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WEIGHT CHARACTERIZATION OF THE BOUNDEDNESS FOR THE RIEMANN-LIOUVILLE DISCRETE TRANSFORM

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Abstract. We establish necessary and sufficient conditions on a weight sequence $\{v_j\}_{j=1}^{\infty}$ governing the boundedness for the Riemann-Liouville discrete transform I_α from $l^p(\mathbb{N})$ to $l^q_{v_j}(\mathbb{N})$ (trace inequality), where $0 < \alpha < 1$. The derived conditions are of D. Adams or Maz'ya–Verbitsky (pointwise) type.

Key words: Riemann–Liouville discrete transform with product kernels, discrete Hardy operator, discrete potentials, weighted inequality, trace inequality.
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1. Introduction

Our aim in this paper is to characterize a weight sequence $\{v_j\}_{j=1}^{\infty}$ for which the operator

$$\{I_\alpha \beta_k\}_n = \sum_{k=1}^{n} \frac{\beta_k}{(n-k+1)^{1-\alpha}}, \qquad n \in \mathbb{N},$$

maps boundedly from $l^p(\mathbb{N})$ to the weighted space $l^q_{v_j}(\mathbb{N})$, where $1 < p \le q < \infty$ and $0 < \alpha < 1$. If $p < q$, then we derive necessary and sufficient condition of D. Adams [1] type, while in the diagonal case $p = q$ we establish Maz'ya–Verbitsky [4] type criteria guaranteeing the trace inequality for I_α.

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Let $1 < p < \infty$. Suppose that $\{v_k\}_{k=1}^{\infty}$ is a sequence of positive numbers (weight sequence). Let $l^p_{v_k}(\mathbb{N})$ be the class of all sequences $\{\beta_k\}_{k=1}^{\infty}$ for which

$$\|\beta_k\|_{l^p_{v_k}(\mathbb{N})} := \left(\sum_{k=1}^{\infty} |\beta_k|^p v_k\right)^{1/p} < \infty\,.$$

If $v_k \equiv 1$, then we denote $l^p_{v_k}(\mathbb{N})$ by $l^p(\mathbb{N})$.

Further, for an a.e. positive function (weight) v on $\mathbb{R}_+ := (0,\infty)$, we denote by $L^p_v(\mathbb{R}_+)$ the class of all measurable functions f on $\mathbb{R}_+$ for which

$$\|f\|_{L^p_v(\mathbb{R}_+)} := \Big(\int\limits_{\mathbb{R}_+} |f(x)|^p v(x)dx\Big)^{1/p} < \infty.$$

If $v \equiv 1$, then we denote $L^p_v(\mathbb{R}_+)$ by $L^p(\mathbb{R}_+)$.

Continuous analog of the operator I_α is the Riemann–Liouvlle transform defined on $\mathbb{R}_+$ given by the formula

$$R_\alpha f(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \quad 0 < \alpha < 1.$$

The $L^p \to L^q_v$ characterization of R_α was studied in the papers [5], [3] (we refer also [7] and the monograph [2], Ch.1). The statements derived by these authors reed as follows:

Theorem A **([5], [7]).** *Let* $1 < p \le q < \infty$, $1/p < \alpha < 1$. *Then the following conditions are equivalent:*

(i) R_α is bounded from $L^p(\mathbb{R}_+)$ into $L^q_v(\mathbb{R}_+)$;

(ii)

$$B \equiv \sup_{t>0} \Big(\int_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)^{1/q} t^{1/p'} < \infty;$$

(iii)

$$B_1 \equiv \sup_{k\in Z} \Big(\int_{2^k}^{2^{k+1}} v(x)dx\Big)^{1/q} 2^{k(\alpha-1/p)q} < \infty.$$

For the case $0 < \alpha < 1/p$, there are known the following statements:

Theorem B **([3]).** *Let* $1 < p < \infty$ *and let* $0 < \alpha < \frac{1}{p}$. *Then the inequality*

$$\int\limits_{\mathbb{R}_+} |R_\alpha f(x)|^p v(x)dx \le c_0 \int\limits_{\mathbb{R}_+} |f(x)|^p dx,\ f \in L^p(\mathbb{R}_+),$$

holds if and only if $W_\alpha v \in L^{p'}_{loc}(\mathbb{R}_+)$ *and*

$$W_\alpha [W_\alpha v]^{p'}(x) \le c W_\alpha v(x) \quad a.e.,$$

where

$$W_\alpha g(t) = \int_t^\infty \frac{g(\tau)}{(\tau-t)^{1-\alpha}}d\tau.$$

Theorem C ([2], p. 131.) *Let $1 < p < q < \infty$, $0 < \alpha < 1/p$. Then the following statements are equivalent:*

(i) There exists a positive constant c such that for all $f \in L^p(\mathbb{R}_+)$,

$$\|R_\alpha f\|_{L^q_v(\mathbb{R}_+)} \le c\|f\|_{L^p(\mathbb{R}_+)};$$

(ii)

$$\sup_{0\le h\le a} (\nu[a; a+h))^{1/q} h^{\alpha-1/p} < \infty;$$

In the paper [3] the authors applied Theorem B to prove the existence of a positive solution for certain non–linear Volterra integral equation.

In the discrete case the following statement holds (see [6], [7]):

Theorem D. *Let $1 \le p \le q < \infty$ and let $1/p < \alpha < 1$. Then the following conditions are equivalent:*

(i) *The operator I_α is bounded from $l^p(\mathbb{N})$ to $l^q_{v_j}(\mathbb{N})$;*

(ii)

$$\sup_{k\in\mathbb{N}} \left(\sum_{m=k}^{\infty} \frac{v_m}{m^{(1-\alpha)q}} \right)^{1/q} k^{1/p'} < \infty;$$

(iii)

$$\sup_{k\in\mathbb{Z}_+} \left(\sum_{m=2^k}^{2^{k+1}} v_m \right)^{1/q} 2^{k(\alpha p-1)} < \infty.$$

Our purpose in this paper is to derive the results similar to Theorems B and C in the discrete case and to derive criteria on a weight sequence $\{v_j\}_j$ guaranteeing the boundedness of I_α from $l^p(\mathbb{N})$ to $l^q_{v_k}(\mathbb{N})$ in the case when $0 < \alpha < 1/p$. As we shall see in this case conditions on $\{v_j\}_{j\in\mathbb{N}}$ are different to the criteria in Theorem D.

Throughout the paper the symbol $\mathbb{N}$ means the set of natural numbers; $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$; $p' := \frac{p}{p-1}$, where $1 < p < \infty$; $\mathbb{R}_+ := [0, \infty)$; the characteristic sequence $\chi_{\{i:a\le i\le b\}}$ (a and b are positive integers) is defined in the usual way:

$$\chi_{\{i:a\le i\le b\}} = \begin{cases} 1 & a \le i \le b; \\ 0 & \text{elsewhere}. \end{cases}$$

The operator formal adjoint to I'_α is given by the formula:

$$\{I'_\alpha \beta_k\}_n = \sum_{k=n}^{\infty} \frac{\beta_k}{(k-n+1)^{1-\alpha}}, \qquad n \in \mathbb{N}.$$

Finally we point out that constants (often different constants in the same series of inequalities) will generally be denoted by c.

2. The main results

Now we formulate the main results of this paper.

Theorem 2.1. *[Adams type characterization] Let* $1 < p < q < \infty$ *and* $0 < \alpha < 1/p$. *Then* I_α *is bounded from* $l^p(\mathbb{N})$ *to* $l^q_{v_k}(\mathbb{N})$ *if and only if*

$$B := \sup_{m,j\in\mathbb{N};\, j\le m} \left(\sum_{k=m}^{m+j} v_k\right)^{1/q} j^{\alpha-1/p} < \infty\,.$$

Theorem 2.2. *[Maz'ya–Verbitsky type characterization] Let* $1 < p < \infty$ *and let* $0 < \alpha < 1/p$. *Then the inequality*

$$\sum_{i=1}^{\infty} \Big(I_\alpha g_j\Big)_i^p v_i \le c \sum_{i=1}^{\infty} g_i^p \tag{1}$$

holds for all non-negative sequences $\{g_i\}_i$ *if and only if* $\{I'_\alpha v_i\}_i < \infty$ *for all* $i \in \mathbb{N}$ *and there is a positive constant* c *such that*

$$\Big\{I'_\alpha [I'_\alpha v_j]^{p'}\Big\}_i \le c \Big\{I'_\alpha v_j\Big\}_i \tag{2}$$

for all $i \in \mathbb{N}$.

3. Proof of the main results

Let $(X, \mathcal{U}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces with ν being σ- finite. Suppose that $k(x, y)$ is a non–negative real–valued $\mathcal{U} \times \mathcal{B}$– measurable function and that

$$Kf(y) = \int_X k(x, y) f(x) d\mu(x)$$

is the kernel operator.

Denote:

$$e_\lambda(x) := \{y \in Y : k(x, y) > \lambda\}, \;\; e_\lambda(y) := \{x \in X : k(x, y) > \lambda\},$$

where λ is a positive number;

$$M_r(\mu)(y) := \sup_{\lambda>0} \lambda^r \mu\big(e_\lambda(y)\big); \;\; M_s(\nu)(x) := \sup_{\lambda>0} \lambda^s \nu\big(e_\lambda(x)\big),$$

where r and s are real numbers.

To prove Theorem 2.1 we use the following statement which is a corollary of part (ii) of Theorem A in [1].

Theorem E. *Suppose that* $1 < p < q < \infty$, $\frac{s}{q} = \frac{r}{p} + 1 - r$, *where* $r > 0$. *If* $M_r(\mu)(y) \le A < \infty$ *for all* $y \in Y$*;* $M_s(\nu)(x) \le B < \infty$ *for all* $x \in X$, *then the operator* K *is bounded from* $L^p(X, \mu)$ *to* $L^q(Y, \nu)$, *where* $L^p(X, \mu)$ $L^q(Y, \nu)$ *are Lebesgue spaces defined with respect to the measures* μ *and* ν *respectively.*

Proof of Theorem 2.1. *Sufficiency.* Suppose that $X = Y = \mathbb{N}$, μ is the counting measure on $\mathbb{N}$ and that $d\nu(n) = v_n d\mu(n)$, where $\{v_n\}_{n=1}^{\infty}$ is the weight sequence. In our case the kernel operator is given by

$$\{I_\alpha\{g_m\}\}_n = \sum_{m=1}^{\infty} k(m,n) g_m, \quad n \in \mathbb{N},$$

where

$$k(m,n) = \chi_{\{m\in\mathbb{N}:1\le m\le n\}}(n-m+1)^{\alpha-1}.$$

Let $r = \frac{1}{1-\alpha}$ and let $\frac{s}{q} = \frac{r}{p} + 1 - r$. That is $s = \frac{q(\alpha-1/p)}{\alpha-1} > 0$. We have

$$\begin{aligned}
\sup_{n\in\mathbb{N}} M_r(\mu)(n) &= \sup_{\lambda\le 1, n\in\mathbb{N}} \lambda^r \mu\{m\in\mathbb{N} : m\le n; (n-m+1)^{\alpha-1} > \lambda\} \\
&= \sup_{\lambda\ge 1, n\in\mathbb{N}} \lambda^{r(\alpha-1)} \mu\{m\in\mathbb{N} : m\le n; n-m+1 < \lambda\} \\
&\le \sup_{k,n\in\mathbb{N}} k^{-1} \sum_{m=\max\{n-k,1\}}^{n} 1 \le c.
\end{aligned}$$

Further,

$$\begin{aligned}
\sup_{m\in\mathbb{N}} M_s(\nu)(m) &= \sup_{\lambda\le 1, m\in\mathbb{N}} \lambda^s \nu\{n\in\mathbb{N} : m\le n; (n-m+1)^{\alpha-1} > \lambda\} \\
&= \sup_{\lambda\ge 1, m\in\mathbb{N}} \lambda^{s(\alpha-1)} \nu\{n\in\mathbb{N} : m\le n; n-m+1 < \lambda\} \\
&\le \sup_{k,m\in\mathbb{N}} k^{s(\alpha-1)} \sum_{n=m}^{m+k} v_n \le c \sup_{n\le j;\, n,j\in\mathbb{N}} S_{n,j},
\end{aligned}$$

where

$$S_{n,j} := (j-n+1)^{q(\alpha-1/p)} \sum_{m=n}^{j} v_m.$$

Further, let m, n be positive integers satisfying the condition $1 \le m \le n$. Then there exists a non-negative integer k_0 such that $2^{k_0} \le n-m+1 \le 2^{k_0+1}$. Therefore by using the fact that $0 < \alpha < 1/p$, we obtain that

$$\begin{aligned}
S_{m,n} &= \left(\sum_{k=m}^{n} v_k\right)(n-m+1)^{(\alpha-1/p)q} \\
&\le \sum_{l=1}^{k_0+2} \left(\sum_{k=m+2^{l-1}}^{m+2^l} v_k\right)(2^{k_0+1})^{(\alpha-1/p)q}
\end{aligned}$$

$$\le \sum_{l=1}^{k_0+2} \left(\sum_{k=m+2^{l-1}}^{m+2^l} v_k\right) 2^{l(\alpha-1/p)q} 2^{l(1/p-\alpha)q} (2^{k_0+1})^{(\alpha-1/p)q} \le cB^q.$$

Necessity: Let

$$\beta^{(m,j)} = \begin{cases} 1 & \text{if } m-j<k\le m; \\ 0 & \text{otherwise,} \end{cases}$$

where m, j are positive integers such that $1 \le j \le m$. Then we have

$$\begin{aligned} \left(\sum_{n=1}^{\infty} v_n \left(\sum_{k=1}^{n} \frac{(\beta^{(m,j)})_k}{(n-k+1)^{1-\alpha}}\right)^q\right)^{1/q} &\ge \left(\sum_{n=m}^{m+j} v_n \left(\sum_{k=m-j}^{m} \frac{1}{(n-k+1)^{1-\alpha}}\right)\right)^{1/q} \\ &\ge c\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha}. \end{aligned}$$

Therefore, by the boundedness of I_α we conclude that

$$\left(\sum_{n=m}^{m+j} v_n\right)^{1/q} j^{\alpha-1/p} \le c, \quad 1 \le j \le m.$$

□

To prove Theorem 2.2 we need to prove some auxiliary statements.

Proposition 3.1. *Let $1<p<\infty$, and let $0<\alpha<1/p$. If I_α is bounded from $l^p(\mathbb{N})$ to $l^p_{v_i}(\mathbb{N})$ then there exist a positive constant c such that*

$$\sum_{i=m}^{m+h} v_i \le c\, h^{1-\alpha p} \tag{3}$$

holds for all positive integers m and h.

Proof. First we suppose that $h \le m$. For the sequence $g^{(m,h)} = \chi_{\{k:\, m-h<k\le m\}}$ we have

$$\begin{aligned} \left(\sum_{i=1}^{\infty} v_i \Big(\sum_{j=1}^{i} \frac{\big(g^{(m,h)}\big)_j}{(i-j+1)^{1-\alpha}}\Big)^p\right)^{1/p} &\ge \left(\sum_{i=m}^{m+h} v_i \Big(\sum_{j=m-h+1}^{m} (i-j+1)^{\alpha-1}\Big)^p\right)^{1/p} \\ &\ge c\Big(\sum_{i=m}^{m+h} v_i\Big)^{1/p} h^{\alpha}. \end{aligned}$$

Therefore, by the boundedness of I_α we get

$$\Big(\sum_{i=m}^{m+h} v_i\Big)^{1/p} \le c h^{1/p-\alpha}.$$

Hence (3) holds for all positive integers m and h satisfying $h \le m$. Now let $m<h$. Then there exist a positive integer k such that $2^k \le h \le 2^{k+1}$.

Therefore taking into account the condition $0<\alpha<1/p$ we obtain

$$\begin{aligned}\sum_{i=m}^{m+h} v_i &\le \sum_{i=1}^{k}\Big(\sum_{j=m+2^{i-1}}^{m+2^i} v_j\Big)\\ &= \sum_{i=1}^{k}\Big[\Big(\sum_{j=m+2^{i-1}}^{m+2^i} v_j\Big)2^{i(1-\alpha p)}2^{i(\alpha p-1)}\Big]\\ &\le c\sum_{i=1}^{k}2^{i(1-\alpha p)} \le c\,2^{k(1-\alpha p)} \le c\,h^{1-\alpha p}.\end{aligned}$$

□

Proof of necessity of Theorem 2.2. Let us first show that, from (1) it follows that $\{I'_\alpha v_j\}_k<\infty$ for all $k\in\mathbb{N}$. By the duality arguments (1) is equivalent to the inequality

$$\sum_{i=1}^{\infty}\Big(I'_\alpha g_j\Big)_i^{p'} \le c\sum_{i=1}^{\infty} g_i^{p'} v_i^{1-p'}. \tag{4}$$

Let $v_i^{(1)}=v_i\chi_{\{i:\, m\le i<m+2h\}}$ and $v_i^{(2)}=v_i\chi_{\{i:\, 1\le i<m \text{ or } i\ge m+2h\}}$, where m and h are positive integers.
Note that for $k\ge m+2h-1$ and $m\le i\le m+h$, we have $k-m+1\le 2(k-i+1)$. Further, by using (3), we come to the estimates:

$$\begin{aligned}\{I'_\alpha v_j^{(2)}\}_i &\le \sum_{k=m+2h-1}^{\infty} v_k(k-i+1)^{\alpha-1}\\ &\le c\sum_{k=m+h}^{\infty} v_k(k-m+1)^{\alpha-1}\\ &\le c\sum_{k=m+h}^{\infty} v_k\Big(\sum_{j=k-m+1}^{\infty} j^{\alpha-2}\Big)\\ &= c\sum_{j=h+1}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{j+m-1} v_k\Big)\\ &\le c\sum_{j=h+1}^{\infty} j^{\alpha-2}.j^{1-\alpha p}<\infty.\end{aligned}$$

Therefore $\big(I'_\alpha v_j^{(2)}\big)_i<\infty$. The fact that $\big(I'_\alpha v_j^{(1)}\big)_i<\infty$ is obvious. Thus, $\big(I'_\alpha v_j\big)_i<\infty$ for every $i\in\mathbb{N}$ because m and h are taken arbitrarily.

Now we prove that (1) yields (2). For this we need some lemmas.

Lemma 3.2. *Let $0<\alpha<1$. Then there are positive constants $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ depending only on α such that for all $m\in\mathbb{N}$ the inequality*

$$(I'_\alpha\beta_s)_m \le c_\alpha^{(1)}\sum_{j=1}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{m+j-1}\beta_k\Big)\le c_\alpha^{(2)}\,(I'_\alpha\beta_s)_m$$

holds, where $\beta_m\ge 0$.

Proof. The proof follows easily if we observe that there are positive constants $b_\alpha^{(1)}$ and $b_\alpha^{(2)}$ independent of k and m such that

$$\sum_{j=k-m+1}^{\infty} j^{\alpha-2}\le b_\alpha^{(1)}(k-m+1)^{\alpha-1}\le b_\alpha^{(2)}\sum_{j=k-m+1}^{\infty} j^{\alpha-2}.$$

It remains to change the order of summation.

□

Corollary 3.3. *Let $0<\alpha<1$, $\beta_m\ge 0$. Then there are positive constants $c_\alpha^{(1)}$ and $c_\alpha^{(2)}$ depending only on α such that for all $m\in\mathbb{N}$ the inequality*

$$\Big\{I'_\alpha[I'_\alpha\beta_s]^{p'}\Big\}_m\le c_\alpha^{(1)}\sum_{j=1}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{m+j-1}\{I'_\alpha\beta_s\}_k^{p'}\Big)\le c_\alpha^{(2)}\Big\{I'_\alpha[I'_\alpha\beta_s]^{p'}\Big\}_m$$

holds.

Let $\{v_i^{(1)}\}$ and $\{v_i^{(2)}\}$ be defined as above. Then by using (4) we have that

$$\sum_{i=m}^{m+h}\Big(I'_\alpha v_j^{(1)}\Big)_i^{p'}\le c\sum_{i=m}^{m+h} v_i.$$

Thus, by Corollary 3.3 we conclude that

$$\Big\{I'_\alpha[I'_\alpha v_k^{(1)}]^{p'}\Big\}_i\le c\sum_{j=1}^{\infty} j^{\alpha-2}\Big(\sum_{k=i}^{i+2(j-1)} v_k\Big)\le c\Big\{I'_\alpha v_s\Big\}_i.$$

For the estimate of $\Big\{I'_\alpha[I'_\alpha v_k^{(2)}]^{p'}\Big\}_i$, we need some auxiliary statements.

Lemma 3.4. *Let $0<\alpha<1$. Then there is a positive constant c such that for all natural numbers m,k and j with $m\le k\le m+j-1$, the inequality*

$$\Big\{I'_\alpha v_s^{(2)}\Big\}_k\le c\sum_{s=j}^{\infty} s^{\alpha-2}\Big(\sum_{t=m}^{m+s-1} v_t\Big)$$

holds, where $v_s^{(2)}=v_s\chi_{\{s:\,1\le s<m \text{ or } s\ge m+2j\}}$.

Proof. Using the arguments of the proof of Lemma 3.2 and the fact that

$$\Big(I'_\alpha v_s^{(2)}\Big)_k=\sum_{s=m+2j}^{\infty} v_s(s-k+1)^{\alpha-1}$$

we have

$$\begin{aligned}\left(I'_\alpha v_s^{(2)}\right)_k &\le c\sum_{s=m+2j}^{\infty} v_s(s-m+1)^{\alpha-1}\\ &\le c\sum_{s=m+2j}^{\infty} v_s \sum_{t=s-m+1}^{\infty} t^{\alpha-2}\end{aligned}$$

$$\le c\sum_{t=j}^{\infty} t^{\alpha-2}\Big(\sum_{s=m}^{m+t-1} v_s\Big).$$

□

Lemma 3.5. *Let $0<\alpha<1$. Then there is a positive constant c such that*

$$\Big\{I'_\alpha[I'_\alpha v_i^{(2)}]^{p'}\Big\}_m \le c\sum_{t=1}^{\infty} t^{\alpha-1}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\Big(\sum_{j=m}^{m+s-1} v_j\Big)\Big)^{p'}$$

Proof. Using Lemma 3.4 in Corollary 3.3 we have that

$$\begin{aligned}\Big\{I'_\alpha[I'_\alpha v_i^{(2)}]^{p'}\Big\}_m &\le c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{k=m}^{m+t-1}\{I'_\alpha v_k\}^{p'}\Big)\\ &\le c\sum_{t=1}^{\infty} t^{\alpha-2}\sum_{k=m}^{m+t-1}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\epsilon=m}^{m+s-1} v_\epsilon\Big)^{p'}\\ &\quad \text{(the inner sum does not depend on k)}\\ &= c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\epsilon=m}^{m+s-1} v_\epsilon\Big)^{p'}\Big(\sum_{k=m}^{m+t-1} 1\Big)\\ &= c\sum_{t=1}^{\infty} t^{\alpha-2}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\epsilon=m}^{m+s-1} v_\epsilon\Big)^{p'}.\end{aligned}$$

□

Lemma 3.6. *Let $0<\alpha<1$. Then there is a positive constant c such that*

$$\Big\{I'_\alpha[I'_\alpha v_i^{(2)}]^{p'}\Big\}_m \le c\sum_{t=1}^{\infty} t^{\alpha}\Big(\sum_{s=t}^{\infty} s^{\alpha-2}\sum_{\epsilon=m}^{m+s+1} v_\epsilon\Big)^{p'-1}\Big(t^{\alpha-2}\sum_{j=m}^{m+t+1} v_j\Big).$$

Proof. We will deduce the discrete case to the continuous case. Let $v(x)=v_j$, $j\le x<j+1$. Then $\int_j^{j+1} v(x)dx=v_j$. Hence, by using lemmas proved above, the Lebesgue differentiation theorem and integration by parts, we

find that

$$\begin{aligned}\Big\{I'_\alpha[I'_\alpha v_i^{(2)}]^{p'}\Big\}_m &\leq c\sum_{n=1}^{\infty} n^{\alpha-1}\Big(\sum_{j=n}^{\infty} j^{\alpha-2}\Big(\sum_{k=m}^{m+2j} v_k\Big)\Big)^{p'}\\ &\leq c\sum_{n=1}^{\infty}\int_n^{n+1} x^{\alpha-1}\Big(\sum_{i=2n}^{\infty}\int_i^{i+1} y^{\alpha-2}\Big(\sum_{k=m}^{m+y} v_k\Big)dy\Big)^{p'}dx\\ &\leq c\int_1^{\infty} x^{\alpha-1}\Big(\int_x^{\infty} y^{\alpha-2}\Big(\sum_{k=m}^{m+y} v_k\Big)dy\Big)^{p'}dx\end{aligned}$$

$$\begin{aligned} &= c\Big[\frac{x^\alpha}{\alpha}\Big(\int_x^{\infty}\cdots\Big)^{p'}\Big|_1^{\infty} + \int_1^{\infty} x^\alpha\Big(\int_x^{\infty}\cdots\Big)^{p'-1} x^{\alpha-2}\Big(\sum_{k=m}^{m+x} v_k\Big)dx\Big]\\ &= c\Big[-\frac{1}{\alpha}\Big(\int_1^{\infty}\cdots\Big)^{p'} + \int_1^{\infty} x^\alpha\Big(\int_x^{\infty}\cdots\Big)^{p'-1} x^{\alpha-2}\Big(\sum_{k=m}^{m+x} v_k\Big)dx\Big]\\ &\leq c\int_1^{\infty} x^\alpha\Big(\int_x^{\infty}\cdots\Big)^{p'-1} x^{\alpha-2}\Big(\sum_{k=m}^{m+x} v_k\Big)dx\\ &= c\sum_{n=1}^{\infty}\int_n^{n+1} x^\alpha\Big(\int_x^{\infty}\cdots\Big)^{p'-1} x^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big)dx\\ &\leq c\sum_{n=1}^{\infty} n^\alpha\Big(\int_n^{\infty}\cdots\Big)^{p'-1} n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big)\\ &\leq c\sum_{n=1}^{\infty} n^\alpha\Big(\sum_{k=n}^{\infty}\int_k^{k+1} k^{\alpha-2}\Big(\sum_{i=m}^{m+k+1} v_i\Big)dy\Big)^{p'-1} n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big)\\ &= c\sum_{n=1}^{\infty} n^\alpha\Big(\sum_{k=n}^{\infty} k^{\alpha-2}\Big(\sum_{i=m}^{m+k+1} v_i\Big)\Big)^{p'-1} n^{\alpha-2}\Big(\sum_{k=m}^{m+n+1} v_k\Big).\end{aligned}$$

□

Now necessity of Theorem 2.2 follows easily because we know that the trace inequality implies (see Proposition 3.1)

$$\sum_{k=m}^{m+j} v_k \leq c\, j^{1-\alpha p},$$

where the positive constant c is independent of positive integers m and j. Indeed, by using this inequality in Lemma 3.6 we have that

$$\begin{aligned}\Big\{I'_\alpha[I'_\alpha v_j^{(2)}]^{p'}\Big\}_m &\leq c\sum_{n=1}^{\infty} n^{\alpha}\Big(\sum_{k=n}^{\infty} k^{\alpha-2}(k+2)^{1-\alpha p}\Big)^{p'-1}\Big(n^{\alpha-2}\sum_{k=m}^{m+n+1} v_k\Big)\\ &\leq c\sum_{n=1}^{\infty} n^{\alpha-2}\sum_{k=m}^{m+n+1} v_k\\ &\leq c\sum_{n=1}^{\infty}(3n)^{\alpha-2}\sum_{k=m}^{m+n+1} v_k\\ &\leq c\sum_{j=3}^{\infty}[3(j-2)]^{\alpha-2}\sum_{k=m}^{m+j-1} v_k\\ &\leq c\sum_{j=3}^{\infty} j^{\alpha-2}\sum_{k=m}^{m+j-1} v_k\\ &\leq c\Big\{I'_\alpha v_j\Big\}_m.\end{aligned}$$

In the last inequality we used Lemma 3.2, in particular, the estimate from below.

Necessity of Theorem 2.2 is proved.

Now we prove *sufficiency* of Theorem 2.2. We will need some auxiliary statements.

Lemma 3.7. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Then there exists a positive constant c such that for all non-negative sequences $\{g_i\}_{i\in\mathbb{Z}}$ and all $i \in \mathbb{N}$, the following inequality holds*

$$\{I_\alpha g_k\}_i^p \leq c\{I_\alpha[I_\alpha g_k]_j^{p-1} g_m\}_i, \tag{5}$$

Proof. First we assume that $\{V_\alpha g_j\}_i := \{I_\alpha[I_\alpha g_k]^{p-1}g_j\}_i$ and

$$\{V_\alpha g_j\}_i \leq \{I_\alpha g_j\}_i^p;$$

otherwise (5) is obvious for $c = 1$. Now let us assume that $1 < p \le 2$. Then we have

$$\begin{aligned}\{I_\alpha g_k\}_i^p &= \sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}\\ &\le \sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=1}^{k}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}\\ &\quad+\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1}\\ &\equiv I_i^{(1)}+I_i^{(2)}.\end{aligned}$$

It is obvious that if $j \le k \le i$, then $k-j+1 \le i-j+1$. Consequently,

$$I_i^{(1)} \le \sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=1}^{k}(k-j+1)^{\alpha-1}g_j\Big)^{p-1} = \{V_\alpha g_k\}_i.$$

Now we use Hölder's inequality with respect to the exponents $\frac{1}{p-1}$, $\frac{1}{2-p}$ and measure $d\mu(k) = (i-k+1)^{\alpha-1}g_k\,d\mu_c(k)$ (here μ_c is the counting measure). We have

$$\begin{aligned}I_i^{(2)} &\le \Big(\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big)^{2-p}\Big(\sum_{k=1}^{i}\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)(i-k+1)^{\alpha-1}g_k\Big)^{p-1}\\ &= \{I_\alpha g_k\}_i^{2-p}(J_i)^{p-1},\end{aligned}$$

where

$$J_i \equiv \sum_{k=1}^{i}\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)(i-k+1)^{\alpha-1}g_k.$$

Using Fubini's theorem we find that

$$J_i = \sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\Big(\sum_{k=1}^{j}(i-k+1)^{\alpha-1}g_k\Big).$$

Further, it is obvious that the following estimates

$$\begin{aligned}\sum_{k=1}^{j}(i-k+1)^{\alpha-1}g_k &\le \Big(\sum_{k=1}^{j}(i-k+1)^{\alpha-1}g_k\Big)^{p-1}\Big\{I_\alpha g_k\Big\}_i^{2-p}\\ &\le \{I_\alpha g_k\}_j^{p-1}\{I_\alpha g_k\}_i^{2-p}\end{aligned}$$

hold, where $j \le i$. Taking into account the last estimate, we obtain

$$\begin{aligned}J_i &\le \Big(\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\,\{I_\alpha g_k\}_j^{p-1}\Big)\Big\{I_\alpha g_k\Big\}_i^{2-p}\\ &= \{V_\alpha g_k\}_i\{I_\alpha g_k\}_i^{2-p}.\end{aligned}$$

Thus,

$$\begin{aligned} I_i^{(2)} &\leq \{I_\alpha g_k i\}_i^{2-p}\{I_\alpha g_k\}_i^{(2-p)(p-1)}\{V_\alpha g_k\}_i^{p-1} \\ &= \{I_\alpha g_k\}_i^{p(2-p)}\{V_\alpha g_k\}_i^{p-1}. \end{aligned}$$

Combining the estimate for $I^{(1)}$ and $I^{(2)}$ we derive

$$\{I_\alpha g_k\}_i^p \leq \{V_\alpha g_k\}_i + \{I_\alpha g_k\}_i^{p(2-p)}\{V_\alpha g_k\}_i^{p-1}.$$

As we have assumed that $\{V_\alpha g_k\}_i \leq \{I_\alpha g_k\}_i^p$, we obtain

$$\{V_\alpha g_k\}_i = \{V_\alpha g_k\}_i^{2-p}\{V_\alpha g_k\}_i^{p-1} \leq \{V_\alpha g_k\}_i^{p-1}\{I_\alpha g_k\}_i^{p(2-p)}.$$

Hence

$$\begin{aligned} \{I_\alpha g_k\}_i^p &\leq \{V_\alpha g_k\}_i^{p-1}\{I_\alpha g_k\}_i^{p(2-p)} + \{V_\alpha g_k\}_i^{p-1}\{I_\alpha g_k\}_i^{p(2-p)} \\ &= 2\{V_\alpha g_k\}_i^{p-1}\{I_\alpha g_k\}_i^{p(2-p)}. \end{aligned}$$

Applying the fact $\big(I_\alpha g_j\big)_i < \infty$ we find that

$$\{I_\alpha g_k\}_i^p \leq 2^{\frac{1}{p-1}}\{V_\alpha g_k\}_i.$$

Now we deal with the case $p > 2$. Let us assume again that

$$\{V_\alpha g_j\}_i \leq \{I_\alpha g_j\}_i^p.$$

Since $p > 2$ we have

$$\begin{aligned} \{I_\alpha g_k\}_i^p &= \sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1} \\ &\leq 2^{p-1}\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=1}^{k}(i-j+1)^{\alpha-1}g_j\Big)^{p-1} \\ &\quad +2^{p-1}\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1} \\ &=: 2^{p-1}I_i^{(1)} + 2^{p-1}I_i^{(2)}. \end{aligned}$$

It is clear that if $j \leq k \leq i$, then $(i-j+1)^{\alpha-1} \leq (k-j+1)^{\alpha-1}$. Therefore like the case $1 < p \leq 2$ we have that $I_i^{(1)} \leq \{V_\alpha g_k\}_i$.

Now we estimate $I_i^{(2)}$. We obtain

$$\begin{aligned} \Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-1} &= \Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{p-2}\Big(\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j\Big) \\ &\leq \Big\{I_\alpha g_j\Big\}_i^{p-2}\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j. \end{aligned}$$

Using Fubini's theorem and the last estimate we have

$$\begin{aligned} I_i^{(2)} &\le \Big\{I_\alpha g_j\Big\}_i^{p-2}\sum_{k=1}^{i}(i-k+1)^{\alpha-1}g_k\sum_{j=k}^{i}(i-j+1)^{\alpha-1}g_j \\ &= \Big\{I_\alpha g_j\Big\}_i^{p-2}\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=1}^{j}(i-k+1)^{\alpha-1}g_k \\ &\le \Big\{I_\alpha g_j\Big\}_i^{p-2}\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=1}^{j}(j-k+1)^{\alpha-1}g_k. \end{aligned}$$

Due to Hölder's inequality with respect to the exponents $\{p-1, \frac{p-1}{p-2}\}$ and the measure $d\mu(j) = (i-j+1)^{\alpha-1}g_j d\mu_c(j)$ (μ_c is the counting measure) we derive

$$\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\sum_{k=1}^{j}(j-k+1)^{\alpha-1}g_k \le \Big(\sum_{j=1}^{i}(i-j+1)^{\alpha-1}g_j\Big)^{\frac{p-2}{p-1}}$$

$$\times\Bigg(\sum_{j=1}^{i}\Big(\sum_{k=1}^{j}(j-k+1)^{\alpha-1}g_k\Big)^{p-1}(i-j+1)^{\alpha-1}g_j\Bigg)^{\frac{1}{p-1}}$$

$$= \{I_\alpha g_j\}_i^{\frac{p-2}{p-1}}\{V_\alpha g_j\}_i^{\frac{1}{p-1}}.$$

Combining these estimates we obtain

$$\{I_\alpha g_j\}_i^p \le 2^{p-1}\{V_\alpha g_j\}_i + 2^{p-1}\{I_\alpha g_j i\}_i^{\frac{p(p-2)}{p-1}}\{V_\alpha g_j\}_i^{\frac{1}{p-1}}.$$

By virtue of the inequality $\{V_\alpha g_i\}_i \le \{I_\alpha g_j\}_i^p$ it follows that

$$\{V_\alpha g_j\}_i = \{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{V_\alpha g_j\}_i^{\frac{p-2}{p-1}} \le \{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.$$

Hence

$$\{I_\alpha g_j\}_i^p \le 2^{p-1}\Big(\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}} + \{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}\Big)$$

$$= 2^p\{V_\alpha g_j\}_i^{\frac{1}{p-1}}\{I_\alpha g_j\}_i^{\frac{p(p-2)}{p-1}}.$$

Further, from the last estimate we conclude that

$$\{I_\alpha g_j\}_i^p \le 2^{p(p-1)}\{V_\alpha g_j\}_i,$$

where $2 < p < \infty$. □

Lemma 3.8. *Let $1 < p < \infty$, $0 < \alpha < 1$ and v_i be a sequence of positive numbers on $\mathbb{N}$. Let there exist a constant $c > 0$ such that the inequality*

$$\|I_\alpha\{g_i\}\|_{l^p_{v_i^{(1)}}(\mathbb{N})} \le c_1\|g_i\|_{l^p(\mathbb{N})}, \quad \{v_s^{(1)}\}_i = \{I'_\alpha v_s\}_i^{p'}$$

holds for all sequences $g_i \in l^p(\mathbb{N})$. *Then*

$$\|I_\alpha\{g_i\}\|_{l^p_{v_i}(\mathbb{N})} \le c_2 \|g_i\|_{l^p(\mathbb{N})}, \quad g_i \in l^p(\mathbb{N}),$$

where $c_2 = c_1^{1/p'} c^{1/p}$.

Proof. Let $g_i \ge 0$. Using Lemma 3.7, Fubini's theorem and Hölder's inequality we derive

$$\begin{aligned}
\sum_{k=1}^{\infty}\{I_\alpha g_s\}_k^p v_k &\le c\sum_{k=1}^{\infty}\sum_{i=1}^{k}\{I_\alpha g_j\}_i^{p-1} g_i (k-i+1)^{\alpha-1} v_k \\
&= c\sum_{i=1}^{\infty}\{I_\alpha g_j\}_i^{p-1} g_i \{I'_\alpha v_j\}_i \\
&\le c\Big(\sum_{i=1}^{\infty} g_i^p\Big)^{1/p}\Big(\sum_{i=1}^{\infty}\{I_\alpha g_j\}_i^p v_i^{(1)}\Big)^{1/p'} \\
&= c\,\|g_i\|_{l^p(\mathbb{N})}\,\|I_\alpha g_i\|_{l^p_{v_i^{(1)}}(\mathbb{N})}^{p-1} \\
&\le c_1^{p-1} c\,\|g_i\|_{l^p(\mathbb{N})}\,\|g_i\|_{l^p(\mathbb{N})}^{p-1} \\
&= c_1^{p-1} c\,\|g_i\|_{l^p(\mathbb{N})}^{p}
\end{aligned}$$

Hence,

$$\|I_\alpha g_j\|_{l^p_{v_i}(\mathbb{N})} \le c_1^{1/p'} c^{1/p} \|g_j\|_{l^p(\mathbb{N})}.$$

□

Lemma 3.9. *Let* $0 < \alpha < 1$ *and* $1 < p < \infty$. *Suppose that* $\{I'_\alpha v_s\}_i < \infty$ *and*

$$\Big\{I'_\alpha [I'_\alpha v_s]^{p'}\Big\}_i \le c\Big\{I'_\alpha v_i\Big\}_i$$

for all $i \in \mathbb{N}$. *Then we have*

$$\|I_\alpha\{g_i\}\|_{l^p_{v_i^{(1)}}(\mathbb{N})} \le c\,\|g_i\|_{l^p(\mathbb{N})}, \quad g_i \in l^p(\mathbb{N}), \tag{6}$$

where $\{v_s^{(1)}\}_i = \{I'_\alpha v_s\}_i^{p'}$.

Proof. Let $g_i \ge 0$ and let g_i be supported on the set $E_m := \{i : 1 \le i \le m\}$, where m is a natural number. Let $t_{i,j}^{(n)} = \chi_{\{j:1\le j\le i\}} \min\{(i-j+1)^{\alpha-1}, n\}$, $n \in \mathbb{N}$. Then using Lemma 3.7 (which is true also for the kernel $t_{i,j}^{(n)}$), Fubini's theorem and Hölder's inequality we obtain the following chain of inequalities:

$$\begin{aligned}\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{i} t_{i,j}^{(n)} g_j\Big)^p v_i^{(1)} &\leq c\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{i} t_{i,j}^{(n)}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^{p-1} g_j\Big) v_i^{(1)} \\ &\leq c\sum_{j=1}^{\infty} g_j\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^{p-1}\Big(\sum_{i=j}^{\infty} t_{i,j}^{(n)} v_i^{(1)}\Big) \\ &\leq c\,\|g_i\|_{l^p(\mathbb{N})}\Big(\sum_{j=1}^{m}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^p\Big\{I'_\alpha[I'_\alpha v_s]^{p'}\Big\}_j^{p'}\Big)^{1/p'} \\ &\leq c\,\|g_i\|_{l^p(\mathbb{N})}\Big(\sum_{j=1}^{m}\Big(\sum_{k=1}^{j} t_{j,k}^{(n)} g_k\Big)^p\Big\{I'_\alpha v_s\Big\}_j^{p'}\Big)^{1/p'}.\end{aligned}$$

Since $\sum_{k=1}^{j} t_{j,k}^{(n)} g_k < \infty$ and $\{I'_\alpha v_s\}_j < \infty$ for all j, therefore we have that

$$\Big(\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{i} t_{i,j}^{(n)} g_j\Big)^p v_i^{(1)}\Big)^{1/p} \leq c\,\|g_i\|_{l^p(\mathbb{N})}\,.$$

Passing now by m and n to $+\infty$ we derive (6). □

Combining these lemmas we have also sufficiency of Theorem 2.2. Theorem 2.2 is completely proved.

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