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WEAKENED CONDITION FOR THE STABILITY TO SOLUTIONS OF PARABOLIC EQUATIONS WITH "MAXIMA"

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Abstract. A class of reaction-diffusion equations with nonlinear reaction terms perturbed with a term containing "maxima" under initial and boundary conditions is studied. The similar problems that have no "maxima" have been studied during the last decade by many authors. It would be of interest the standard conditions for the reaction function to be weakened in the sense that the partial derivative of the reaction function, w.r.t. the unknown, to be bounded from above by a rational function containing $(1+t)^{-1}$, where t is the time. When we slightly weaken the standard condition imposed on the reaction function then the solution still decays to zero not necessarily in exponential order. Then we have no exponential stability for the solution of the considered problem. We establish a criterion for the nonexponential stability. The asymptotic behavior of the solutions when $t \to +\infty$ is discussed as well. The parabolic problems with "maxima" arise in many areas as the theory of automation control, mechanics, nuclear physics, biology and ecology.

Key words: Reaction-diffusion equation, stability, "maxima".
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1. Introduction

We investigate an initial and boundary value problem (IBVP) for a reaction diffusion equation $u_t - Lu = F$ with a reaction function

$$F = F(t, x, u(t, x), \max_{s\in[t-\sigma,t]} u(s, x)),$$

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where F has the form $F \equiv f(t,x,u(t,x)) + R(t,x, \max_{s\in[t-\sigma,t]} u(s,x))$, and the functions f, R are sufficiently smooth. Such equations belong to the class of parabolic partial differential equations (PDE) with "maxima" or may say in more general sense PDE with functional argument. Here F depends not only on u taken in the instantaneous time t and space x, but also on the function $\max_{s\in[t-\sigma,t]} u(s,x)$ defined in the time interval $[t-\sigma,t]$ that begins at $t-\sigma$ to t, and $t \in [0,T)$ with some positive number T that in some cases could be replaced by infinity. Then the domain of existence of the solution must be taken as $[-\sigma,T) \times \Omega$, where Ω is a bounded domain with a smooth boundary.

The study of differential equations with "maxima" or both delay and "maxima" starts many years ago with the pioneer works of A. R. Magomedov [10], [11], where were studied linear differential equations with "maxima" in the connection with the theory of the automatic control to different physical systems. In lots of applications the "maxima" is applied when the control corresponds to the maximal deviation of the regulated quantity that could be for instance temperature, heat, current density, pressure and so on. Meanwhile, the study of differential equations with "maxima" continue in several directions - existence and uniqueness of the solutions, oscillation, stability, asymptotic behavior of the solutions etc. The oscillation properties for the solutions of ODEs with "maxima" were studied by D. Bainov and his associates [1], [3] - [6], and of parabolic PDEs in [2] and [12]. The theory of neutral partial differential equations of hyperbolic and parabolic type with "maxima" was represented for the first time in [2] (see also [12]). Results of existence, oscillation and blow up for functional PDEs can be seen in [9], [13]-[15]. However, above stated parabolic and hyperbolic PDEs with "maxima" are not profoundly studied. The stability of the solutions of parabolic PDEs with "maxima" are investigated only in particular cases. In the present paper we study the stability of the solutions for parabolic PDEs in the form similar to this in [7].

As a typical example we may consider the mathematical model with "maxima" describing a dynamical hit system with feed back so that as the temperature reaches certain critical value then a sensor transfers the information to the heat source and a regulation effect hold.

2. Preliminary results and notations

In the parabolic equations which arise in some mathematical models describing phenomena in physics and biology the time evolution of the system is expressed by the partial derivative $\dfrac{\partial u}{\partial t} \equiv u_t$ of the unknown density function $u = u(t,x)$, that means concentration, temperature, population, etc. In most cases the evolutionary process is described by an initial and boundary value problem (IBVP), where the unknown density function u starts at some fixed initial moment t_0 and after passing a finite period of time describes the changes in the quantity $u(t,x)$. The basic question here is whether, as time t increases, the time-dependent unknown function u remains in a neighborhood of a steady-state solution $u_s = u_s(x)$, that is, a solution that does not depend on the time t. Other question is whether the solution $u(t,x)$ converges to the steady state (steady-state solution) as $t \to +\infty$. It is important to know for a given steady state u_s what is the set of initial functions whose corresponding time-dependent solutions converge to u_s as $t \to +\infty$. This leads to the problem of stability, often called Lyapunov stability, and asymptotic stability of a steady-state solution and its stability region. More details about this can be seen in [13].

Introduce the notations:

$$D_T \equiv (0,T] \times \Omega, \quad S_T \equiv (0,T] \times \partial\Omega, \quad D_{-\sigma} \equiv [-\sigma, 0] \times \Omega,$$
$$D_\infty \equiv (0,+\infty) \times \Omega, \quad E_T \equiv [-\sigma, T] \times \overline{\Omega}.$$

In the work [15] the parabolic equation studied there has a reaction function of the form $F(t,x,u(t,x),u(t-\sigma,x))$ with delay $\sigma > 0$. A similar problem is studied in [13]. The following functional equation with delay under the initial and boundary conditions has the form (see [2], [13], [14]):

$$\begin{aligned} &(a) \quad u_t - Lu = F(t,x,u(t,x),u(t-\sigma,x)) \text{ in } D_T,\\ &(b) \quad Bu = h(x) \quad \text{on} \quad S_T, \\ &(c) \quad u(t,x) = \eta_0(t,x) \text{ in } D_{-\sigma}. \end{aligned} \tag{1}$$

Here $T > 0$, $\sigma > 0$, the initial function η_0 is Hölder continuous in $D_{-\sigma}$ with $\eta_0(0,x) \in C^\theta(\bar{\Omega})$, $h(x)$ is assumed in the class $C^{1+\theta}(\partial\Omega)$, and the operator

$$L \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j} \tag{2}$$

is uniformly elliptic in the sense that the matrix $\{a_{ij}(x)\}$ is positive definite on $\bar{\Omega}$. We assume that the coefficients of L are in the class $C^{1+\theta}(\bar{\Omega})$ $(0 < \theta < 1)$. The boundary operator B is defined by $B \equiv \alpha_0(x)\frac{\partial}{\partial \nu} + \beta_0(x)$, where $\alpha_0(x)$ and $\beta_0(x)$ are nonnegative functions also in $C^{1+\theta}(\partial\Omega)$ and not identically zero on $\partial\Omega$; $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. We assume that L and B are self-adjoint. The source F is Hölder continuous in $D_T \times \mathbb{R} \times \mathbb{R}$.

In the present paper we consider the case when the reaction function F in (1) (a) has the form $F = f(t, x, u(t, x)) + R(t, x, \max_{s\in[t-\sigma,t]} u(s, x))$,

$$\begin{aligned} &(a)\ \ u_t - Lu = f(t, x, u(t, x)) + R(t, x, \max_{s\in[t-\sigma,t]} u(s, x)) && \text{in } D_T,\\ &(b)\ \ Bu = h(x) && \text{on } \ S_T, \qquad (3)\\ &(c)\ \ u(t, x) = \eta_0(t, x) && \text{in } D_{-\sigma}, \end{aligned}$$

where σ is a given positive constant representing the delay by which is determined the third argument $\max_{s\in[t-\sigma,t]} u(s, x)$ of the function R. Both functions $f(\cdot, \cdot, \eta)$ and $R(\cdot, \cdot, \eta)$ are assumed to be C^1-functions in η, and Hölder continuous with respect to t and x; $R(t, x, \cdot)$ is a monotone nondecreasing function.

Recall some basic definitions.

Definition 1. *A function $\tilde{u} \in Lip^\theta(E_T) \cap C^{1,2}(D_T)$ is called:*

i) a solution of IBVP (3) if it satisfies (3)(a)-(c) for every (t, x) in their domains

ii) an upper solution of IBVP if it satisfies (3) with $=$ replaced by $\geq$ in (a), (b) and (c)

iii) a lower solution of IBVP if it satisfies (3) with $=$ replaced by $\leq$ in (a), (b) and (c)

Introduce the notation $Lip^\theta(E_T)$ - the space of all Lipschitz continuous functions which are at the same time $C^{0,\theta}$ (Hölder continuous) and defined on the bounded set E_T.

The following proposition is obvious:

Proposition 1. *$\tilde{u} \in Lip^\theta(E_T) \cap C^{1,2}(D_T)$ is a solution of IBVP (3) if and only if it is simultaneously upper and lower solution.*

Definition 2. *A function u is said to be a solution of (3) if it is simultaneously upper and lower solution. Evidently every solution of (3) satisfies this equation.*

Definition 3. *A pair $\tilde{u} = \tilde{u}(t, x)$, $\hat{u} = \hat{u}(t, x)$ is called ordered if $\tilde{u} \geq \hat{u}$ in E_T. Then the set of all functions $z = z(t, x)$ such that $\hat{u} \leq z \leq \tilde{u}$ in E_T is denoted by $\langle \hat{u}, \tilde{u}\rangle$ and is called sector.*

We note that the problem (3) may have some solution $u_s = u_s(x)$ that does not depend on the time t. Such a solution is called a steady-state solution or steady-state.

Definition 4. *A steady state solution $u_s(x)$ of (3) is said to be stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|u(t, x) - u_s(x)| < \varepsilon \text{ in } D_T \qquad (4)$$

whenever $|\eta_0(0,x) - u_s(x)| < \delta$ *in* Ω.

$u_s(\cdot)$ *is said to be asymptotically stable if it is stable for each* T*, including* $T = \infty$*, and*

$$\lim_{t\to\infty} |u(t,x) - u_s(x)| = 0, \quad x \in \bar{\Omega}, \tag{5}$$

The steady-state solution u_s *is called unstable if (4) does not hold.*

The above definition implies that if u_s is asymptotically stable then it is an isolated steady-state solution in the sense that there is a neighborhood $\mathcal{U}_s$ of u_s in $C(\bar{\Omega})$ such that u_s is the only steady-state solution in $\mathcal{U}_s$.

Definition 5. *The set*

$$\mathcal{A} \equiv \{\eta_0(t,x) \in \textstyle\bigcup_{\theta\in(0,1)} Lip^\theta, \text{ the corresponding solution } u(t,x) \text{ satisfying (4) and (5)}\}$$

is called stability region of (3). IBVP (3) is said to be globally asymptotically stable if $\mathcal{A} \equiv \bigcup_{\theta\in(0,1)} Lip^\theta$.

Assume that following hypotheses are satisfied:

(H1) $f(t,x,0) = R(t,x,0) = 0$ for $(t,x) \in D_T$, $h(x) \equiv 0$ and $\beta_0(x) \neq 0$ for $x \in \partial\Omega$.

Let λ_0 and $\Phi(x)$ in Ω be the principal eigenvalue and corresponding normalized eigenfunction, respectively, of the elliptic problem

$$\begin{aligned} -Lu &= \lambda u \text{ in } \Omega, \\ Bu &= 0, \text{ on } \Omega. \end{aligned} \tag{6}$$

We remind that $0 < \Phi(x) \leq 1$ and $\lambda_0 > 0$.

Let α $(\lambda_0 > \alpha)$ and ρ be some positive constants.

(H2) The partial derivative $f_\eta(t,x,\eta)$ satisfies the estimate

$$f_\eta(t,x,\eta) \leq \lambda_0 - \alpha \text{ for } |\eta| \leq \rho, \ ((t,x) \in D_T). \tag{7}$$

Let $\alpha \geq A > 0$ for some $A > 0$ and $1 > \sigma > 0$. Define the continuous function $m = m(t)$ in the interval $[0,T)$ $(T > 0)$, where T can be infinity and then $m : [0,+\infty) \to \mathbb{R}$,

$$m(t) = (1+t)^{-1}\beta(t), \tag{8}$$

and the function $\beta(t)$ satisfies the inequality

$$0 < \beta(t) \leq A(1-\sigma)^{\alpha-A} \text{ for all } t \in [0,T). \tag{9}$$

Obviously, it makes sense that

$$0 < \beta(t) \leq A\left(1 - \frac{\sigma}{1+t}\right)^{\alpha-A} \quad \text{for } t \in [0,T). \tag{10}$$

(H3) Suppose that the partial derivative $R_\xi(t,x,\xi)$ of the function $R(t,x,\xi)$ satisfies the inequality

$$\begin{aligned} &R_\xi(t,x,\xi) \leq m(t) \\ &\text{for } |\xi| \leq \rho, \quad ((t,x) \in D_T). \end{aligned} \tag{11}$$

Next consider the inequality

$$\frac{dz}{dt} \geq -\alpha(1+t)^{-1}z + m(t) \max_{s\in[t-\sigma,t]} z(s), \quad t \in [0,T). \tag{12}$$

Lemma 2. *Let the condition (8), (9) be satisfied. Then the function $z = \rho(1+t)^{-\alpha+A}$ satisfies the differential inequality (12).*

Proof. We have $\dfrac{dz}{dt} = (A-\alpha)\rho(1+t)^{-\alpha+A-1}$. Thus the differential inequality (12) becomes:

$$(A-\alpha)\rho(1+t)^{-\alpha+A-1} \geq -\rho\alpha(1+t)^{-1}(1+t)^{-\alpha+A} + \rho m(t)(1+t-\sigma)^{-\alpha+A}, \ t \in [0,T).$$

Suppose (12) is not true, hence

$$A(1+t)^{-\alpha+A-1} < m(t)(1+t-\sigma)^{-\alpha+A}, \quad \text{and}$$

$$m(t) > A(1+t)^{-1}\left(1 - \frac{\sigma}{1+t}\right)^{\alpha-A}.$$

Having in mind this and also (8) - (10) it turns out that the latter contradicts to (10). □

3. Main results

First, we consider the regularity of $U(t,x) \equiv \max_{s\in[t-\sigma,t]} u(s,\ x)$.

Lemma 3. *If $u \in Lip^\theta(E_T)$ then $(t,x) \to \max_{s\in[t-\sigma,t]} u(s,x)$ is in $Lip^\theta(\overline{D}_T)$.*

Proof. Let us choose any points $(t_1,x_1),\ (t_2,x_2) \in [0,T] \times \overline{\Omega}$. Since

$$|u(t_1+s,x_1) - u(t_2+s,x_2)| + u(t_2+s,x_2) \geq u(t_1+s,x_1),$$

we have

$$\max_{s\in[-\sigma,0]}|u(t_1+s,x_1)-u(t_2+s,x_2)|+\max_{s\in[-\sigma,0]}u(t_2+s,x_2)\geq\max_{s\in[-\sigma,0]}u(t_1+s,x_1).$$

Similarly we have

$$\max_{s\in[-\sigma,0]}|u(t_1+s,x_1)-u(t_2+s,x_2)|+\max_{s\in[-\sigma,0]}u(t_1+s,x_1)\geq\max_{s\in[-\sigma,0]}u(t_2+s,x_2).$$

Then obtain

$$\begin{aligned}&\max_{s\in[-\sigma,0]}|u(t_1+s,\ x_1)-u(t_2+s,\ x_2)|\geq\\&\geq|\max_{s\in[-\sigma,0]}u(t_1+s,\ x_1)-\max_{s\in[-\sigma,0]}u(t_2+s,\ x_2)|.\end{aligned}\tag{13}$$

Next we have by admission that

$$|u(t_1,\ x_1)-u(t_2,\ x_2)|\leq H(|t_1-t_2|+|x_1-x_2|)^\theta,$$

where H is the Hölderian constant which is independent of $t_1,\ t_2,\ x_1$ and x_2.

Due to (13) one has that

$$\frac{|U(t_1,\ x_1)-U(t_2,x_2)|}{(|t_1-t_2|+|x_1-x_2|)^\theta}\leq\max_{s\in[-\sigma,0]}\frac{|u(t_1+s,\ x_1)-u(t_2+s,x_2)|}{(|t_1-t_2|+|x_1-x_2|)^\theta}\leq H,$$
$$\text{for}\ \ t_1,\ t_2\in[0,T],\quad x_1,x_2\in\overline{\Omega}.$$

Hence $U(t,x)$ is in $Lip^\theta(D_T)$. □

Remark 1. *Evidently Lemma 3 is true when Hölder is replaced with Lipschitz. However* $(t,x)\to\max_{s\in[t-\sigma,t]}u(s,x)$ *is not continuously differentiable (but only Lipschitz) even for analitic* $u(\cdot,\cdot)$. *Indeed let* $u(x)=x^2$. *If* $t<\frac{\sigma}{2}$, *then* $\max_{s\in[x-\sigma,x]}u(s)=(x-\sigma)^2$. *When* $x\geq\frac{\sigma}{2}$, *then* $\max_{s\in[x-\sigma,x]}u(s)=x^2$. *The left derivative of* $\max_{s\in[x-\sigma,x]}u(s)$ *at* $\frac{\sigma}{2}$ *is* $-\sigma$ *while the right one is* σ.

We notice that f and R are C^1-functions in the sector $\langle\hat{u},\tilde{u}\rangle$. For a given pair of ordered upper and lower solutions $\tilde{u},\ \hat{u}$, we use $u^{(0)}=\tilde{u}$ and $u^{(0)}=\hat{u}$ as two independent initial iterations and construct their respective sequences from the iteration process

$$\begin{aligned}&u_t^{(k)}-Lu^{(k)}+\underline{c}u^{(k)}=\underline{c}u^{(k-1)}+f(t,x,u^{(k-1)})+\\&\qquad\qquad+R(t,x,\max_{s\in[t-\sigma,t]}u^{(k-1)}(s,x))\ \text{ in }\ D_T,\\&Bu^{(k)}=h(t,x)\qquad\qquad\text{on}\ \ S_T,\\&u^{(k)}(t,x)=\eta_0(t,x)\qquad\quad\text{in}\ \ D_{-\sigma},\end{aligned}$$

where $\underline{c}(t,x) = \sup\{-f_u(t,x,u);\ \hat{u} \le u \le \tilde{u}\}$. Denote these two sequences by $\{\bar{u}^{(k)}\}$ and $\{\underline{u}^{(k)}\}$, respectively, and refer to them as upper and lower sequences.

The following statement is known as Theorem 1 of [7].

Theorem 4. *Under the above assumptions, the sequences $\{\bar{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ converge monotonically to a unique solution u to (3), and $\hat{u} \le u \le \tilde{u}$ in E_T.*

We have the following.

Theorem 5. *Let the hypotheses (H1)-(H3) be satisfied. Then a unique solution $u = u(t,x)$ of (3) exists and satisfies the inequality*

$$|u(t,x)| \le \rho(1+t)^{-\alpha+A}\Phi(x), \quad (t,x) \in E_T, \tag{14}$$

whenever $|\eta_0(t,x)| \le \rho(1+t)^{-\alpha+A}\Phi(x)$ in $D_{-\sigma}$. And the steady-state solution $u \equiv 0$ is asymptotically stable.

Proof. We should prove that $u \equiv 0$ is the steady-state solution to (3)(a),(b) and that there exist upper and lower solutions of (3) i.e., $\tilde{u} \equiv z(t)\Phi(x)$ and $\hat{u} \equiv -z(t)\Phi(x)$ respectively. Here $z(t) = \rho(1+t)^{-\alpha+A}$ and $\Phi(x)$ is defined in $\overline{\Omega}$ (see (7)). Obviously, the first statement is trivial. We substitute the function $\tilde{u}$ in the left side of (3) and obtain the inequality

$$\tilde{u}_t - L\tilde{u} = z'(t)\Phi(x) - z(t)L\Phi(x) = \{z'(t) + \lambda_0 z(t)\}\Phi(x) \ge$$
$$\ge \{\lambda_0 - \alpha(1+t)^{-1}\}\Phi(x)z(t) + m(t)\Phi(x) \max_{s\in[t-\sigma,t]} z(s),$$

which follows from Lemma 2. Since $|z(t)\Phi(x)| \le \rho$ in D_T, the hypotheses (H1) and (H2) assert that

$$f(t,x,z(t)\Phi(x)) = f(t,x,z(t)\Phi(x)) - f(t,x,0) = f_\eta(t,x,\eta^*(t,x))z(t)\Phi(x) \le$$
$$\le (\lambda_0 - \alpha)z(t)\Phi(x) \le (\lambda_0 - \alpha(1+t)^{-1})z(t)\Phi(x) \quad \text{in} \quad D_T,$$

where $0 \le \eta^*(t,x) \le z(t)\Phi(x)$. Thus we similarly obtain from (H1) and (H3) that

$$R(t,x,\max_{s\in[t-\sigma,t]} z(s)\Phi(x)) = R(t,x,\max_{s\in[t-\sigma,t]} z(s)\Phi(x)) - R(t,x,0) =$$
$$= R_\xi(t,x,\xi^*(t,x)) \max_{s\in[t-\sigma,t]} z(s)\Phi(x) \le m(t)\Phi(x) \max_{s\in[t-\sigma,t]} z(s),$$

where $0 \le \xi^*(t,x) \le \Phi(x) \max_{s\in[t-\sigma,t]} z(s)$. Next, we get

$$f(t,x,z(t)\Phi(x)) + R(t,x,\max_{s\in[t-\sigma,t]} z(s)\Phi(x)) \le$$
$$\le (\lambda_0 - \alpha(1+t)^{-1})z(t)\Phi(x) + m(t)\Phi(x) \max_{s\in[t-\sigma,t]} z(s).$$

This shows that

$$\tilde{u}_t - L\tilde{u} \ge f(t,x,z(t)\Phi(x)) + R(t,x,\max_{s\in[t-\sigma,t]} z(s)\Phi(x)).$$

For the boundary and initial conditions we have.

$$B\tilde{u}(t,x) = z(t)B\Phi(x) = 0 \quad \text{on } S_T,$$

$$\tilde{u}(t,x) \equiv z(t)\Phi(x) \geq \eta_0(t,x) \ \text{ in } \ D_{-\sigma}.$$

Therefore we verified that $\tilde{u}$ is a desired upper solution.

Furthermore, we show that $\hat{u}(t,x) = -z(t)\Phi(x)$ is a lower solution when $|\eta_0(t,x)| \leq \rho(1+t)^{-\alpha+A}\Phi(x)$. For this purpose substitute the function $\hat{u}$ in the left hand side of (3). Thus we obtain

$$\hat{u}_t - L\hat{u} = -z'(t)\Phi(x) + z(t)L\Phi(x) = -\{z'(t) + \lambda_0 z(t)\}\Phi(x) \leq$$
$$\leq -\{\lambda_0 - \alpha(1+t)^{-1}\}\Phi(x)z(t) - m(t)\Phi(x)\max_{s\in[t-\sigma,t]} z(s) \leq$$
$$\leq \{\lambda_0 - \alpha(1+t)^{-1}\}\Phi(x)\{-z(t)\} + m(t)\Phi(x)\max_{s\in[t-\sigma,t]}\{-z(s)\}.$$

Since $|z(t)\Phi(x)| \leq \rho$ in D_T, the hypotheses (H1) and (H2) assert that

$$f(t,x,-z(t)\Phi(x)) = f(t,x,-z(t)\Phi(x)) - f(t,x,0) =$$
$$= f_\eta(t,x,\eta^*(t,x))(-z(t))\Phi(x) \geq (\lambda_0 - \alpha)(-z(t))\Phi(x) \geq$$
$$\geq (\lambda_0 - \alpha(1+t)^{-1})(-z(t))\Phi(x) \ \text{ in } \ D_T,$$

where $0 \geq \eta^*(t,x) \geq -z(t)\Phi(x)$. And we similarly obtain from (H1) and (H3) that

$$R(t,x,\max_{s\in[t-\sigma,t]}\{-z(s)\}\Phi(x)) = R(t,x,\max_{s\in[t-\sigma,t]}\{-z(s)\}\Phi(x)) - R(t,x,0) =$$
$$= R_\xi(t,x,\xi^*(t,x))\Phi(x)\max_{s\in[t-\sigma,t]}\{-z(s)\} \geq m(t)\Phi(x)\max_{s\in[t-\sigma,t]}\{-z(s)\}.$$

Hence

$$f(t,x,-z(t)\Phi(x)) + R(t,x,\max_{s\in[t-\sigma,t]}\{-z(s)\}\Phi(x)) \geq$$
$$\geq (\lambda_0 - \alpha(1+t)^{-1})\{-z(t)\}\Phi(x) + m(t)\Phi(x)\max_{s\in[t-\sigma,t]}\{-z(s)\}.$$

This shows that

$$\hat{u}_t - L\hat{u} \leq f(t,x,-z(t)\Phi(x)) + R(t,x,\max_{s\in[t-\sigma,t]}\{-z(s)\}\Phi(x)).$$

For the boundary and initial conditions we have

$$B\tilde{u}(t,x) = -z(t)B\Phi(x) = 0 \ \text{ on } \ S_T,$$

$$\tilde{u}(t,x) \equiv -z(t)\Phi(x) \leq \eta_0(t,x) \ \text{ in } \ D_{-\sigma}.$$

Thus we verified that $\hat{u}$ is a desired lower solution. Therefore, by Theorem 4, we have a unique solution $u = u(t,x)$ to (3) and it satisfies the inequality

$$|u(t,x)| \leq \rho(1+t)^{-\alpha+A}\Phi(x), \quad (t,x) \in E_T, \tag{15}$$

whenever $|\eta_0(t,x)| \leq \rho(1+t)^{-\alpha+A}\Phi(x)$ in $D_{-\sigma}$. This suggests that the steady-state solution $u \equiv 0$ is asymptotically stable.

□

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References

[1] V. G. Angelov, D. D. Bainov: *On the functional differential equations with "maximums"*, Appl. Anal. 16(1983) 187–194.

[2] D. D. Bainov, D. P. Mishev: Oscillation Theory for Neutral Differential Equations with Delay, Adam Hilger, Bristol, Philadelphia and New York, IOP Publishing Ltd 1991.

[3] D. Bainov, V. Petrov, V. Proytcheva: *Asymptotic behaviour of second order neutral differential equations with "maxima"*, Tamkang Journal of Mathematics, 25(1995), 267–275.

[4] D. Bainov, V. Petrov, V. Proytcheva: *Oscillatory and asymptotic behaviour of second order neutral differential equations with "maxima"*, Dynamic Systems and Applications, 4(1995), 137-146.

[5] D. Bainov, V. Petrov, V. Proytcheva, *Oscillation and nonoscillation of first order neutral differential equations with maxima*, SUT Journal of Mathematics, 31(1995), 17-28.

[6] D. Bainov, A. Zahariev: *Oscillating and asymptotic properties of a class of functional differential equations with maxima*, Czechslovak Math. Journal, 34(1984), 247-251.

[7] T. Donchev, N. Kitanov, D. Kolev: *Stability for the solutions of parabolic equations with "maxima"*, PanAmer. Math. J., 20(2010), 1–19.

[8] A. Friedman: Partial Differential Equatioms of Parabolic Type, Prentice Hall, Englewood Cliffs, N.J., 1964.

[9] M. Kolev, A. Nosheen: *Blow-up to reaction-diffusion equation with delay*, IJPAM, 81(2012), 885–899.

[10] A. R. Magomedov: *On some questions about differential equations with maxima*, Izv. Akad. Nauk Azerbaidzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk, 1(1977), 104–108 (in Russian).

[11] A. R. Magomedov: *Theorem of existence and uniqueness of solutions of linear differential equations with maxima*, Izv. Akad. Nauk Azerbaidzhan. SSR Ser. Fiz.–Tekhn. Mat. Nauk, 5(1979), 116–118 (in Russian).

[12] D. P. Mishev: *Oscillatory properties of the solutions of parabolic differential equations of neutral type with maxima*, Godishnik Vissh. Uchebn. Zaved. Prilozhna Mat. 2(1989), 19–28.

[13] C. V. Pao: Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.

[14] N. Yoshida: *Forced oscillations of solutions of parabolic equations*, Bull. Austral. Math. Soc., 36(1987), 289–294.

[15] N. Yoshida: *Oscillation of nonlinear parabolic equation with functional arguments*, Hiroshima Mathematical Journal, 16(1986).