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WEIGHTED HOMOGENEOUS POLYNOMIALS WITH ISOMORPHIC MILNOR ALGEBRAS

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Abstract. We recall first some basic facts on weighted homogeneous functions and filtrations in the ring A of formal power series. We introduce next their analogues for weighted homogeneous diffeomorphisms and vector fields. We show that the Milnor algebra is a complete invariant for the classification of weighted homogeneous polynomials with respect to right-equivalence, i.e. change of coordinates in the source and target by diffeomorphism.

Key words: Milnor algebra, right-equivalence, weighted homogeneous polynomial.
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1. Introduction

Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a weighted homogeneous polynomial of degree d w.r.t weights $(w_1, \dots, w_n)$ and $f_i = \frac{\partial f}{\partial x_i}$, $i = 1, \dots, n$ its partial derivatives. The Milnor algebra of f is defined by

$$M(f) = \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathcal{J}_f},$$

where $\mathcal{J}_f = \langle f_1, \dots, f_n \rangle$ is the Jacobian ideal.

We say that two weighted homogeneous polynomials $f, g : \mathbb{C}^n \to \mathbb{C}$ are $\mathcal{R}$-equivalent if there exists a diffeomorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ such that $f \circ \psi = g$.

We recall first some basic facts on weighted homogeneous functions and filtrations in the ring A of formal power series. We introduce next their analogues for weighted homogeneous diffeomorphisms and vector fields.

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In Theorem 2 we show that two weighted homogeneous polynomials f and g having isomorphic Milnor algebras are right-equivalent. The Example of Gaffney and Hauser, in [3], suggests us that we can not extend this result for arbitrary analytic germs.

2. Preliminary Results

We recall first some basic facts on weighted homogeneous functions and filtrations in the ring A of formal power series. We introduce next their analogues for weighted homogeneous diffeomorphisms and vector fields. For a more complete introduction see [1], Chap. 1, §3.

A holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ (defined on the complex space $\mathbb{C}^n$) is a weighted homogeneous function of degree d with weights $w_1, \ldots, w_n$ if

$$f(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^d f(x_1, \ldots, x_n) \,\forall\, \lambda > 0.$$

In terms of the Taylor series $\sum f_{\underline{k}} x^{\underline{k}}$ of f, the weighted homogeneity condition means that the exponents of the nonzero terms of the series lie in the hyperplane

$$L = \{\underline{k} : w_1 k_1 + \ldots + w_n k_n = d\}.$$

Any weighted homogeneous function f of degree d satisfies Euler's identity

$$\sum_{i=1}^{n} w_i x_i \frac{\partial f}{\partial x_i} = d.f \tag{1}$$

It implies that a weighted homogeneous function f belongs to its Jacobean ideal $\mathcal{J}_f$. The necessary and sufficient conditions of a function-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ to be equivalent to a weighted homogeneous function-germ are $f \in \mathcal{J}_f$, which is the well known result of Saito [5].

Consider $\mathbb{C}^n$ with a fixed coordinate system $x_1, \ldots, x_n$. The algebra of formal power series in the coordinates will be denoted by $A = \mathbb{C}[[x_1, \ldots, x_n]]$. We assume that a weighted homogeneity type $\underline{w} = (w_1, \ldots, w_n)$ is fixed. With each such $\underline{w}$ there is associated a filtration of the ring A, defined as follows.

The monomial $\mathbf{x}^{\underline{k}}$ is said to have degree d if $< \underline{w}, \underline{k} >= w_1 k_1 + \ldots + w_n k_n = d$.

The order d of a series (resp. polynomial) is the smallest of the degrees of the monomials that appear in that series (resp. polynomial).

The series of order larger than or equal to d form a subspace $A_d \subset A$. The order of a product is equal to the sum of the orders of the factors. Consequently, A_d is an ideal in the ring A. The family of ideals A_d constitutes a decreasing filtration of A: $A_{\acute{d}} \subset A_d$ whenever $\acute{d} > d$. We let A_{d+} denote the ideal in A formed by the series of order higher than d.

The quotient algebra A/A_{d+} is called the algebra of d-weighted jets, and its elements are called d-weighted jets.

Several Lie groups and algebras are associated with the filtration defined in the ring A of power series by the type of weighted homogeneity $\underline{w}$. In the case of ordinary homogeneity these are the general linear group, the group of k-jets of diffeomorphisms, its subgroup of k-jets with $(k-1)$-jet equal to the identity, and their quotient groups. Their analogues for the case of a weighted homogeneous filtration are defined as follows.

A formal diffeomorphism $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a set of n power series $g_i \in A$ without constant terms for which the map $g^* : A \to A$ given by the rule $g^* f = f \circ g$ is an algebra isomorphism.

The diffeomorphism g is said to have order d if for every s

$$(g^* - 1)A_s \subset A_{s+d}.$$

The set of all diffeomorphisms of order $d \geq 0$ is a group G_d. The family of groups G_d yields a decreasing filtration of the group G of formal diffeomorphisms; indeed, for $\acute{d} > d \geq 0$, $G_{\acute{d}} \subset G_d$ and is a normal subgroup in G_d.

The group G_0 plays the role in the weighted homogeneous case that the full group of formal diffeomorphisms plays in the homogeneous case. We should emphasize that in the weighted homogeneous case $G_0 \neq G$, since certain diffeomorphisms have negative orders and do not belong to G_0.

The group of d-weighted jets of type $\underline{w}$ is the quotient group of the group of diffeomorphisms G_0 by the subgroup G_{d+} of diffeomorphisms of order higher than d: $J_d = G_0/G_{d+}$.

Note that in the ordinary homogeneous case our numbering differs from the standard one by 1: for us J_0 is the group of 1-jets and so on.

J_d acts as a group of linear transformations on the space A/A_{d+} of d-weighted jets of functions. A special importance is attached to the group J_0, which is the weighted homogeneous generalization of the general linear group.

A diffeomorphism $g \in G_0$ is said to be weighted homogeneous of type $\underline{w}$ if each of the spaces of weighted homogeneous functions of degree d (and type $\underline{\mathrm{w}}$) is invariant under the action of g^*.

The set of all weighted homogeneous diffeomorphisms is a subgroup of G_0. This subgroup is canonically isomorphic to J_0, the isomorphism being provided by the restriction of the canonical projection $G_0 \to J_0$.

The infinitesimal analogues of the concepts introduced above look as follows.

A formal vector field $v = \sum v_i \partial_i$, where $\partial_i = \partial/\partial x_i$, is said to have order d if differentiation in the direction of v raises the degree of any function by at least d: $L_v A_s \subset A_{s+d}$.

We let $\mathfrak{g}_d$ denote the set of all vector fields of order d. The filtration arising in this way in the Lie algebra $\mathfrak{g}$ of vector fields (i.e., of derivations of the algebra A) is compatible with the filtrations in A and in the group of diffeomorphisms G:

1. $f \in A_d, v \in \mathfrak{g}_s \Rightarrow fv \in \mathfrak{g}_{d+s}, L_v f \in A_{d+s}$
2. The module $\mathfrak{g}_d$, $d \geq 0$, is a Lie algebra w.r.t. the Poisson bracket of vector fields.
3. The Lie algebra $\mathfrak{g}_d$ is an ideal in the Lie algebra $\mathfrak{g}_0$.
4. The Lie algebra $\mathfrak{j}_d$ of the Lie group J_d of d-weighted jets of diffeomorphisms is equal to the quotient algebra $\mathfrak{g}_0/\mathfrak{g}_{d+}$.
5. The weighted homogeneous vector fields of degree 0 form a finite-dimensional Lie subalgebra of the Lie algebra $\mathfrak{g}_0$; this subalgebra is canonically isomorphic to the Lie algebra $\mathfrak{j}_0$ of the group of 0-jets of diffeomorphisms.

The support of a weighted homogeneous function of degree d and type $\underline{w}$ is the set of all points $\underline{k}$ with nonnegative integer coordinates on the diagonal

$$L = \{\underline{k} : \langle \underline{k}, \underline{w} \rangle = d\}.$$

Weighted homogeneous functions can be regarded as functions given on their supports: $\sum f_{\underline{k}} x^{\underline{k}}$ assumes at $\underline{k}$ the value $f_{\underline{k}}$. The set of all such functions is a linear space $\mathbb{C}^r$, where r is the number of points in the support. Both the group of weighted homogeneous diffeomorphisms (of type $\underline{w}$) and its Lie algebra $\mathfrak{a}$ act on this space.

The Lie algebra $\mathfrak{a}$ of a weighted homogeneous vector field of degree 0 is spanned, as a $\mathbb{C}$-linear space, by all monomial fields $x^{\underline{P}}\partial_i$ for which $< \underline{P}, \underline{w} >= w_i$. For example, the n fields $x_i\partial_i$ belong to $\mathfrak{a}$ for any $\underline{w}$.

Example 1. *Consider the weighted homogeneous polynomial* $f = x^2y + z^2$ *of degree* $d = 6$ *w.r.t. weights* $(2, 2, 3)$. *Note that the Lie algebra of weighted homogeneous vector fields of degree* 0 *is spanned by*

$$\mathfrak{a} = \langle x^{\underline{P}}\partial_i :< \underline{P}, \underline{w} >= w_i, i = 1, 2, 3\rangle = \langle x\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\rangle$$

3. Main Results

We recall first Mather's lemma providing effective necessary and sufficient conditions for a connected submanifold (in our case the path P) to be contained in an orbit.

Lemma 1. *([4]) Let* $m : G \times M \to M$ *be a smooth action and* $P \subset M$ *a connected smooth submanifold. Then* P *is contained in a single* G*-orbit if and only if the following conditions are fulfilled:*
(a) $T_x(G.x) \supset T_xP$, *for any* $x \in P$.
(b) $\dim T_x(G.x)$ *is constant for* $x \in P$.

For arbitrary (i.e. not necessary with isolated singularities) weighted homogeneous polynomials we establish the following result.

Theorem 2. *Let f, g be two weighted homogeneous polynomials of degree d w.r.t. weights $(w_1, \ldots, w_n)$ such that $\mathcal{J}_f = \mathcal{J}_g$. Then $f \overset{\mathcal{R}}{\sim} g$, where $\overset{\mathcal{R}}{\sim}$ denotes the right equivalence.*

Proof. Let $H^d_{\underline{w}}(n, 1; \mathbb{C})$ be a space of weighted homogeneous polynomials from $\mathbb{C}^n$ to $\mathbb{C}$ of degree d w.r.t weights $(w_1, \ldots, w_n)$. Let $f, g \in H^d_{\underline{w}}(n, 1; \mathbb{C})$ such that $\mathcal{J}_f = \mathcal{J}_g$. Set $f_t = (1-t)f + tg \in H^d_{\underline{w}}(n, 1; \mathbb{C})$. Consider the $\mathcal{R}$-equivalence action on $H^d_{\underline{w}}(n, 1; \mathbb{C})$ under the group of 1-jets J_0, we have

$$T_{f_t}(J_0.f_t) = \mathbb{C}\langle x^{\underline{P}} \frac{\partial f_t}{\partial x_i};\ i = 1, \ldots, n \text{ and } <\underline{P}, \underline{w}>= w_i\rangle \tag{2}$$

Note that $T_{f_t}(J_0.f_t) \subset \mathcal{J}_{f_t} \cap H^d_{\underline{w}}$. But $\mathcal{J}_{f_t} \cap H^d_{\underline{w}} \subset \mathcal{J}_f \cap H^d_{\underline{w}}$ since

$$\frac{\partial f_t}{\partial x_i} = (1-t)\frac{\partial f}{\partial x_i} + t\frac{\partial g}{\partial x_i} \in (1-t)\mathcal{J}_f + t\mathcal{J}_g = \mathcal{J}_f \quad \text{(because } \mathcal{J}_f = \mathcal{J}_g)$$

Therefore, we have the inclusion of finite dimensional $\mathbb{C}$-vector spaces

$$T_{f_t}(J_0.f_t) = \mathbb{C}\langle x^{\underline{P}} \frac{\partial f_t}{\partial x_i};\ i = 1, \ldots, n \text{ and } <\underline{P}, \underline{w}>= w_i\rangle \subset \mathcal{J}_f \cap H^d_{\underline{w}} \tag{3}$$

with equality for $t = 0$ and $t = 1$.
Let's show that we have equality for all $t \in [0, 1]$ except finitely many values. Take $\dim(\mathcal{J}_f \cap H^d_{\underline{w}}) = m$ (say). Let's fix $\{e_1, \ldots, e_m\}$ a basis of $\mathcal{J}_f \cap H^d_{\underline{w}}$. Consider the m polynomials corresponding to the generators of the space (2):

$$\alpha_i(t) = x^{\underline{P}} \frac{\partial f_t}{\partial x_i} = x^{\underline{P}}[(1-t)\frac{\partial f}{\partial x_i} + t\frac{\partial g}{\partial x_i}], \text{ where } <\underline{P}, \underline{w}>= w_i \text{ and } \underline{P} = (P_1, \ldots, P_n)$$

We can express each $\alpha_i(t)$, $i = 1, \ldots, m$ in terms of above mentioned fixed basis as

$$\alpha_i(t) = \phi_{i1}(t)e_1 + \ldots + \phi_{im}(t)e_m,\ \forall\ i = 1, \ldots, m \tag{4}$$

where each $\phi_{ij}(t)$ is linear in t. Consider the matrix of transformation corresponding to the eqs. (4)

$$(\phi_{ij}(t))_{m\times m} = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) & \ldots & \phi_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1}(t) & \phi_{m2}(t) & \ldots & \phi_{mm}(t) \end{pmatrix}$$

having rank at most m. Note that the equality

$$\mathbb{C}\langle x^{\underline{P}} \frac{\partial f_t}{\partial x_i};\ i = 1, \ldots, n \text{ and } <\underline{P}, \underline{w}>= w_i\rangle = \mathcal{J}_f \cap H^d_{\underline{w}}$$

holds for those values of t in $\mathbb{C}$ for which the rank of above matrix is precisely m. We have the $m \times m$-matrix whose determinant is a polynomial of degree m in t and by the fundamental theorem of algebra it has at most m roots in $\mathbb{C}$ for which rank of the matrix of transformation will be less than m. Therefore,

the above-mentioned equality does not hold for at most finitely many values, say $t_1, \ldots, t_q$ where $1 \leq q \leq m$.
It follows that the dimension of the space (2) is constant for all $t \in \mathbb{C}$ except finitely many values $\{t_1, \ldots, t_q\}$.
For an arbitrary smooth path

$$\alpha : \mathbb{C} \longrightarrow \mathbb{C} \backslash \{t_1, \ldots, t_q\}$$

with $\alpha(0) = 0$ and $\alpha(1) = 1$, we have the connected smooth submanifold

$$P = \{f_t = (1 - \alpha(t))f(x) + \alpha(t)g(x) : t \in \mathbb{C}\}$$

of $H^d_{\underline{w}}$. By the above, it follows $\dim T_{f_t}(J_0.f_t)$ is constant for $f_t \in P$.
Now, to apply Mather's lemma, we need to show that the tangent space to the submanifold P is contained in that to the orbit $J_0.f_t$ for any $f_t \in P$. One clearly has

$$T_{f_t}P = \{\dot{f}_t = -\dot{\alpha}(t)f(x) + \dot{\alpha}(t)g(x) : \forall\, t \in \mathbb{C}\}$$

Therefore, by Euler formula 1, we have

$$T_{f_t}P \subset T_{f_t}(J_0.f_t)$$

By Mather's lemma the submanifold P is contained in a single orbit. Hence the result. □

Corollary 3. *Let f, g be two weighted homogeneous polynomials of degree d w.r.t. weights $(w_1, \ldots, w_n)$. If $M(f) \simeq M(g)$ (isomorphism of graded $\mathbb{C}$-algebra) then $f \overset{\mathcal{R}}{\sim} g$.*

Proof. We show firstly that an isomorphism of graded $\mathbb{C}$-algebras

$$\varphi : M(g) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{J}_g} \xrightarrow{\simeq} M(f) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{J}_f}$$

is induced by an isomorphism $u : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ such that $u^*(\mathcal{J}_g) = \mathcal{J}_f$.
Consider the following commutative diagram.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathcal{J}_g & \overset{u^*}{\dashrightarrow} & \mathcal{J}_f \\
\downarrow i & & \downarrow j \\
\mathbb{C}[x_1, \dots, x_n] & \overset{u^*}{\dashrightarrow} & \mathbb{C}[x_1, \dots, x_n] \\
\downarrow p & & \downarrow q \\
M(g) & \underset{\simeq}{\overset{\varphi}{\longrightarrow}} & M(f) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Define the morphism $u^* : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}[x_1, \dots, x_n]$ by

$$u^*(x_i) = L_i(x_1, \dots, x_n) = \sum_{j=1}^{n} a_{ij} x_j^{\alpha_j} + \sum a_{ik_1 \dots k_n} x_{k_1}^{\beta_1} \dots x_{k_n}^{\beta_n}\, ;\ i = 1, \dots, n \quad (5)$$

where $k_l \in \{1, \dots, n\}$ and $w_{k_1}\beta_1 + \dots + w_{k_n}\beta_n = deg_{\underline{w}}(x_i) = w_j\alpha_j$, which is well defined by commutativity of diagram below.

$$\begin{array}{ccc}
x_i & \overset{u^*}{\mapsto} & L_i \\
\downarrow p & & \downarrow q \\
\widehat{x}_i & \underset{\simeq}{\overset{\varphi}{\longmapsto}} & \widehat{L}_i
\end{array}$$

Note that the isomorphism φ is a degree preserving map and is also given by the same morphism u^*. Therefore, u^* is an isomorphism.

Now, we show that $u^*(\mathcal{J}_g) = \mathcal{J}_f$. For every $G \in \mathcal{J}_g$, we have $u^*(G) \in \mathcal{J}_f$ by commutative diagram below.

$$\begin{array}{ccc}
G & \overset{u^*}{\longmapsto} & F = u^*(G) \\
\downarrow p & & \downarrow q \\
\widehat{0} & \overset{\varphi}{\longmapsto} & \widehat{F} = \widehat{0}
\end{array}$$

It implies that $u^*(\mathcal{J}_g) \subset \mathcal{J}_f$. As u^* is an isomorphism, therefore it is invertible and by repeating the above argument for its inverse, we have $u^*(\mathcal{J}_g) \supset \mathcal{J}_f$.

Thus, u^* is an isomorphism with $u^*(\mathcal{J}_g) = \mathcal{J}_f$. By eq. (5), the map $u : \mathbb{C}^n \to \mathbb{C}^n$ can be defined by

$$u(z_1, \ldots, z_n) = (L_1(z_1, \ldots, z_n), \ldots, L_n(z_1, \ldots, z_n))$$

where $L_i(z_1, \ldots, z_n) = \sum_{j=1}^n a_{ij} x_j^{\alpha_j} + \sum a_{ik_1 \ldots k_n} x_{k_1}^{\beta_1} \ldots x_{k_n}^{\beta_n}$; $i = 1, \ldots, n$, $k_l \in \{1, \ldots, n\}$ and $w_{k_1}\beta_1 + \ldots + w_{k_n}\beta_n = deg_{\underline{w}}(x_i) = w_j\alpha_j$. Note that u is an isomorphism by Prop. 3.16 see [2], p.23.

In this way, we have shown that the isomorphism φ is induced by the isomorphism $u : \mathbb{C}^n \to \mathbb{C}^n$ such that $u^*(\mathcal{J}_g) = \mathcal{J}_f$.

Consider $u^*(\mathcal{J}_g) =< g_1 \circ u, \ldots, g_n \circ u >= \mathcal{J}_{g\circ u}$, where $g_j = \frac{\partial g}{\partial x_j}$. Therefore, $\mathcal{J}_{g\circ u} = \mathcal{J}_f \Rightarrow g \circ u \stackrel{\mathcal{R}}{\sim} f$, by Theorem 2. But $g \circ u \stackrel{\mathcal{R}}{\sim} g$. It follows that $g \stackrel{\mathcal{R}}{\sim} f$. □

Remark 1. *The converse implication, namely*

$$f \stackrel{\mathcal{R}}{\sim} g \Rightarrow M(f) \simeq M(g)$$

always holds(even for analytic germs f, g defining IHS), see [2], *p90.*

The following Example of Gaffney and Hauser [3], suggests us that we can not extend the Theorem 2 for arbitrary analytic germs.

Example 2. *Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be any function satisfying $h \notin \mathcal{J}_h \subseteq \mathcal{O}_n$ i.e. $h \notin H^d_{\underline{w}}(n, 1; \mathbb{C})$. Define a family $f_t : (\mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ by $f_t(x, y, z) = h(x) + (1 + z + t)h(y)$, and let $(X_t, 0) \subseteq (\mathbb{C}^{2n+1}, 0)$ be the hypersurface defined by f_t. Note that*

$$\mathcal{J}_{f_t} = \langle \frac{\partial h}{\partial x_i}(x), \frac{\partial h}{\partial y_j}(y), h(y) \rangle, \ t \in \mathbb{C}.$$

On the other hand, the family $\{(X_t, 0)\}_{t\in\mathbb{C}}$ is not trivial i.e. $(X_t, 0) \not\cong (X_0, 0)$: For, if $\{f_t\}_{t\in\mathbb{C}}$ were trivial, we would have by Proposition 2, §1, [3]

$$\frac{\partial f_t}{\partial t} = h(y) \in (f_t) + m_{2n+1}\mathcal{J}_{f_t} = (f_t) + m_{2n+1}\mathcal{J}_{h(x)} + m_{2n+1}\mathcal{J}_{h(y)} + m_{2n+1}(h(y))$$

Solving for $h(y)$ implies either $h(y) \in \mathcal{J}_{h(y)}$ or $h(x) \in \mathcal{J}_{h(x)}$ contradicting the assumption on h.
It follows that f_t is not $\mathcal{R}$-equivalent to f_0.

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