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COVERING COVER PEBBLING NUMBER FOR SQUARE OF A CYCLE

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Abstract. Let G be a connected graph. Let p be the number of pebbles distributed on the vertices of G. A pebbling move is defined by removing two pebbles from one vertex and put a pebble on an adjacent vertex. The *covering cover pebbling number*, $\sigma(G)$, is the least p such that after a sequence of pebbling moves, the set of vertices should form a covering for G from every configuration of p pebbles on the vertices of G. In this paper, we determine the covering cover pebbling number for square of a cycle.

Key words: graph pebbling, cover pebbling, covering, square of a cycle.
AMS subject: 05C70, 05C99.

1. Introduction

Pebbling, one of the latest evolutions in graph theory proposed by Lakarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hulbert published a survey of graph pebbling [3]. Given a connected graph G, distribute certain number of pebbles on its vertices in some configuration. Precisely, a configuration on a graph G, is a function from $V(G)$ to $N \cup \{0\}$ representing a placement of pebbles on G. The *size* of the configuration is the total number of pebbles placed on the vertices. *Support vertices* of a configuration C are those on which there is at least one pebble on C. In any configuration, if all the pebbles are placed on a single vertex, it is called a *simple configuration.* A pebbling move is the removal of two pebbles from one vertex and the addition of one pebble to an adjacent vertex. In (regular) pebbling, the target is selected and the aim is to move a pebble to the target vertex. The minimum number of pebbles, such that regardless of their initial placement and regardless of the target vertex, we

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can pebble that target vertex is called the pebbling number of G, denoted by $f(G)$. In cover pebbling, the aim is to cover all the vertices with pebbles, that is, to move a pebble to every vertex of the graph simultaneously. The minimum number of pebbles required such that regardless of their initial placement on G, there is a sequence of pebbling moves, at the end of which, every vertex has at least one pebble on it, is called the *cover pebbling number* of G. In [2], Crull et al. determine the cover pebbling number for complete graphs, paths, and trees. Hulbert and Munyan [4], determine the cover pebbling number of the d-cube. A set $K \subseteq V(G)$ is a covering if every edge of G has at least one end in K. The *covering cover pebbling number* of G, denoted by $\sigma(G)$, is the smallest number of pebbles, such that, however the pebbles are initially placed on the vertices of the graph, after a sequence of pebbling moves, the set of vertices with pebbles forms a covering of G. In [5], Lourdusamy and Punitha Tharani determine the covering cover pebbling number for complete graphs, paths, wheel graphs, complete r-partite graphs and binary trees.

In the next section, we determine the covering cover pebbling number of square of a cycle, that is $\sigma(C_n^2)$, for $n \geq 4$. For this we need the following theorem:

Theorem 1. [6] *The covering cover pebbling number of P_n^2 is as follows: (i) $\sigma(P_3^2) = 3$; (ii) $\sigma(P_4^2) = 4$; (iii) $\sigma(P_5^2) = 7$; and (vi) for $n \geq 6$,*
$\sigma(P_n^2) = \sum_{i \in I, j \in J} \left(2^{\lfloor \frac{i}{2} \rfloor} + 2^{\lfloor \frac{j}{2} \rfloor}\right) + \begin{cases} 1 & \text{if } \alpha = 2 \\ 0 & \text{if } \alpha = 0 \text{ or } 1 \end{cases}$ *where* $n \equiv \alpha (mod\ 3), I = \{\alpha+1, \alpha+4, \alpha+7, ..., n-5, n-2\}$ *and* $J = \{\alpha+2, \alpha+5, \alpha+8, ..., n-4, n-1\}$.

2. The Covering Cover Pebbling Number for Square of a Cycle

Definition 1. [7] *Let $G = (V(G), E(G))$ be a connected graph. The n^{th} power of G, denoted by G^p is the graph obtained from G by adding the edge uv to G whenever $2 \leq d(u,v) \leq p$ in G, that is, $G^p = (V(G), E(G) \cup uv : 2 \leq d(u,v) \leq p \text{ in } G)$. If $p = 1$, we define $G^1 = G$. We know that if p is large enough, that is, $p \geq n-1$, then $G^p = K_n$.*

Notation 1. *The labeling of C_n^2 is $C_n^2 : v_1v_2...v_nv_1$.*

First note that, $\sigma(C_4^2) = 5$ and $\sigma(C_5^2) = 7$ since $C_4^2 = K_4$, and $C_5^2 = K_5$[5].

Lemma 2. *The value of $\sigma(C_n^2)$ is attained when the configuration is a simple configuration.*

Proof. Assume first that a worst configuration consists of more than one set of consecutively support vertices ("islands"). The cardinality of any such island is at most two, for, were it to be three or more, one could relocate the pebbles to the inner one or two vertices, thereby causing a larger number of pebbles to be needed to cover the edges - a contradiction. Thus each island consists of at most two vertices. Now consider the effect of relocating all the pebbles on to a single island. Once again one reaches a contradiction to the assumption that there could be more than one island since after relocating all the pebbles to a single island, one would now require more pebbles than $\sigma(C_n^2)$ to cover the edges of the graph.

Next assume that the island consists of exactly two vertices. Clearly, a worst initial configuration of pebbles is obtained by placing $\sigma(C_n^2) - 1$ pebbles on one vertex, say v_1, and a single pebble on the other vertex, say v_2, since it would now cost more pebbles to cover the edges of the v_2 side of the square of a cycle. Had, however, all the pebbles been at v_1, we would need at least two more pebbles to cover the edges, raising a contradiction. The statement follows. □

Theorem 3. *For $m = 2n$ or $2n + 1$ $(n \geq 3)$, let $n + 1 \equiv \alpha' (mod\ 3)$.*
(i) Let $m = 2n (n \geq 3)$. Then the covering cover pebbling number of C_m^2 is given by $\sigma(C_m^2) = 2\sigma(P_{n+1}^2) - \begin{cases} 1 & \text{if } \alpha' = 0 \text{ or } 2 \\ 0 & \text{if } \alpha' = 1 \end{cases}$.
(ii) Let $m = 2n + 1$ $(n \geq 3)$.
If n is even, then the covering cover pebbling number of C_m^2 is given by $\sigma(C_m^2) = 2\left[\sigma(P_n^2) + 2^{\frac{n}{2}}\right] - \begin{cases} 1 & \text{if } \alpha' = 0 \text{ or } 1 \\ 0 & \text{if } \alpha' = 2 \end{cases}$
and if n is odd, then the covering cover pebbling number of C_m^2 is given by $\sigma(C_m^2) = \sigma(P_{n+2}^2) + \sigma(P_{n+1}^2) - \begin{cases} 1 & \text{if } \alpha' = 2 \\ 0 & \text{if } \alpha' = 0 \text{ or } 1 \end{cases}$.

Proof. By Lemma 2, fix a vertex, say v_1.

Case 1: Let $m = 2n$, where $n \geq 3$.
Consider the paths $P_A : v_1 v_2 v_3 ... v_{n+1}$ and $P_B : v_1 v_{2n} v_{2n-1} ... v_{n+1}$. So we need $2\sigma(P_{n+1}^2)$ pebbles at v_1 to cover the edges of C_m^2. Also note that v_1 may be pebbled twice. This will happen only when $\alpha' =$ 0 or 2 where $n + 1 \equiv \alpha' (\text{mod } 3)$. Thus we need at least $2\sigma(P_{n+1}^2) - \begin{cases} 1 & \text{if } \alpha' = 0 \text{ or } 2 \\ 0 & \text{if } \alpha' = 1 \end{cases}$ pebbles from v_1 to cover the edges of C_m^2.

Case 2: Let $m = 2n + 1$, where $n \geq 3$.
There exist two cases, namely, 'n is odd' and 'n is even'.

Subcase 2.1: n is odd
If n is odd then consider the paths $P_A : v_1 v_2 v_3 ... v_{n+2} v_{n+3}$ and $P_B : v_1 v_{2n+1} v_{2n} ... v_{n+4} v_{n+3}$. So we need $\sigma(P_{n+2}^2) + \sigma(P_{n+1}^2)$ pebbles to cover the edges of C_m^2. Also note that the vertex v_1 may be pebbled twice. This will happen only when $\alpha' = 2$, where

$n+1 \equiv \alpha^{'}(\text{mod } 3)$. Therefore, we need $\sigma(P_{n+2}^2)+\sigma(P_{n+1}^2) - \begin{cases} 1 & \text{if } \alpha^{'}=2 \\ 0 & \text{if } \alpha^{'}=0 \text{ or } 1 \end{cases}$ pebbles from v_1 to cover the edges of C_m^2.

Subcase 2.2: n is even

If n is even then consider the paths $P_A : v_1v_2v_3...v_{n+1}$ and $P_B : v_1v_{2n+1}v_{2n}...v_{n+1}$. So we need $2\left[\sigma(P_n^2)+2^{\frac{n}{2}}\right]$ pebbles to cover the edges of C_m^2. Also note that the vertex v_1 may be pebbled twice. This will happen only when $\alpha^{'}=0$ or 1. Thus we need $2\left[\sigma(P_n^2)+2^{\frac{n}{2}}\right] - \begin{cases} 1 & \text{if } \alpha^{'}=0 \text{ or } 1 \\ 0 & \text{if } \alpha^{'}=2 \end{cases}$ pebbles from v_1 to cover the edges of C_m^2.

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