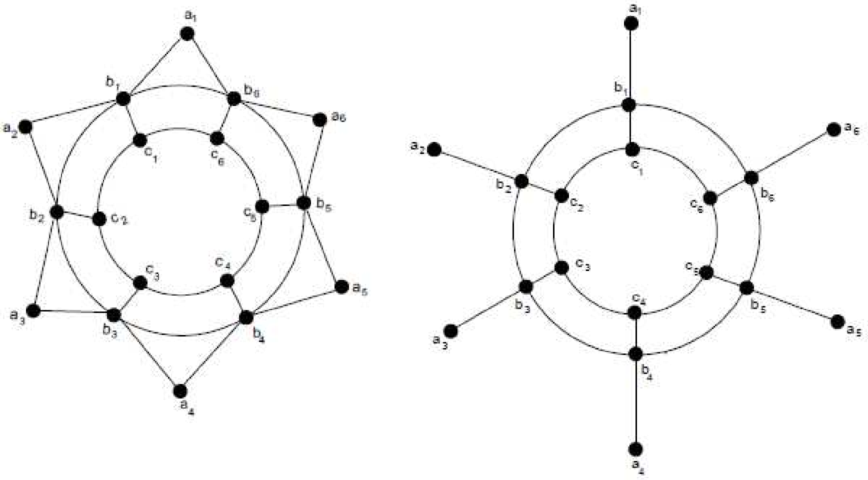

FIGURE 1. Graphs D_6^* and D_6^p

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & \text{for } i = 1; \\ (i, i-1, k+2-i), & \text{for } 2 \le i \le k+1; \\ (2k+2-i, 3+2k-i, i-k), & \text{for } k+2 \le i \le n. \end{cases}$$

$$r(a_i|W) = \begin{cases} (2, 3, k+2), & i = 1; \\ (2, 2, k+1), & i = 2; \\ (i, i-1, k+3-i), & 3 \le i \le k+1; \\ (k+1, k+1, 2), & i = k+2; \\ (2k+3-i, 2k+4-i, i-k), & k+3 \le i \le n. \end{cases}$$

We note that there are no two vertices having the same representation implying that $dim(D_n^*) \le 3$.

Now we show that $dim(D_n^*) \ge 3$, by proving that there is no resolving set W with $|W| = 2$. We have the following possibilities:

(1). Both vertices of W are on the inner cycle. Without loss of generality we suppose that one resolving vertex is c_1, and the other is c_t, $(2 \le t \le k+1)$.
For $2 \le t \le k$, we have
$r(c_n|W) = r(b_1|W) = (1, t)$.
And for $t = k+1$, we get
$r(c_2|W) = r(c_n|W) = (1, k-1)$, a contradiction.

(2). Both vertices of W are on the outer cycle. Without loss of generality we suppose that one resolving vertex is b_1, and the other is b_t, $(2 \le t \le k+1)$.
For $2 \le t \le k+1$, we have
$r(c_1|W) = r(a_1|W) = (1, t)$,

a contradiction.
(3). Both vertices of W are adjacent to outer cycle. We suppose that one resolving vertex is a_1, and the other is a_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k$, we have
$r(c_n|W) = r(a_n|W) = (2, t+1)$.
And for $t = k+1$, we get
$r(b_1|W) = r(b_n|W) = (1, k)$, a contradiction.
(4). One vertex on the inner cycle and the other is on the outer cycle. Consider one resolving vertex is c_1, and the other is b_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k$, we have
$r(a_1|W) = r(b_n|W) = (2, t)$.
And for $t = k+1$, we deduce
$r(b_n|W) = r(b_2|W) = (1, k-1)$, a contradiction.
(5). One vertex on the inner cycle and the other is the adjacent vertices to outer cycle. Consider one resolving vertex is c_1, and the other is a_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k-1$, we have
$r(a_n|W) = r(b_{n-1}|W) = (3, t+1)$.
And for $t = k$, we get
$r(b_1|W) = r(c_2|W) = (1, k-1)$, similarly for $t = k+1$, the representation is
$r(b_1|W) = r(c_2|W) = (1, k-1)$ a contradiction.
(6). One vertex on the outer cycle and the other is adjacent to outer cycle. Consider one resolving vertex is b_1, and the other is a_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k$, we have
$r(a_n|W) = r(c_n|W) = (2, t+1)$.
And for $t = k+1$, the representation is
$r(c_k|W) = r(a_{k+2}|W) = (k, 2)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(D_n^*)$ implying that $dim(D_n^*) = 3$.

Case(2). Suppose $n = 2k+1$, $k \geq 3$, $k \in I\!N$. Consider the set $W = \{c_1, c_2, c_{k+1}\} \subset V(D_n^*)$. We show that W is a resolving set for $V(D_n^*)$. For this we take the representations of vertices of $V(D_n^*) \setminus W$ with respect to W:

$$r(c_i|W) = \begin{cases} (i-1, i-2, k+1-i), & \text{for } 3 \leq i \leq k; \\ (2k+2-i, 2k+2-i, 1), & \text{for } i = k+2; \\ (2k+2-i, 2k+3-i, i-k-1), & \text{for } k+3 \leq i \leq n. \end{cases}$$

$$r(b_i|W) = \begin{cases} (1, 2, k+1), & \text{for } i = 1; \\ (i, i-1, k+2-i), & \text{for } 2 \leq i \leq k+1; \\ (k+1, k+1, 2), & \text{for } i = k+2; \\ (2k+3-i, 2k+4-i, i-k), & \text{for } k+3 \leq i \leq n. \end{cases}$$

$$r(a_i|W) = \begin{cases} (2,3,k+2), & i=1; \\ (2,2,k+1), & i=2; \\ (i,i-1,k+3-i), & 3 \le i \le k+1; \\ (k+2,k+1,2), & i=k+2; \\ (2k+4-i,2k+5-i,i-k), & k+3 \le i \le n. \end{cases}$$

Proceeding on same line as in (1) we observe that there are no two vertices having the same representations, implying that $dim(D_n^*) \le 3$.
Now we show that $dim(D_n^*) \ge 3$. Consider that $dim(D_n^*) = 2$, then there are the same possibilities as in case(1) and contradiction can be deduced analogously. This implies that $dim(D_n^*) \ge 3$ in this case . Finally from case(1) and (2), we get $dim(D_n^*) = 3$. Which completes the proof. □

Theorem 2. *For* $n \ge 3$

$$dim(D_n^p) = \begin{cases} 2, & \text{if } n = 2k+1; \\ 3, & n = 2k. \end{cases}$$

Proof. **Case(1)**. When $n = 2k+1$, $k \in I\!N$. Suppose $W = \{c_1, c_{k+1}\} \subset V(D_n^p)$, we show that W is resolving set for $V(D_n^p)$. For this we take the representation of any vertex of $V(D_n^p)\backslash W$ with respect to W:

$$r(c_i|W) = \begin{cases} (i-1,k+1-i), & 2 \le i \le k; \\ (2k+2-i,i-k-1), & k+2 \le i \le n. \end{cases}$$

$$r(b_i|W) = \begin{cases} (i,k-i+2), & 1 \le i \le k+1; \\ (2k+3-i,i-k), & k+2 \le i \le n. \end{cases}$$

$$r(a_i|W) = \begin{cases} (i+1,k-i+3), & 1 \le i \le k+1; \\ (2k+4-i,i-k+1), & k+2 \le i \le n. \end{cases}$$

Since these representations are pair wise distinct it follow,s that $dim(D_n^p) \le 2$ By [2] it is clear that $dim(D_n^p) \ge 2$. Which implies that $dim(D_n^p) = 2$, for odd n.
Case(2). When $n = 2k$, $k \in I\!N$. Suppose $W = \{c_1, c_2, c_{k+1}\} \subset V(D_n^p)$, we show that W is resolving set for $V(D_n^p)$. The representation of any vertex of $V(D_n^p)\backslash W$ with respect to W:

$$r(c_i|W) = \begin{cases} (i-1,i-2,k+1-i), & 3 \le i \le k; \\ (2k+1-i,2k+2-i,i-k-1), & k+2 \le i \le n. \end{cases}$$

$$r(b_i|W) = \begin{cases} (1,2,k+1), & \text{for } i=1; \\ (i,i-1,k+2-i), & \text{for } 2 \le i \le k+1; \\ (2k+2-i,2k+3-i,i-k), & \text{for } k+2 \le i \le n. \end{cases}$$

$$r(a_i|W) = \begin{cases} (2,3,k+2), & \text{for } i=1; \\ (i+1,i,k+3-i), & \text{for } 2 \le i \le k+1; \\ (2k-i+3,2k-i+4,i-k+1), & \text{for } k+2 \le i \le n. \end{cases}$$

We note that there are no two vertices having the same representations implying that $dim(D_n^p) \leq 3$.
Now we show that $dim(D_n^p) \geq 3$, by proving that there is no resolving set W with $|W| = 2$, then there are the following possibilities to be discussed,
(1). Both vertices of W are on the inner cycle. Without loss of generality we suppose that one resolving vertex is c_1, and the other is c_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k$, we have
$r(c_n|W) = r(b_1|W) = (1, t)$.
And for $t = k+1$,
$r(c_2|W) = r(c_n|W) = (1, k-1)$, a contradiction.
(2). Both vertices of W are on the outer cycle. Without loss of generality we suppose that one resolving vertex is b_1, and the other is b_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k+1$, we have
$r(c_1|W) = r(a_1|W) = (1, t)$.
a contradiction.
(3). Both vertices of W are pendant to the outer cycle. We suppose that one resolving vertex is a_1, and the other is a_t, $(2 \leq t \leq k+1)$. For $2 \leq t \leq k$, we have
$r(c_1|W) = r(b_n|W) = (2, t+1)$.
And for $t = k+1$,
$r(c_2|W) = r(a_n|W) = (3, k+1)$, a contradiction.
(4). One vertex on the inner cycle and the other is on the outer cycle. Consider one resolving vertex is c_1, and the other is b_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k$, we have
$r(a_1|W) = r(b_n|W) = (2, t)$.
And for $t = k+1$,
$r(b_n|W) = r(b_2|W) = (2, k-1)$, a contradiction.
(5). One vertex on the inner cycle and the other is the pendant vertex to outer cycle. Consider one resolving vertex is c_1, and the other is a_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k-1$, we have
$r(a_n|W) = r(b_{n-1}|W) = (3, t+2)$.
And for $t = k$,
$r(b_1|W) = r(c_2|W) = (1, k)$, similarly for $t = k+1$,
$r(b_1|\{c_1, a_t\}) = r(c_2|\{c_1, a_t\}) = (1, k+1)$ a contradiction.
(6). One vertex on the outer cycle and the other is pendant vertex to exterior cycle. Consider one resolving vertex is b_1, and the other is a_t, $(1 \leq t \leq k+1)$. For $1 \leq t \leq k$, we have
$r(a_n|\{b_1, a_t\}) = r(c_n|\{b_1, a_t\}) = (2, t+2)$.
And for $t = k+1$,
$r(b_k|\{b_1, a_t\}) = r(b_{k+2}|\{b_1, a_t\}) = (k-1, 2)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices

for $V(D_n^p)$ implying that $dim(D_n^p) = 3$.

□

2.1. **Acknowledgement.** The authors are indebted to the anonymous referee for his/her many valuable comments and suggestion on the earlier version of this paper.

References

[1] J. Caceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of some families of graphs. *Electronic Notes in Disc. Math.*, 22(2005), 129-133.

[2] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and metric dimension of a graph, *Disc. Appl. Math.*, 105(2000), 99-113.

[3] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combin.*, 2(1976), 191-195.

[4] M. Imran, S. A. Bokhary, A. Q. Baig, On families of convex polytopes with constant metric dimension, *Comput. Math. Appl.*, 60(9)(2010),2629-2638.

[5] I. Javaid, M. T. Rahim, K. Ali, Families of regular graphs with constant metric dimension, *Utilitas Math.*, 75(2008), 21-33.

[6] I. Javaid, On the connected partition dimension of unicyclic graphs. *J. Comb. Math. Comput.*, 65(2008), 71-77 .

[7] I. Javaid, S. Shokat, On the partition dimension of some wheel related graph. *J. Prim Res. Math.*, 4(2008), 154-164 .

[8] I. Javaid, M. Salman, M. A. Chaudhry, S. Shokat , Fault-tolerance in resolvability. *Util. Math.*, 80(2009), 263-275.

[9] K. Karliraj, V. J. Vivin, On equatable coloring of helm and gear graphs, International J. Math. Combin., 4(2010), 32-37.

[10] Murtaza Ali, Gohar Ali, Muhammad Imran, A. Q. Baig, Muhammad Kashif Shafiq, On the metric dimension of Mobius ladders.(Pre-print)

[11] Murtaza Ali, Gohar Ali, Usman Ali, M. T. Rahim, On Cycle Related Graphs with Constant Metric Dimension, Open Journal of Discrete Mathematics, 2, 2012, 21-23

[12] I. Tomescu, M. Imran, On metric and partition dimensions of some infinite regular graphs, *Bull. Math. Soc. Sci. Math. Roumanie*, 52(100),4(2009), 461-472.

[13] I. Tomescu, I. Javaid, On the metric dimension of the Jahangir graph, *Bull. Math. Soc. Sci. Math. Roumanie*, 50(98),4(2007),371-376.

[14] I. Tomescu, I. Javaid, Slamin, On the partition dimension and connected partition dimension wheels. *Ars. Comb.*, 84(2007), 311-317.