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ON GROTHENDIECK-LIDSKIǏ TRACE FORMULAS AND APPLICATIONS TO APPROXIMATION PROPERTIES

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ABSTRACT. The purpose of this short note is to consider the questions in connection with famous the Grothendieck-Lidskiǐ trace formulas, to give an alternate proof of the main theorem from [10] and to show some of its applications to approximation properties:
Theorem: Let $r \in (0,1], 1 \leq p \leq 2, u \in X^* \widetilde{\otimes}_{r,p} X$ and u admits a representation $u = \sum \lambda_i x_i' \otimes x_i$ with $(\lambda_i) \in \ell_r, (x_i')$ bounded and $(x_i) \in \ell_{p'}^w(X)$. If $1/r + 1/2 - 1/p = 1$, then the system (μ_k) of all eigenvalues of the corresponding operator $\widetilde{u}$ (written according to their algebraic multiplicities), is absolutely summable and $trace(u) = \sum \mu_k$.

Key words: eigenvalue distributions, approximation properties, trace formulas, r-nuclear operators.
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1. Introduction

Alexander Grothendieck [3], in 1955, has proved that for any 2/3 nuclear operator $T \in L(X, X)$, acting on a Banach space X, the trace is well defined, the system of its eigenvalues is absolutely summable and the (nuclear) trace is equal to the sum of the eigenvalues. On the other hand Lidskiǐ [8] has proved in 1959 that for any Hilbert space H and every operator $S \in S_1(H)$, the same is true. We have proved in [9] that if $1 \leq p \leq \infty$ and $0 < s \leq 1$ be such that $1/s = 1 + |1/2 - 1/p|$ then for any subspace Y of any L_p space,

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the trace on $N_{r,p}(Y)$ is a well defined, the system of eigenvalues is absolutely summable and moreover the trace is equal to the spectral trace i.e. the sum of its eigenvalues. And from this the two classical results, of Grothendieck and Lidskiĭ, also follow.

We provide another proof of the main result of [10] mentioned below as **Theorem 1** using main theorem of our recent paper [9]. Reecall (from [9]): Let Y be a subspace of an $L_p(\mu)$, $1 \leq p \leq \infty$. If $T \in N_s(Y)$, $1/s = 1 + |1/2 - 1/p|$, then; **(1)** the (nuclear) trace of T is well defined, **(2)** $\sum_{n=1}^{\infty} |\lambda_n(T)| < \infty$, where $\{\lambda_n(T)\}$ is the system of eigenvalues of the operator T (written according in according to their multiplicities), and $trace(T) = \sum_{n=1}^{\infty} \lambda_n(T)$.

2. Preliminaries

All the terminology and facts used here are standard and can be found in [1], [2], [3] and [4]. Let X be a Banach space, we denote its closed unit ball with B_X, its dual by X^* and for $x \in X$, $x' \in X^*$, we use the notation $< x', x >$ for $x'(x)$. For $0 < q \leq \infty$, $\ell_q^{strong}(X)$ and $\ell_q^{weal}(X)$ together with $\|(x_i)\|_{\ell_q^{strong}(X)} := (\sum_{i=1}^{\infty} \|x_i\|^q)^{1/q}$ and $\|(X_i)\|_{\ell_q^{weak}(X)} := \sup_{x' \in B_{X^*}} (\sum_{i=1}^{\infty} | < x', x_i > |^q)^{1/q}$ respectively are Quasi-Banach spaces for $0 < q < 1$ and Banach spaces for $1 \leq q \leq \infty$ of respectively the strongly q-summing and weakly q-summing sequences. In this article we are going to consider $\ell_r^{strong}(X^*)$ and $\ell_{p'}^{weak}(X)$, where $0 < r \leq 1$, $1 \leq p \leq 2$ and p' is the conjugate exponent of p.

For Banach spaces X and Y, consider the tensor product $X^* \otimes Y$ and a linear map $\widetilde{j}_{r,p} : X^* \otimes Y \to L(X, Y)$ where $\widetilde{j}_{r,p}(u = \sum_{i=1}^{N} x_i' \otimes x_i)(x) = \widetilde{u}(x) = \sum_{i=1}^{N} < x_i', x > x_i$. The image of this linear map consists of all the finite rank operator from X to Y. Let $0 < r \leq 1$ and $1 \leq p \leq 2$ be such that $1/r + 1/2 - 1/p = 1$ and we fix them throughout the remaining of this article. Define a function $\mathfrak{n}_{r,p}$ on $X^* \otimes X$ by:

$$\mathfrak{n}_{r,p}(u) = \inf_{u = \sum_{i=1}^{N} x_i' \otimes y_i} (\|(x_i')_{i=1}^{N}\|_{\ell_r^{strong}} \cdot \|(y_i)_{i=1}^{N}\|_{\ell_{p'}^{weak}(X)})$$

The linear space $X^* \otimes X$ together with $\mathfrak{n}_{r,p}$ is a Quasi-normed space(see for example [1]). We denote the completion of it by $X^* \widetilde{\otimes}_{r,p} Y$. A typical element u of this space $X^* \widetilde{\otimes}_{r,p} Y$ has one of its representation $u = \sum_{i=1}^{\infty} x_i' \otimes y_i$ where $(x_i')_{i=1}^{\infty} \in \ell_r^{strong}(X)$ and $(y_i)_{i=1}^{\infty} \in \ell_{p'}^{weak}(Y)$ or equivalently $u = \sum_{i=1}^{\infty} \lambda_i \widetilde{x_i}' \otimes \widetilde{y_i}$ where $(\lambda_i)_{i=1}^{\infty} \in \ell_r, (\widetilde{x_i}')_{i=1}^{\infty} \in \ell_{\infty}^{strong}(X^*)$ and $(\widetilde{y_i})_{u=1}^{\infty} \in \ell_{p'}^{weak}(X)$, moreover it can also be assumed that λ_i's are non-negative and decreasing. The map $\widetilde{j}_{r,p} : X^* \otimes Y \to L(X, Y)$ is continuous, therefore, it can be continuously and uniquely extended to the completion $X^* \widetilde{\otimes}_{r,p} Y$. We use the same symbol for the extension as well. Each tensor corresponds to an operator which is compact, has countable number of eigenvalues counted with multiplicities and each eigenvalue has finite algebraic multiplicity.

The projective space $X^*\widehat{\otimes}X$ is the completion of the finite rank operators with the cross norm $\pi(u) = \inf\{\sum_{i=1}^{N} \|x_i'\|\|x_i\| : u = \sum_{i=1}^{N} x_i' \otimes x_i\}$ on it(see for example [1], [5]). It is the largest space among all those which we are considering, with the greatest cross norm π on it [1]. That is, the space $Y^*\widetilde{\otimes}_{r,p}X$ can be embedded in $Y^*\widehat{\otimes}X$, ($j_{r,p} : Y^*\widetilde{\otimes}_{r,p}X \to Y^*\widehat{\otimes}X$ is a continuous injection and moreover $\|j_{r,p}\| \leq 1$). There is a continuous *trace* on the projective space defined by: For $u = \sum_{i=1}^{\infty} x_i' \otimes x_i$,

$$trace(u) = \sum_{i=1}^{\infty} < x_i', x_i > \tag{1}$$

Therefore there is a continuous *trace* on $X^*\widetilde{\otimes}_{r,p}X$ as well and it is defined in the same way. The expression in (1) is independent of any representation of u, see for example [1].

The Fredholm determinant $\delta(z, u)$ for $u \in X^*\widehat{\otimes}X$ is an entire funtion,

$$\delta(z, u) = 1 - trace(u)z + \alpha_2(u)z^2 + \ldots + (-1)^n\alpha_n(u)z^n + \ldots$$

where,

$$\alpha_n(u) = \sum_{i_1<i_2\ldots<i_n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n} \det(< x_{i'_\alpha}, x_{i_\beta} >)_{1\leq\alpha,\beta\leq n}$$

with the property that if $(z_i)_{i=1}^{\infty}$ be the system of zeros of $\delta(z, u)$ and $(\mu_i)_{i=1}^{\infty}$ be the system of non-zero eigenvalues counted with multiplicities of the associated operator $\widetilde{u}$ then $z_i = 1/\mu_i$, see for example [2] about it.

Two operators $S \in L(X, X)$ and $T \in L(Y, Y)$ are said to be related if there exists $A \in L(X, Y)$ and $B \in L(Y, X)$ such that $S = B \circ A$ and $T = A \circ B$. The related operators have many properties in common. More can be seen in [1] and [2] about them but all we need here is a partial case of the principle of related operators proposed by A. Pietsch in 1963 which says, with notation as above, that if S is a Riesz operator then so is T and moreover they have the same non-zero eigenvalues even with the same algebraic multiplicities. The space $N_{r,p} := \widetilde{j}_{r,p}(X^*\widetilde{\otimes}_{r,p}X)$ consists of compact operators and hence Riesz, see for example [1] or [2] for more about it.

For $u = \sum_{i=1}^{\infty} \lambda_i x_i \otimes x_i \in X^*\widetilde{\otimes}_{r,p}X$, the associated operator $\widetilde{u}$ factors through ℓ_p space in the following way: $\widetilde{u} : X \xrightarrow{A_0} \ell_\infty \xrightarrow{\Delta_{1-r}} c_0 \xrightarrow{\Delta_r} \ell_1 \hookrightarrow \ell_p \xrightarrow{B} X$

and hence it is related to an operator $T := \iota\Delta_r\Delta_{1-r}A_0B : \ell_p \to \ell_p$. Where Δ_s is a diagonal operator with the diagonal $(\lambda_i^s)_{i=1}^\infty$ for $s \in \{r, 1-r\}$, $A_0(x) = (< x'_i, x >)_{i=1}^\infty$ and $B(c_i)_{i=1}^\infty = \sum_{i=1}^\infty c_i x_i$ It is easy to see that T is s-nuclear operator and hence the main theorem from [9] applies to it.

The projective space $Y^*\widehat{\otimes}X$, also sometimes called as classical Grothendieck space, has the dual space $L(X, Y^{**})$ with the duality given by *trace*, see for example [5]. The operator $\widetilde{j} : Y^*\widehat{\otimes}X \to L(Y, X)$ is not injective in general. A space X has classical approximation property (due to Grothendieck) if $\widetilde{j}$ is injective. It was proved by A. Grothendieck, and later a short proof was provided by Oleg Reinov [7], that every Banach space has $AP_{2/3,1}$ (in our notation). We introduce the following approximation property:

Definition. Let $0 < r \leq 1,\ 1 \leq p \leq 2$ be such that $1/r + 1/2 - 1/p = 1$. A Banach space X has the approximation property of type (r, p), for short $AP_{r,p}$, iff for every Banach space Y the map $\widetilde{j}_{r,p} : Y^*\widetilde{\otimes}_{r,p}X \to L(Y, X)$ is injective.

Let's mention here that $\widetilde{j}_{r,p} = \widetilde{j} \circ \widetilde{j}_{r,p}$. It is clear from the definition that if a Banach space has classical AP then it also has $AP_{r,p}$.

3. Theorem and its proof

The following theorem is from [10]. Another proof is provided in this note using the main theorem of one of our recent paper [9], prinicple of related operators and Hadamard type theorem due to Lindelöf 1905.

Theorem 1. *Let $r \in (0, 1], 1 \leq p \leq 2, u \in X^*\widetilde{\otimes}_{r,p}X$ and u admits a representation $u = \sum \lambda_i x'_i \otimes x_i$ with $(\lambda_i) \in \ell_r, (x'_i)$ bounded and $(x_i) \in \ell^w_{p'}(X)$. If $1/r + 1/2 - 1/p = 1$, then the system (μ_k) of all eigenvalues of the corresponding operator $\widetilde{u} = \widetilde{j}_{r,p}(u)$ (written according to their algebraic multiplicities), is absolutely summable and moreover*

$$trace(u) = \sum_{k=1}^\infty \mu_k.$$

Proof. It is enough to consider the representation $u = \sum_{i=1}^\infty \lambda_i x'_i \otimes x_i$ such that $(x'_i)_{i=1}^\infty \in B_{\ell^{weak}_{p'}(X)}$. Using the triangle inequality,

$$\delta(z, u) \leq 1 + |trace(u)||z| + |\alpha_2(u)||z|^2 + \ldots + |\alpha_n(u)||z|^n + \ldots$$

where,

$$|\alpha_n(u)| \leq \sum_{i_1<i_2\ldots<i_n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}|\det(< x_{i'_\alpha}, x_{i_\beta} >)_{1\leq\alpha,\beta\leq n}|.$$

Now we use the Hadamard's inequality for the determinants(which can be found, for example, in [4], p.1018) together with the trivial embedding of $\iota : \ell_{p'}^n \hookrightarrow \ell_2^n$ with $\|\iota\| = n^{\frac{1}{p}-\frac{1}{2}}$ implies that

$$|\det(< x_{i'_\alpha}, x_{i_\beta} >)_{1\leq\alpha,\beta\leq n}| \leq \prod_{\alpha=1}^{n}(\sum_{\beta=1}^{n} | < x'_{i_\alpha}, x_{i_\beta} > |^2)^{\frac{1}{2}} \leq n^{n(\frac{1}{p}-\frac{1}{2})}$$

Therefore,

$$\begin{aligned}
|\delta(z,u)| &\leq \sum_{n=0}^{\infty} \sum_{i_1<i_2<\ldots<i_n} n^{n(\frac{1}{p}-\frac{1}{2})} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_n}|z|^n \\
&\leq \prod_{n=0}^{\infty}(1 + n^{(\frac{1}{p}-\frac{1}{2})}\lambda_n|z|) \\
&\leq \exp(\sum_{n=0}^{\infty} n^{\frac{1}{p}-\frac{1}{2}}\lambda_n|z|) \\
&= \exp(\sum_{n=0}^{\infty} n^{\frac{1}{r}-1}\lambda_n^{1-r}\lambda_n^r|z|) \\
&= \exp(\sum_{n=0}^{\infty}(n\lambda_n^r)^{\frac{1-r}{r}}\lambda_n^r|z|) \\
&\leq \exp(\sum_{n=0}^{\infty} c\lambda_n^r|z|) \qquad \text{where } c = \sup_k (k\lambda_k^r)^{\frac{1-r}{r}}
\end{aligned}$$

Now for given $\varepsilon > 0$, there exist $n_\varepsilon \in \mathbb{N}$ such that: $\sum_{n=n_\varepsilon+1}^{\infty} \lambda_n^r < \frac{\varepsilon}{2(c+1)}$ And now choose $M := \sum_{n=0}^{\infty} c\lambda_n^r$ and therefore we have,

$$|\delta(z,u)| \leq \exp(M|z|) \qquad \text{for all} \qquad z \in \mathbb{C}$$

The Fredholm determinant $\delta(z,T)$ is an entire function and moreover has zeros at $(\mu_n^{-1})_1^\infty$ where $(\mu_n)_{n=0}^\infty$ is the system of eigenvalues of operator $\tilde{u}$. Moreover, since $\tilde{u}$ factors through ℓ_p spaces, therefore using the principle of related operator and main theorem form [9], we observe that $(\mu_n)_1^\infty \in \ell_1$. Now use Theorem 4.8.9, p. 225 from [2] due to Lindelöf 1905 which is related to Hadamard's factorization theorem, to conclude:

$$\begin{aligned}
\delta(z,u) &= \prod_{n=0}^{\infty}(1 - \mu_n z) \\
&= 1 - (\sum_{n=0}^{\infty}\mu_n)z + \ldots
\end{aligned}$$

And now comparing these coefficients with the those in the definition of $\delta(z,u)$ we obtain:

$$\text{trace}(u) = \sum_{n=0}^{\infty} \lambda_n < x'_n, x_n >= \sum_{n=0}^{\infty} \mu_n$$

$\square$

Corollary 2. *Every Banach space X has $AP_{r,p}$.*

Proof. Consider $u \in Y^* \widetilde{\otimes}_{r,p} X$ such that $\widetilde{u} = 0$. Elements of the $L(X, Y^{**})$ defines a continuous linear functional on $Y^* \widehat{\otimes} X$ through the *trace*, i.e. for every continuous linear functional $\varphi \in (Y^* \widehat{\otimes} X)^*$, there exists $T \in L(X, Y^{**})$ such that $\varphi(u) = trace(T \circ u)$ (see for example [1] or [5]). But it is clear that for such an operator, the trace is not only well defined but is equal to the sum of the eigenvalues of the corresponding operator. Coming back to our supposition, if we insists that $u \neq 0$ then by Hahn-Banach theorem, there exists $T \in L(X, Y^{**})$ such that $trace(T \circ u) \neq 0$, where the tensor $T \circ u = \sum_{i=1}^{\infty} y'_i \otimes T(x_i) \in Y^* \widehat{\otimes}_{r,p} Y^{**} \subset Y^{***} \widehat{\otimes}_{r,p} Y^{**}$, then we arrive at a contradiction since **Theorem 1** forces $trace(T \circ u) = \underbrace{\sum_{i=1}^{\infty}}\mu_i$ where $(\mu_i)_{i=1}^{\infty}$ is the system of eigenvalues of the associated operator $\widetilde{T \circ u} = T \circ \widetilde{u}$, because all the eigenvalues μ_i's of $\widetilde{T \circ u}$ are zero due to the operator being identically zero on Y. Thus $u \neq 0$ and hence $\widetilde{j}_{r,p}$ is injective. $\square$

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