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COMAXIMAL FACTORIZATION GRAPHS IN INTEGRAL DOMAINS

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Abstract. In [1], I. Beck introduced the idea of a zero divisor graph of a commutative ring and later in [2], J. Coykendall and J. Maney generalized this idea to study factorization in integral domains. They defined irreducible divisor graphs and used these irreducible divisor graphs to characterize UFDs. We define comaximal factorization graphs and use these graphs to characterize UCFDs defined in [3]. We also study that, in certain cases, comaximal factorization graph is formed by joining r copies of the complete graph K_m with one copy of complete graph K_n in common.

Key words: factorization, integral domain, complete graph, windmill graph.
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A recent subject of study linking commutative ring theory with graph theory has been the concept of the zero-divisor graph of a commutative ring. Let R be a commutative ring with identity and denote the set of (nonzero) zero-divisors of R as $Z(R)$. The zero-divisor graph of R, denoted by $\Gamma(R)$, is the graph with vertex set $Z(R)$ and if $a, b \in Z(R)$, $ab \in E(\Gamma(R))$ if and only if $ab = 0$. Zero-divisor graphs were introduced by I. Beck in [1].

Recently in [2], J. Coykendall and J. Maney stretched this idea in a different direction, applied it to factorization in integral domains. They introduced the following concept in an integral domain D:

Let $x \in D$ be a nonzero nonunit that can be factored into irreducibles. The irreducible divisor graph of x is said to be the graph $G(x) = (V, E)$ where $V = \{y \in \overline{Irr}(D) | \ y|x\}$, and given $y_1, y_2 \in \overline{Irr}(D)$, $y_1y_2 \in E$ if and only if $y_1y_2|x$. Further, attach n-1 loops to the vertex y if y^n divides x.

In [3], S. Mcadam and R.G. Swan studied the analogue to unique factorization domains. They introduced the following concepts in an integral domain D:

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(1) Let b be a nonzero nonunit element of D. Then b is called a ***pseudo-atom*** if it is impossible to factor b as $b = cd$ with c and d comaximal (i.e., $(c, d) = D$) non-units.
(2) Let b be a nonzero nonunit element of D. Then $b = b_1b_2...b_m$ is called a ***complete comaximal factorization*** of b if the b_is are pairwise comaximal pseudo-atoms.
(3) D is called a ***comaximal factorization domain (CFD)*** if every nonzero nonunit element b of D has a complete comaximal factorization.
(4) D is called a ***unique comaximal factorization domain (UCFD)*** if D is a CFD in which complete comaximal factorizations are unique (up to order and units).

In this paper, we consider the complete comaximal factorization, define comaximal factorization graphs and use these graphs to characterize UCFDs (Theorem 10). We also study that, in certain cases, comaximal factorization graph is a join of r copies of the complete graph K_m with one copy of complete graph K_n in common (Proposition 5).

For the sake of convenience, some graph theoretic definitions are given. Recall that a graph G is called a ***simple graph*** if it has no loops or multiple edges. If G is a graph with a subgraph H, then H is an ***induced subgraph*** of G if for all $u, v \in H$, $uv \in E(G)$ implies that $uv \in E(H)$. For $v \in V$, denote the ***neighborhood of*** v in $\mathcal{G}(x)$ by $N(v) = N_{\mathcal{G}}(v) = \{u \in V | uv \in E\}$. A graph G is said to be ***complete*** if each vertex is adjacent to all the other vertices. The ***windmill graph*** W_n^m is the graph obtained by taking m copies of the complete graph K_n with a vertex in common. For $n = 3$, the windmill graph W_3^m is also called a ***friendship graph*** F_m, a graph obtained by taking m copies of the cycle graph C_3 with a vertex in common.

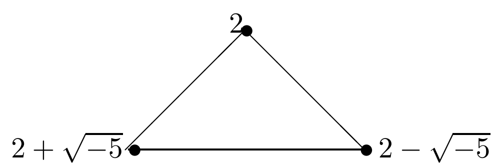
Definitions 1. *Let D be a domain and let x be a nonzero nonunit that has complete comaximal factorization.*

(1) *Call a pseudo-atom a* **comaximal factor** *of x if it appears in some complete comaximal factorization of x.*
(2) *A* **comaximal factorization graph** *of x is a simple graph $\mathcal{G}(x) = (V, E)$ with $V = \{a |\ a$ is a comaximal factor of $x\ \}$ and $E = \{y_1y_2 \mid y_1y_2$ appears in some complete comaximal factorization of $x\}$.*
(3) *Call a pseudo-atom a* **common comaximal factor** *of x if it appears in every complete comaximal factorization of x.*
(4) *If $A \subseteq V(\mathcal{G}(x))$, then by $\mathcal{G}_A(x)$, we mean an induced subgraph of $\mathcal{G}(x)$ on $N(A)$.*
(5) *By* **disjoint complete comaximal factorizations** *we mean the complete comaximal factorizations such that no comaximal factor is common in any two of these factorizations.*

Example 2. Let $D = \mathbb{Z}[\sqrt{-5}]$. In order to draw the comaximal factorization graph $\mathcal{G}(18)$ of 18 in D, we need to find all complete comaximal factorizations of 18. First of all we write the ideal (18) as a product of prime ideals in D, i.e., $(18) = (2, 1+\sqrt{-5})^2(3, 1+\sqrt{-5})^2(3, 1-\sqrt{-5})^2$. If $I = (2, 1+\sqrt{-5})$, $J = (3, 1+\sqrt{-5})$ and $K + (3, 1-\sqrt{-5})$, then I^2, J^2, K^2, IJ, JK, and KI are all principal in D. Also note that $IJ+JK \subseteq J, JK+KI \subseteq K$ and $IJ+JI \subseteq I$. So by avoiding the pairs which are not comaximal, we get that the only complete comaximal factorization of 18 is

$$18 = 2(2+\sqrt{-5})(2-\sqrt{-5})$$

So, $\mathcal{G}(18)$ is as follows

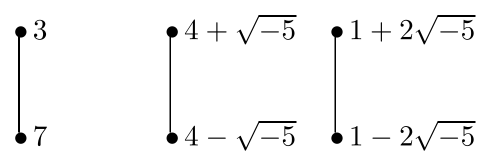


Remark 3. *Let $x \in D$ be a nonzero nonunit element of a domain D. Then $\mathcal{G}(x)$ is a disjoint union of complete graphs if and only if every two complete comaximal factorizations of x are disjoint.*

Example 4. *Let $D = \mathbb{Z}[\sqrt{-5}]$. To draw the comaximal factorization graph $\mathcal{G}(21)$ of 21 in D, we need to find all the complete comaximal factorizations of 21. We write the ideal (21) as a product of prime ideals in D, i.e., $(21) = (3, 1+\sqrt{-5})(3, 1-\sqrt{-5})(7, 3+\sqrt{-5})(7, 3-\sqrt{-5})$. If $I = (3, 1+\sqrt{-5}), J = (3, 1-\sqrt{-5}), K = (7, 3+\sqrt{-5})$ and $P = (7, 3-\sqrt{-5})$, then $IJ = (3), KP = (7), IK = (1-2\sqrt{-5}), JP = (1+2\sqrt{-3}), JK = (1+4\sqrt{-5})$ and $IP = (1-4\sqrt{-5})$. So, by rearrangement of prime ideals in above product, we get that the only complete comaximal factorizations of 21 are*

$$21 = 3.7 = (1+2\sqrt{-5})(1-2\sqrt{-5}) = (4+\sqrt{-5})(4-\sqrt{-5})$$

Hence, $\mathcal{G}(21)$ is as follows



Denote by $rK_m + k_n$ the graph obtained by joining r copies of the complete graph K_m with one copy of complete graph K_n in common. For example, $rK_m + K_1$ is a windmill graph W^r_{m+1}, $rK_2 + K_1$ is a friendship graph F_r and $2K_2 + K_1$ is a Butterfly graph F_2.

Proposition 5. *Let x be a nonzero nonunit element of a domain D such that any complete comaximal factorization of x have length m. Then $\mathcal{G}(x) = rK_{m-s} + K_s$ if and only if $x = a_1a_2\cdots a_sy$, where $a_1, a_2, .., a_s$ are common comaximal factors of x (see Definitions 1) and y has r number of disjoint complete comaximal factorizations.*

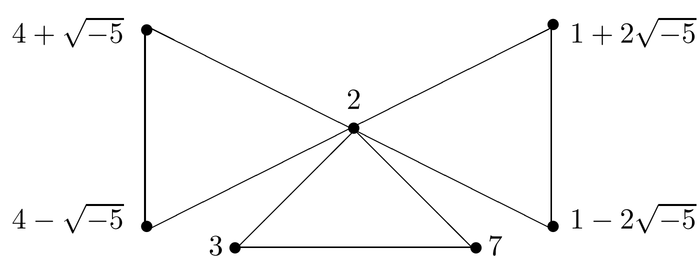
Proof. Suppose that $\mathcal{G}(x) = rK_{m-s} + K_s$. Then the complete comaximal factorizations of x will be $x = a_1a_2\cdots a_sb_{11}\cdot b_{12}\cdots b_{1(m-s)} = a_1a_2\cdots a_sb_{21}\cdot b_{22}\cdots b_{2(m-s)} = \cdots\cdots = a_1a_2\cdots a_sb_{r1}\cdots b_{r(m-s)}$. So one can write $x = a_1a_2\cdots a_sy$ where $y = b_{11}\cdot b_{12}\cdots b_{1(m-s)} = b_{21}\cdot b_{22}\cdots b_{2(m-s)} = \cdots\cdots = b_{r1}\cdot b_{r2}\cdots b_{r(m-s)}$.

Conversely if $x = a_1a_2\cdots a_sy$, where $a_1, a_2, , , , a_s$ are common comaximal factors of x and y has r number of disjoint complete comaximal factorizations. Then $\mathcal{G}(x) = rK_{m-s} + K_s$. □

Example 6. *Let $D = \mathbb{Z}[\sqrt{-5}]$. To draw the comaximal factorization graph $\mathcal{G}(42)$ of 42 in D, we need all the complete comaximal factorizations of 42. For this write the ideal (42) as a product of prime ideals in D, i.e., $(42) = (2, 1+\sqrt{-5})^2(3, 1+\sqrt{-5})(3, 1-\sqrt{-5})(7, 3+\sqrt{-5})(7, 3-\sqrt{-5})$. By permuting and multiplying the prime ideals, we get that the only complete comaximal factorizations of 42 are*

$$42 = 2.3.7 = 2(1 + 2\sqrt{-5})(1 - 2\sqrt{-5}) = 2(4 + \sqrt{-5})(4 - \sqrt{-5})$$

Hence, $\mathcal{G}(42)$ is as follows



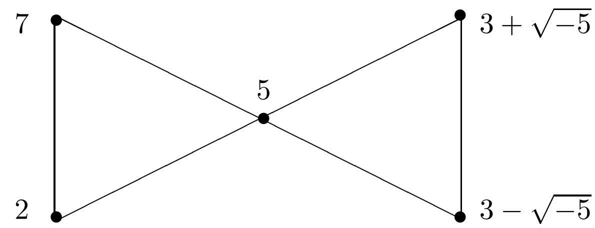
Friendship graph F_3

Example 7. *Let $D = \mathbb{Z}[\sqrt{-5}]$. To draw the comaximal factorization graph $\mathcal{G}(70)$ of 70 in D, we need all the complete comaximal factorizations of 70. We write the ideal (70) as a product of prime ideals in D, i.e.,*

$(70) = (2, 1+\sqrt{-5})^2(\sqrt{-5})^2(7, 3+\sqrt{-5})(7, 3-\sqrt{-5})$. *By using the technique as in previous examples, we get that the only complete comaximal factorizations of* 70 *are*

$$70 = 2 \cdot 7 \cdot 5 = (3 + \sqrt{-5})(3 - \sqrt{-5})5$$

Hence, $\mathcal{G}(70)$ *is as follows*



Butterfly graph, $3K2 + K1$

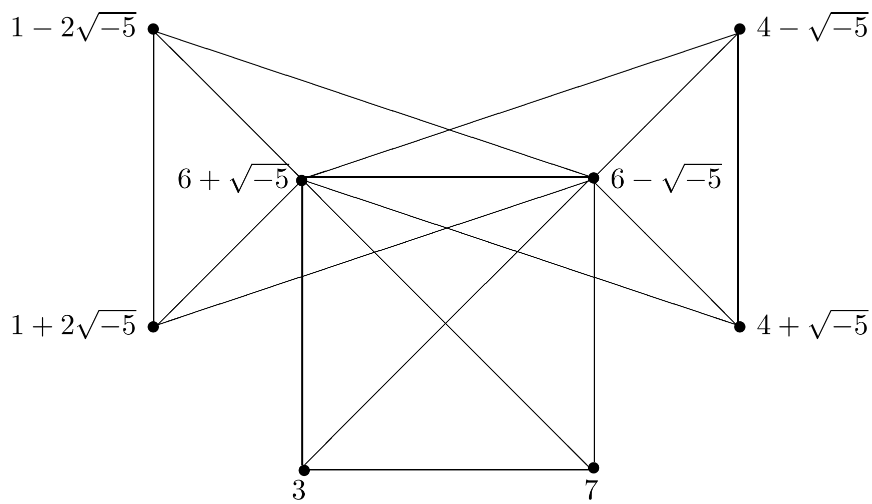
Example 8. *Let* $D = \mathbb{Z}[\sqrt{-5}]$. *To draw the comaximal factorization graph* $\mathcal{G}(861)$ *of* 861 *in* D, *we need all complete comaximal factorizations of* 861 *in* D. *We write* (861) *as a product of prime ideals in* D, *i.e.*

$$(861) = (3, 1-\sqrt{-5})(3, 1+\sqrt{-5})(7, 3-\sqrt{-5})(7, 1-\sqrt{-5})(41, 6-\sqrt{-5})(41, 6+\sqrt{-5})$$

Adopting the same procedure as in previous examples, we get that the only complete comaximal factorizations of 861 *are*

$$\begin{aligned} 861 &= 3 \cdot 7 \cdot (6 + \sqrt{-5})(6 - \sqrt{-5}) \\ &= (1 + 2\sqrt{-5})(1 - 2\sqrt{-5})(6 + \sqrt{-5})(6 - \sqrt{-5}) \\ &= (4 + \sqrt{-5})(4 - \sqrt{-5})(6 + \sqrt{-5})(6 - \sqrt{-5}) \end{aligned}$$

So, $\mathcal{G}(861)$ *is as follows*



$\mathcal{G}(861) = 3K2 + K2$

Recall that If G is a graph with a subgraph H, then H is an induced subgraph of G if for all $u, v \in H$, $uv \in E(G)$ implies that $uv \in E(H)$. For $v \in V$, denote the neighborhood of v in $\mathcal{G}(x)$ by $N(v) = N_{\mathcal{G}}(v) = \{u \in V | uv \in E\}$. If $A \subseteq V(\mathcal{G}(x))$, then by $\mathcal{G}_A(x)$, we mean an induced subgraph of $\mathcal{G}(x)$ on $N(A)$.

Lemma 9. *Let D be a domain and $x \in D$ be a nonzero nonunit having complete comaximal factorization and let a be comaximal factor of x. Then the vertex set of $\mathcal{G}_a(x)$ and $\mathcal{G}(x/a)$ coincide. In other words the vertices of $\mathcal{G}(x/a)$ are precisely those in the neighborhood of a. Moreover, the edge set of $\mathcal{G}(x/a)$ contained in edge set of $\mathcal{G}_a(x)$.*

Proof. If x is a pseudo-atom then the result is obviously true. Otherwise, let $b \in N(a)$. Then $ab \in E(\mathcal{G}(x))$, i.e., ab appears in a complete comaximal factorization of x. This implies that $b \in V(\mathcal{G}(x/a))$. Conversely if $b \in V(\mathcal{G}(x/a))$, i.e., b is a comaximal factor of x/a. This implies that ab appears in the complete comaximal factorization of x and hence $b \in N(a)$.

Now let $\alpha\beta \in E(\mathcal{G}(x/a))$. Then $\alpha\beta$ appears in a complete comaximal factorization of x/a. Note that $b_1 b_2 \cdots b_n$ is a complete comaximal factorization of x/a if and only if $ab_1 b_2 \cdots b_n$ is a complete comaximal factorization of x. Therefore, $\alpha\beta a$ is in a complete comaximal factorization of x and thus $\alpha, \beta \in N(a)$, yields that $\alpha\beta \in E(\mathcal{G}_a(x))$. □

At the end, we give a characterization of unique comaximal factorization domains (UCFDs) via comaximal factorization graphs.

Theorem 10. *Let D be a CFD. The following are equivalent:*

(1) *D is a UCFD.*
(2) *For each nonzero nonunit $x \in D$, $\mathcal{G}(x)$ is complete.*
(3) *For each nonzero nonunit $x \in D$, $\mathcal{G}(x)$ is connected.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.

(2) $\Rightarrow$ (1) Let $x \in D$ be a nonzero nonunit and $\mathcal{G}(x) = (V, E)$ be the complete graph with $V = \{x_1, x_2, ..., x_n\}$. Then $x_1, x_2, ..., x_n$ are comaximal factors of x (see Definitnions 1). Since each vertex is adjacent to all the other vertices, so $x_1 x_2 \cdots x_n$ is the only complete comaximal factorization of x. Thus D is a UCFD.

(3) $\Rightarrow$ (1, 2): Suppose that $\mathcal{G}(x)$ is not complete for some nonzero nonunit $x \in D$. Let A be the set of all nonzero nonunit elements x of D for which $\mathcal{G}(x)$ is not complete. Choose $z \in A$ for which $\mathcal{G}(z)$ has a minimal set of vertices. Since $\mathcal{G}(z)$ is connected, so there exist $a, b, c \in V(\mathcal{G}(z))$ such that $ab, bc \in E(\mathcal{G}(z))$ but $ac \notin E(\mathcal{G}(z))$ [otherwise if $ab, bc, ac \in E(\mathcal{G}(z))$ for all $a, b, c \in V(\mathcal{G}(z))$, then $\mathcal{G}(z)$ will be complete]. Now $a, c \in V(\mathcal{G}(z/b))$ and

$ac \notin E(\mathcal{G}(z/b))$ which is a contradiction because $\mathcal{G}(z/b)$ is complete due to the minimality of $V(\mathcal{G}(z))$. Therefore $A = \{\}$.

□

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References

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