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WITHDRAWAL AND DRAINAGE OF GENERALIZED SECOND GRADE FLUID ON VERTICAL CYLINDER WITH SLIP CONDITIONS

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Abstract. This paper investigates the steady thin film flows of an incompressible Generalized second grade fluid under the influence of non-isothermal effects. These thin films are considered for two different problems, namely, withdrawal and drainage problems. The governing continuity and momentum equations are converted into ordinary differential equations. These equations are solved analytically. Expressions for the velocity profile, temperature distribution, volume flux, average velocity and shear stress are obtained in both the cases. Effects of different parameters on velocity and temperature are presented graphically.

Key words: Thin film flow, withdrawal, drainage, Generalized second grade fluid.
AMS SUBJECT: 43.35.Ns, 68.60.Bs, 47.50.Gj.

1. Introduction

In recent years, the world has immense curiosity in the study of non-Newtonian fluids from both fundamental and practical points of view [1-2, 16-20]. The study of non-Newtonian fluids is very important because of its applications in several industrial and engineering processes. The familiar examples of such fluids are paint, shampoo, mud, ketchup, blood, polymer melts, certain oils and greases, clay coatings and many emulsions. Both theoretically and practically the flow analysis of such fluids is very vital. Theoretically speaking, flows of this type are essential in fluid mechanics. From practical

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point of view, these flows have applications in many manufacturing processes in industry. Non-Newtonian fluids are comprehensively studied by researchers primarily involving the analysis of the resultant differential equations. In applied sciences, such as physics or rheology of the atmosphere, approach to fluid mechanics is in an experimental set up leading to the measurement of material coefficients. Due to vast variety in the physical structure of non-Newtonian fluids, it is not easy to propose a single constitutive equation which exhibits all properties of non-Newtonian fluids. Therefore, a number of fluid models have been proposed to predict the non-Newtonian behavior of different types of materials. Amongst these, the Generalized second grade fluid model has attained special attention [9].

Recently, thin film flows have attracted the attention of various researchers. This is because of their applications in many manufacturing processes in industry. But no appropriate attention has been given to thin film flows concerning non-Newtonian fluids even though the literature on such flows is widespread for Newtonian fluids [3]. Siddiqui et al. [4, 7] and Hayat et al. [14, 15] have made few attempts dealing with thin film flows of non-Newtonian fluids.

Many researchers have been attracted by the flow and heat transfer inside thin films [10-12]. This is due to their vast applications in engineering and industry such as food stuff processing, fiber and wire coating, reactor fluidization, transpiration cooling, polymer processing, gaseous diffusion, heat pipes and fluidic cells of many biological and chemical detection systems. Lavrik et al. [13] considered the problem of chambers for chemical and biological detection systems such as fluidic cells for biological and chemical microcantilever. In majority of the problems related to flow and heat transfer studies, the power-law fluid model is taken as the non-Newtonian fluid. Only unpretentious interest has been devoted to the studies where the effects of viscous dissipation are incorporated, although its importance has been shown in many cases such as polymer processing.

In the present manuscript, we discuss thin film flow problems of a Generalized second grade fluid by using a vertically moving cylinder and again down a stationary vertical cylinder. In both cases, the approximate analytical solutions of the resulting differential equations are obtained subject to appropriate boundary conditions. In view of their practical weight, expressions for velocity profile, volume flux, average velocity and shear stress are also calculated. To the best of our knowledge these problems have not yet been reported. At the end graphical results with discussion are given for various physical parameters appearing in the solution. An important observation which we note is that Generalized second grade fluid shows the power law model results for steady case.

2. Basic Equations

The basic equations governing the flow of an incompressible, non-Newtonian fluid including thermal effects are:

$$div\mathbf{V} = 0, \tag{1}$$

$$\rho\frac{D\mathbf{V}}{Dt} = \rho\mathbf{f} - gradp + div\boldsymbol{\tau}, \tag{2}$$

$$\rho C_p\frac{D\Theta}{Dt} = \kappa\nabla^2\Theta + \boldsymbol{\tau}\cdot\mathbf{L}, \tag{3}$$

where $\mathbf{V}$ is velocity vector, $\frac{D}{Dt}$ is material time derivative defined as

$$\frac{D}{Dt}(*) = \left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)(*),$$

ρ is constant density, $\mathbf{f}$ is body force, p is dynamic pressure, Θ is temperature, C_p is specific heat constant, κ is thermal conductivity and $\boldsymbol{\tau}$ is extra stress tensor which is defined differently for different fluids. The extra stress tensor for Generalized second grade fluid is given by

$$\boldsymbol{\tau} = \mu_{eff}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{4}$$

where α_1 and α_2 are normal stress coefficients, μ_{eff} is effective viscosity η is flow consistency index, m is flow behavior index, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and second order Rivlin Ericksen tensors which are defined as

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T,\ \ \mathbf{L} = grad\ \mathbf{V},$$

$$A_2 = \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1.$$

For Generalized second grade fluid, μ_{eff} as a function of the shear rate is defined as

$$\mu_{eff} = \eta\left(\frac{1}{2}tr\mathbf{A}_1^2\right)^{\frac{m}{2}},$$

It is worthwhile to mention here that for $m < 0$, fluid is pseudoplastic or shear thinning, for $m > 0$ fluid is dilatant or shear thickening and for $m = 0$ we obtain second grade fluid model. On the other hand if $\alpha_1 = \alpha_2 = 0$, equation (4) reduces to the power-law model. Furthermore, if $m = \alpha_1 = \alpha_2 = 0$ we obtain the classical Newtonian model. It is important to note that the flow behavior index m has the limits $-1 < m < 1$ [8].

3. Withdrawal Problem

We consider a Generalized second grade fluid falling on the outer surface of an infinitely long vertical cylinder of radius R which moves vertically upward with constant speed w_0 as shown in figure 1(a). The flow is in the form of a thin uniform axisymmetric film of thickness δ in contact with stationary air. Gravity effect is in downward direction. We choose z-axis in the middle of cylinder and r normal to it. We suppose that motion is steady and there is no variation with respect to the component θ. Thus velocity field and temperature distribution are of the form

$$\mathbf{V} = [0, 0, w(r)] \quad \text{and} \quad \Theta = \Theta(r). \tag{5}$$

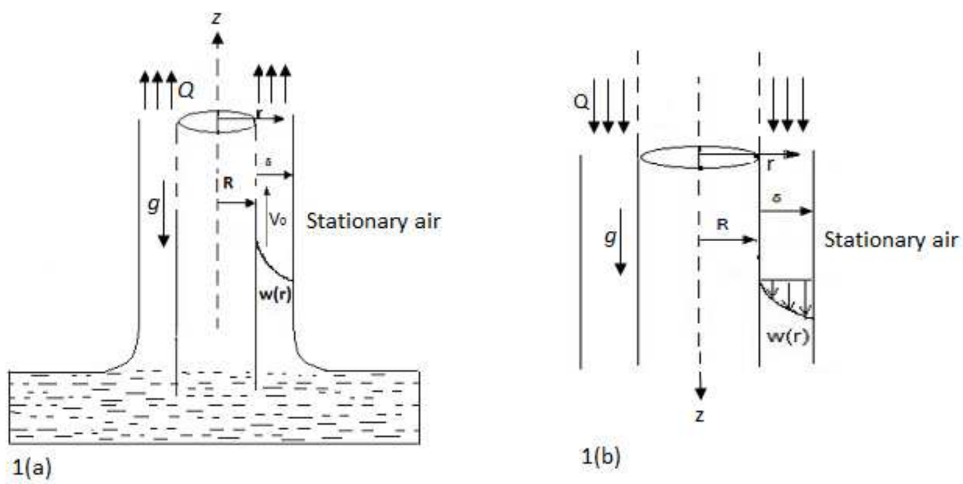

Fig. 1: Geometry of the problems

In view of the above profile (5), equation (1) is identically satisfied and equation (2) gives

$$r \quad -\text{component of momentum:} \quad 0 = -\frac{\partial p}{\partial r} + (2\alpha_1 + \alpha_2)\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{dw}{dr}\right)^2\right] \tag{6}$$

$$\theta \quad -\text{component of momentum:} \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \tag{7}$$

$$z \quad -\text{component of momentum:} 0 = -\rho g - \frac{\partial p}{\partial z} + \frac{\eta}{r}\frac{d}{dr}\left[r\left(\frac{dw}{dr}\right)^{m+1}\right]. \tag{8}$$

The velocity profile is obtained from equation (8). If we consider p, the atmospheric pressure (constant), then we can take $\frac{\partial p}{\partial z} = 0$, therefore equation (8) reduces to

$$\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{dw}{dr}\right)^{m+1}\right]=\frac{\rho g}{\eta}, \tag{9}$$

which is highly non-linear ordinary differential equation. Using profile (5) in the energy equation (3), we obtain

$$\kappa\left[\frac{d^2\Theta}{dr^2}+\frac{1}{r}\frac{d\Theta}{dr}\right]+\eta\left[\frac{dw}{dr}\right]^{m+2}=0. \tag{10}$$

The boundary conditions associated with these differential equations are

Free space boundary condition:

$$\tau_{rz}=0 \quad \text{and} \quad \frac{d\Theta}{dr}=0 \quad \text{at} \quad r=R+\delta, \tag{11}$$

Slip boundary condition:

$$w=w_0-\beta\tau_{rz}\,|_{r=R} \quad \text{and} \quad \Theta=\Theta_0 \quad \text{at} \quad r=R, \tag{12}$$

where

$$\tau_{rz}=\eta\left(\frac{dw}{dr}\right)^{m+1}.$$

Integrating equation (9) with respect to r and applying the free space boundary condition (11) we get

$$\frac{dw}{dr}=-\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}}\left[\frac{(R+\delta)^2}{r}-r\right]^{\frac{1}{m+1}}. \tag{13}$$

Using the Binomial series, equation (13) can also be written as

$$\frac{dw}{dr}=-\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}}(R+\delta)^{\frac{2}{m+1}}\sum_{i=0}^{\infty}\binom{\frac{1}{m+1}}{i}\frac{(-1)^i}{(R+\delta)^{2i}}\,r^{2i-\frac{1}{m+1}}. \tag{14}$$

3.1. **Solution of the withdrawal problem.**

3.1.1. *Generalized second grade fluid* ($m\neq 0$). For $m\neq 0$, solutions of equations (14) and (10), when boundary conditions (11) and (12) are applied, are

$$w=w_0+\frac{\beta\rho g}{2}\left(\frac{(R+\delta)^2}{R}-R\right)-\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}}\left(\sum_{i=0}^{\infty}\binom{\frac{1}{m+1}}{i}\frac{(-1)^i\ (R+\delta)^{-2i+\frac{2}{m+1}}}{2i+\frac{m}{m+1}}\right.$$
$$\left.\times\left[r^{2i+\frac{m}{m+1}}-R^{2i+\frac{m}{m+1}}\right]\right), \tag{15}$$

$$\Theta = \Theta_0 + \frac{\eta}{\kappa}\left(\frac{\rho g}{2\eta}\right)^{\frac{m+2}{m+1}} \sum_{i=0}^{\infty} \binom{\frac{m+2}{m+1}}{i} \frac{(-1)^i}{\left(2i + \frac{m}{m+1}\right)(R+\delta)^{2i-2\left(\frac{m+2}{m+1}\right)}}$$
$$\times \left[\frac{R^{2i+\frac{m}{m+1}}}{\left(2i+\frac{m}{m+1}\right)} \left\{ 1 - \left(\frac{r}{R}\right)^{2i+\frac{m}{m+1}} \right\} + (R+\delta)^{2i+\frac{m}{m+1}} \ln\left(\frac{r}{R}\right) \right]. \tag{16}$$

Volume flux, Q, in cylindrical coordinates, is given by

$$Q = \int_{R}^{R+\delta} \int_{0}^{2\pi} r\ w(r)\ d\theta\ dr. \tag{17}$$

Using profile (15), equation (17) becomes

$$Q = \pi w_0 \left((R+\delta)^2 - R^2\right) + \beta\pi \left(\frac{\rho g}{2}\right)\left(\frac{(R+\delta)^2}{R} - R\right)\left((R+\delta)^2 - R^2\right)$$
$$- 2\pi \left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}} \sum_{i=0}^{\infty} \binom{\frac{1}{m+1}}{i} \frac{(-1)^i (R+\delta)^{-2i+\frac{2}{m+1}}}{2i + \frac{m}{m+1}}$$
$$\times \left[\frac{(R+\delta)^{2i+\frac{3m+2}{m+1}} - R^{2i+\frac{3m+2}{m+1}}}{2i + \frac{3m+2}{m+1}} - \frac{R^{2i+\frac{m}{m+1}}}{2}\left((R+\delta)^2 - R^2\right) \right]. \tag{18}$$

The average film velocity, $\overline{V}$, is defined as

$$\bar{V} = \frac{Q}{\pi\left[(R+\delta)^2 - R^2\right]}. \tag{19}$$

Therefore $\bar{V}$ for the upward moving cylinder is given by

$$\bar{V} = w_0 + \beta\ \left(\frac{\rho g}{2}\right)\left(\frac{(R+\delta)^2}{R} - R\right) - 2\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}} \sum_{i=0}^{\infty} \binom{\frac{1}{m+1}}{i} \times$$
$$\frac{(-1)^i (R+\delta)^{-2i+\frac{2}{m+1}}}{2i + \frac{m}{m+1}} \left[\frac{\left((R+\delta)^{2i+\frac{3m+2}{m+1}} - R^{2i+\frac{3m+2}{m+1}}\right)}{\left((R+\delta)^2 - R^2\right)\left(2i + \frac{3m+2}{m+1}\right)} - \frac{R^{2i+\frac{m}{m+1}}}{2} \right]. \tag{20}$$

Shear stress on cylinder is

$$\tau_{rz}|_{r=R} = -\frac{\rho g (R+\delta)^2}{2R}\ \left[1 - \left(\frac{R}{R+\delta}\right)^2\right]. \tag{21}$$

Introducing the non-dimensional parameters

$$r^* = \frac{r}{R},\ w^* = \frac{w}{w_0},\ \delta^* = \frac{\delta}{R},\ \Theta^* = \frac{\Theta - \Theta_0}{\Theta_1 - \Theta_0},\ S_t = \frac{\rho g R^2}{w_0\, \mu_{eff}},\ B_r = \frac{w_0^2 \mu_{eff}}{\kappa(\Theta_1 - \Theta_0)}, \tag{22}$$

where Θ_1 is reference temperature, S_t is Stokes number and B_r is Brinkman number, equations (15) and (16), after dropping $'*'$, become

$$w = 1 + \frac{\beta\,\eta\,S_t}{2\,w_0}\left(\frac{(1+\delta)^2}{R} - R\right) - \left(\frac{S_t}{2}\right)^{\frac{1}{m+1}} \sum_{i=0}^{\infty} \binom{\frac{1}{m+1}}{i} \times \frac{(-1)^i\,(1+\delta)^{\frac{2}{m+1}-2i}}{\left(2i+\frac{m}{m+1}\right)}\left[r^{2i+\frac{m}{m+1}} - 1\right], \tag{23}$$

$$\Theta = B_r\left(\frac{S_t}{2}\right)^{\frac{m+2}{m+1}} \sum_{i=0}^{\infty} \binom{\frac{m+2}{m+1}}{i} \frac{(-1)^i}{\left(2i+\frac{m}{m+1}\right)(1+\delta)^{2i}} \times \left[\frac{1}{\left(2i+\frac{m}{m+1}\right)}\left\{1 - r^{2i+\frac{m}{m+1}}\right\} + (1+\delta)^{2i+\frac{m}{m+1}}\ln r\right]. \tag{24}$$

which are the dimensionless velocity profile and temperature distribution for Generalized second grade fluid, respectively.

3.1.2. *Newtonian fluid ($m = \alpha_1 = \alpha_2 = 0$).* Taking $m = 0$ and using boundary conditions (11) and (12), solutions of equations (14) and (10) are

$$w = w_0 - \frac{\rho g}{4\eta}\left[\left(R^2 - r^2\right) + 2(R+\delta)^2 \ln\left(\frac{r}{R}\right) - \frac{2\eta\beta}{R}\left((R+\delta)^2 - R^2\right)\right]. \tag{25}$$

$$\Theta = \Theta_0 - \frac{\eta}{\kappa}\left(\frac{\rho g}{2\eta}\right)^2 \left[\frac{(R+\delta)^4}{2}\left\{\left((\ln r)^2 - (\ln R)^2\right) - \ln\,(R+\delta)\,\ln\left(\frac{r}{R}\right)\right\} + \frac{1}{16}(r^4 - R^4) - \frac{(R+\delta)^2}{2}\left(r^2 - R^2\right) + \frac{3}{4}(R+\delta)^2 \ln\left(\frac{r}{R}\right)\right]. \tag{26}$$

Volume flux, Q, is calculated from equation (17) by using equation (25), which is

$$Q = w_0\pi\left((R+\delta)^2 - R^2\right) - \frac{\rho g\pi}{8\eta}\left[4(R+\delta)^4 \ln\left(\frac{R+\delta}{R}\right) - 2(R+\delta)^2\left((R+\delta)^2 - R^2\right) - \left((R+\delta)^2 - R^2\right)^2\left(\frac{2\eta\beta}{R} + 1\right)\right]. \tag{27}$$

The average film velocity, $\bar{V}$ is then given by

$$\bar{V} = w_0 - \frac{\rho g\pi}{8\eta}\left[\frac{4(R+\delta)^4 \ln\left(\frac{R+\delta}{R}\right)}{((R+\delta)^2 - R^2)} - 3(R+\delta)^2 + R^2 - \frac{2\eta\beta}{R}\left((R+\delta)^2 - R^2\right)\right]. \tag{28}$$

Shear stress will remain the same as that of the Generalized second grade fluid given by equation (21).

4. Drainage problem

Here we consider, Generalized second grade fluid now falling on the outer surface of a stationary infinitely long vertical cylinder of radius R, as shown in figure 1(b). The flow is in the downward direction due to gravity. The governing equations (2) and (3) become

$$\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{dw}{dr}\right)^{m+1}\right] = -\frac{\rho g}{\eta}, \tag{29}$$

$$\kappa\left[\frac{d^2\Theta}{dr^2} + \frac{1}{r}\frac{d\Theta}{dr}\right] + \eta\left[\frac{dw}{dr}\right]^{m+2} = 0, \tag{30}$$

and the associated boundary conditions are

Free space boundary condition:

$$\tau_{rz} = 0 \quad \text{and} \quad \frac{d\Theta}{dr} = 0 \quad \text{at} \quad r = R + \delta, \tag{31}$$

Slip boundary condition:

$$w = -\beta\tau_{rz}\,|_{r=R} \quad \text{and} \quad \Theta = \Theta_0 \quad \text{at} \quad r = R, \tag{32}$$

Integrating equation (29) with respect to r, and using the free space boundary condition (31), we get

$$\frac{dw}{dr} = \left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}}\left(\frac{(R+\delta)^2}{r} - r\right)^{\frac{1}{m+1}}. \tag{33}$$

By using the Binomial series, equation (33) can be re-written as

$$\frac{dw}{dr} = \left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}} (R+\delta)^{\frac{2}{m+1}} \sum_{i=0}^{\infty} \binom{\frac{1}{m+1}}{i} \frac{(-1)^i}{(R+\delta)^{2i}}\, r^{2i-\frac{1}{m+1}}. \tag{34}$$

4.1. **Solution of the drainage problem.**

4.1.1. *Generalized second grade fluid ($m \neq 0$).* Solving equations (34) and (30) for $m \neq 0$ by using the boundary conditions (31) and (32), we obtain the velocity profile and temperature distribution for the Generalized second grade fluid as

$$
\begin{aligned}
w &= -\frac{\beta\rho g}{2}\left(\frac{(R+\delta)^2}{R}-R\right)+\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}}\sum_{i=0}^{\infty}\begin{pmatrix}\frac{1}{m+1}\\ i\end{pmatrix}\\
&\times \frac{(-1)^i\,(R+\delta)^{-2i+\frac{2}{m+1}}}{2i+\frac{m}{m+1}}\left[r^{2i+\frac{m}{m+1}}-R^{2i+\frac{m}{m+1}}\right], \qquad (35)\\
\Theta &= \Theta_0+\frac{\eta}{\kappa}\left(\frac{\rho g}{2\eta}\right)^{\frac{m+2}{m+1}}\sum_{i=0}^{\infty}\begin{pmatrix}\frac{m+2}{m+1}\\ i\end{pmatrix}\frac{(-1)^i}{\left(2i+\frac{m}{m+1}\right)(R+\delta)^{2i-2\left(\frac{m+2}{m+1}\right)}}\\
&\times \left[\frac{R^{2i+\frac{m}{m+1}}}{\left(2i+\frac{m}{m+1}\right)}\left\{1-\left(\frac{r}{R}\right)^{2i+\frac{m}{m+1}}\right\}+(R+\delta)^{2i+\frac{m}{m+1}}\ln\left(\frac{r}{R}\right)\right]. \qquad (36)
\end{aligned}
$$

These are the explicit expressions for the velocity field and temperature distribution of thin film of a Generalized second grade fluid down a vertical cylinder in case of the drainage problem. From solutions (16) and (36) we observe that the temperature distribution for both withdrawal and drainage problems remain the same even though the velocity profile in each case is different. To calculate volume flux, Q , we use equation (35) in equation (17) to get

$$
\begin{aligned}
Q &= -\frac{\pi\,\beta\rho g}{2}\left(\frac{(R+\delta)^2}{R}-R\right)\left((R+\delta)^2-R^2\right)\\
&\times +2\pi\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}}\sum_{i=0}^{\infty}\begin{pmatrix}\frac{1}{m+1}\\ i\end{pmatrix}\frac{(-1)^i\,(R+\delta)^{-2i+\frac{2}{m+1}}}{2i+\frac{m}{m+1}}\\
&\times\left[\frac{\left((R+\delta)^{2i+\frac{3m+2}{m+1}}-R^{2i+\frac{3m+2}{m+1}}\right)}{2i+\frac{3m+2}{m+1}}-\frac{R^{2i+\frac{m}{m+1}}}{2}\left((R+\delta)^2-R^2\right)\right] \qquad (37)
\end{aligned}
$$

The average film velocity, $\overline{V}$, is then obtained from the formula listed in equation (19) as

$$
\begin{aligned}
\bar{V} &= -\frac{\beta\rho g}{2}\left(\frac{(R+\delta)^2}{R}-R\right)+2\left(\frac{\rho g}{2\eta}\right)^{\frac{1}{m+1}}\sum_{i=0}^{\infty}\begin{pmatrix}\frac{1}{m+1}\\ i\end{pmatrix}(-1)^i\\
&\times\frac{(R+\delta)^{-2i+\frac{2}{m+1}}}{2i+\frac{m}{m+1}}\left[\frac{\left((R+\delta)^{2i+\frac{3m+2}{m+1}}-R^{2i+\frac{3m+2}{m+1}}\right)}{\left((R+\delta)^2-R^2\right)\left(2i+\frac{3m+2}{m+1}\right)}-\frac{R^{2i+\frac{m}{m+1}}}{2}\right] \qquad (38)
\end{aligned}
$$

Shear stress on the surface of cylinder is

$$\tau_{rz}|_{r=R} = \frac{\rho g (R+\delta)^2}{2R}\left[1 - \left(\frac{R}{R+\delta}\right)^2\right]. \tag{39}$$

Introducing the dimensionless parameters defined in equation (22), the non-dimensional velocity profile and temperature distribution for the Generalized second grade fluid in the case of drainage problem are

$$w = -\frac{\beta\,\eta\,S_t}{2\,w_0}\left(\frac{(1+\delta)^2}{R} - R\right) + \left(\frac{S_t}{2}\right)^{\frac{1}{m+1}} \sum_{i=0}^{\infty} \binom{\frac{1}{m+1}}{i} \times \frac{(-1)^i\,(1+\delta)^{\frac{2}{m+1}-2i}}{\left(2i+\frac{m}{m+1}\right)}\left[r^{2i+\frac{m}{m+1}} - 1\right] \tag{40}$$

and

$$\Theta = B_r\left(\frac{S_t}{2}\right)^{\frac{m+2}{m+1}} \sum_{i=0}^{\infty} \binom{\frac{m+2}{m+1}}{i} \frac{(-1)^i}{\left(2i+\frac{m}{m+1}\right)(1+\delta)^{2i}} \times \left[\frac{1}{\left(2i+\frac{m}{m+1}\right)}\left\{1 - r^{2i+\frac{m}{m+1}}\right\} + (1+\delta)^{2i+\frac{m}{m+1}}\ln r\right] \tag{41}$$

respectively.

4.1.2. *Newtonian fluid ($m = \alpha_1 = \alpha_2 = 0$).* Solving equations (33) and (30) for *m=0* with boundary conditions (31) and (32), we obtain

$$w = \frac{\rho g}{4\eta}\left[\left(R^2 - r^2\right) + 2(R+\delta)^2 \ln\left(\frac{r}{R}\right) - \frac{2\eta\beta}{R}\left((R+\delta)^2 - R^2\right)\right], \tag{42}$$

$$\Theta = \Theta_0 - \frac{\eta}{\kappa}\left(\frac{\rho g}{2\eta}\right)^2\left[\frac{(R+\delta)^4}{2}\left\{\left((\ln\, r)^2 - (\ln\, R)^2\right) - \ln\,(R+\delta)\,\ln\left(\frac{r}{R}\right)\right\} + \frac{1}{16}(r^4 - R^4) - \frac{(R+\delta)^2}{2}\left(r^2 - R^2\right) + \frac{3}{4}(R+\delta)^2 \ln\left(\frac{r}{R}\right)\right]. \tag{43}$$

Again it is observed that the temperature distribution for Newtonian fluid in both the withdrawal and drainage problems remain the same. Expressions for volume flux and average film velocity are

$$Q = \frac{\rho g \pi}{8\eta}\left[4(R+\delta)^4 \ln\left(\frac{R+\delta}{R}\right) - 2(R+\delta)^2\left((R+\delta)^2 - R^2\right) - \left((R+\delta)^2 - R^2\right)^2\left(\frac{2\eta\beta}{R} + 1\right)\right] \tag{44}$$

and

$$\bar{V} = \frac{\rho g \pi}{8\eta}\left[\frac{4(R+\delta)^4 \ln\left(\frac{R+\delta}{R}\right)}{((R+\delta)^2 - R^2)} - 3(R+\delta)^2 + R^2 - \frac{2\eta\beta}{R}\left((R+\delta)^2 - R^2\right)\right]. \tag{45}$$

respectively.
Shear stress for Newtonian fluid is also given by equation (39).

5. Results and discussion

In this paper we studied thin film flows for withdrawal and drainage problems using a non-isothermal, incompressible Generalized second grade fluid on cylindrical surfaces. In both problems differential equations and the associated boundary conditions are developed. The approximate analytical solutions of both problems are obtained by using the binomial theorem.

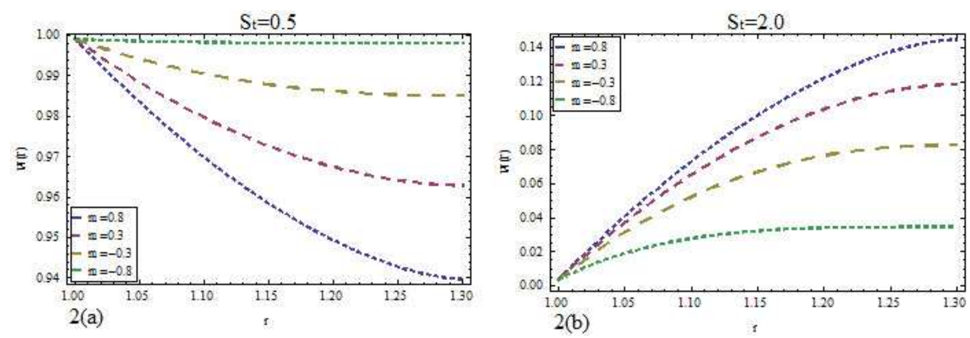

Fig. 2: Effect of m on velocity for withdrawal problem 2(a) and drainage problem 2(b).

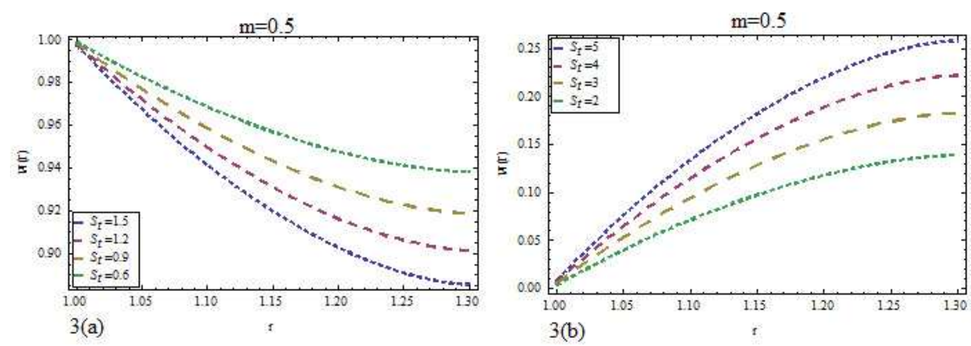

Fig. 3: Effect, on dilatant fluid, of Stokes number S_t on velocity for withdrawal problem 3(a) and drainage problem 3(b).

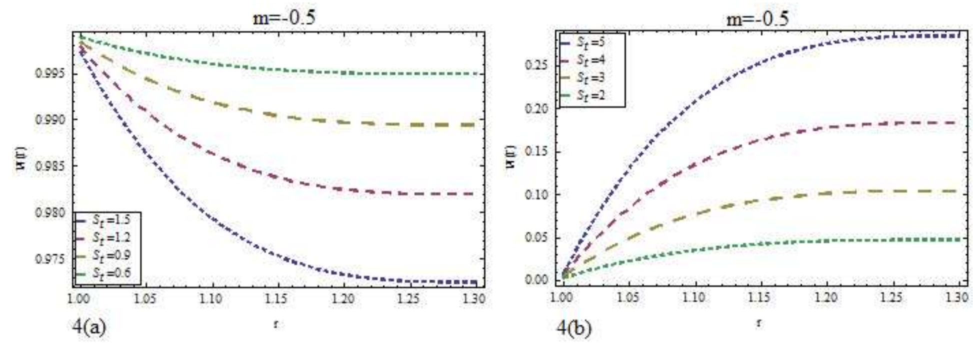

Fig. 4: Effect, on pseudoplastic fluid, of Stokes number S_t on velocity for withdrawal problem 4(a) and drainage problem 4(b).

It is observed that the obtained solutions (23), (24) and (35), (36) are strongly dependent on the non-dimensional parameters, Stokes number S_t, Brinkman number B_r, and the flow behavior index m. The effect of flow behavior index m on velocity profile and temperature distribution for both problems is investigated in figures 2 and 5. In figures 3 and 6, the effects of S_t and B_r numbers on velocity field and temperature distribution are depicted for the shear thickening (dilatant) fluids where as figures 4 and 7 are given for the shear thinning (pseudoplastic) fluids. For withdrawal problem it can be observed in figure 2(a) that magnitude of velocity decreases as the fluid is becoming thicker and vice versa, while figure 2(b) is given for the drainage problem. For withdrawal problem, the effect of Stokes number, S_t, on velocity profile is given in figure 3(a). It is evident that gradient of velocity decreases as S_t increases, while figure 3(b) for the drainage problem shows that there is a direct relation between the Stokes number S_t and velocity of the fluid $w(r)$. Increase in temperature can be seen in figure 5 as the fluid is becoming thicker.

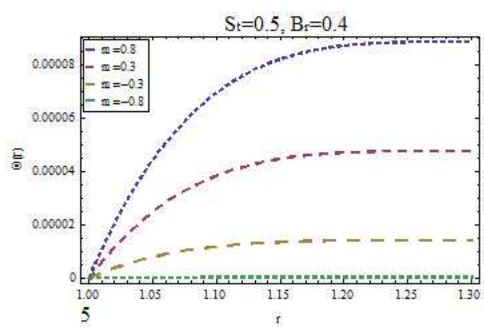

Fig. 5: Effect of parameter m on temperature for both problems.

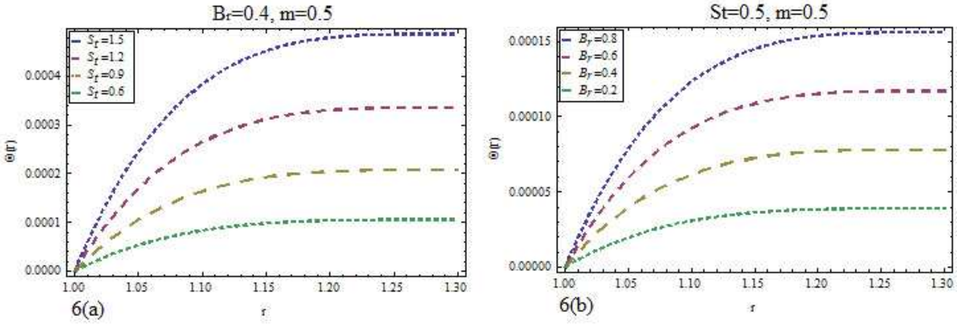

Fig. 6: Effect, on dilatant fluid, of S_t and B_r numbers on temperature for both problems.

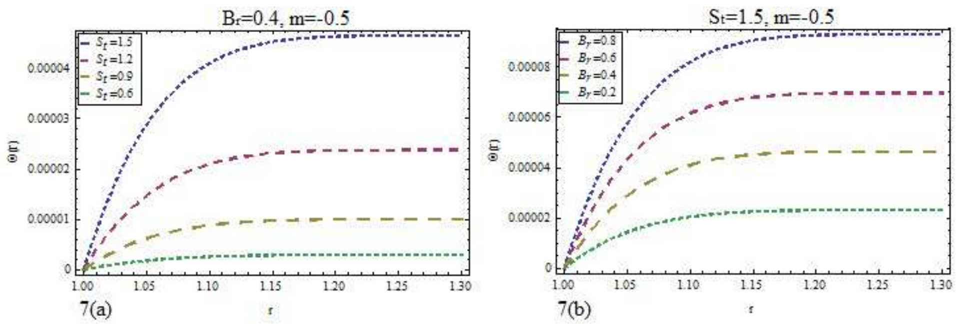

Fig. 7: Effect, on pseudoplastic fluid, of S_t and B_r numbers on temperature for both problems.

The effects of S_t and B_r numbers on heat transfer in dilatant fluids for both withdrawal and drainage problems are shown in figures 6(a) and 6(b) respectively. Raise in the temperature is evident from these figures as the values of the Stokes and Brinkman numbers are incearsing, while graphs for pseudoplastic fluids are given in figures 4 and 7.

6. Conclusion

We have considered steady, incompressible thin film flow for two different problems i.e., withdrawal and drainage problems for the Generalised second grade fluid and obtained series solutions. Explicit expressions for velocity field, temperature distribution, volume flux, average velocity and shear stress are obtained in both problems. It is important to note that normal stresses have no contribution for steady Generalized second grade fluid flow. We do not observe any contribution of power law model verses Generalized second grade fluid model as solutions (15) and (35) for velocity profiles and also solutions (16) and (36) for temperature distributions are same as that of the power law fluid.

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