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CONSTRUCTION OF MIDDLE NUCLEAR SQUARE LOOPS

AMIR KHAN[1], MUHAMMAD SHAH[2], ASIF ALI[2]

ABSTRACT. Middle nuclear square loops are loops satisfying $x(y(zz)) = (xy)(zz)$ for all x, y and z. We construct an infinite family of nonassociative noncommutative middle nuclear square loops whose smallest member is of order 12.

Key words: middle nuclear square loop, construction of loop, C-loops.
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1. INTRODUCTION

A groupoid $(Q, \cdot)$ is a quasigroup if, for each $a, b \in Q$, the equations $ax = b, ya = b$ have unique solutions where $x, y \in Q$ [1]. A loop is a quasigroup with an identity element e. The left nucleus of a loop L is $N_\lambda = \{l \in L : l(xy) = (lx)y$ for every $x, y \in L\}$. The right nucleus of a loop L is the set $N_\rho = \{r \in L : (xy)r = x(yr)$ for every $x, y \in L\}$, and middle nucleus of L is $N_\theta = \{m \in L : (ym)x = y(mx)$ for every $x, y \in L\}$. The nucleus of L is the set $N(L) = N_\rho \cap N_\lambda \cap N_\theta$. A loop $(L, *)$ is termed a middle nuclear square loop if every square element, i.e., every element of the form $x * x$, is in the middle nucleus. In other words, the following identity is satisfied for all $x, y, z \in L$:

$$x * ((y * y) * z)) = (x * (y * y)) * z$$

Every C-loop is a middle nuclear square loop. In this paper we construct a middle nuclear square loop of order 12 which belongs to an infinite family of nonassociative noncommutative middle nuclear square loops constructed here for the first time.

[1]Department of Mathematics and Statistics, University of Swat, Swat, Pakistan. Email: amir.maths@gmail.com,
[2]Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan. Email: shahmaths_problem@hotmail.com, Email: dr_asif_ali@hotmail.com.

2. Construction of middle nuclear square loop

Let H be a multiplicative group with identity element 1, and A be an additively abelian group with identity element 0. Any map

$$\theta : H \times H \to A$$

satisfying

$$\theta(1, g) = \theta(g, 1) = 0 \text{ for every } g \in H$$

is called a factor set. When $\theta : H \times H \to A$ is a factor set, we can define a multiplication on $H \times A$ by

$$(g, a)(h, b) = (gh, a + b + \theta(g, h)). \tag{1}$$

The resulting groupoid is clearly a loop with neutral element $(1, 0)$. It will be denoted by (H, A, θ). Additional properties of (H, A, θ) can be enforced by additional requirements on θ.

Lemma. Let $\theta : H \times H \to A$ be a factor set. Then (H, A, θ) is a middle nuclear square loop if and only if

$$\theta\left(h^2, k\right) + \theta\left(g, h^2k\right) = \theta\left(g, h^2\right) + \theta\left(gh^2, k\right) \text{ for every } g, h, k \in H. \tag{2}$$

Proof. By definition the loop (H, A, θ) is middle nuclear square loop if and only if

$$(g,a)[((h,b)(h,b))(k,c)] = [(g,a)((h,b)(h,b))](k,c)$$
$$\iff (g,a)\left[(h^2, 2b + \theta(h,h))(k,c)\right] = [(g,a)(h^2, 2b + \theta(h,h)](k,c)$$
$$\iff (g,a)[(h^2k, 2b{+}c{+}\theta(h,h){+}\theta\left(h^2,k\right)] = [(gh^2, a{+}2b{+}\theta(h,h){+}\theta(g,h^2)](k,c)$$
$$\iff [g(h^2k), a + 2b + c + \theta(h,h) + \theta\left(h^2,k\right) + \theta\left(g,h^2k\right)] =$$
$$[[(gh^2)k, a + 2b + c + \theta(h,h) + \theta(g,h^2) + \theta(gh^2,k)]$$

comparing both sides we get

$$\theta\left(h^2, k\right) + \theta\left(g, h^2k\right) = \theta\left(g, h^2\right) + \theta\left(gh^2, k\right)$$

We call a factor set θ satisfying (2) a middle nuclear square factor set. □

Proposition. Let A be an abelian group of order n where $n > 2$, and $\beta \in A$ an element of order bigger than 2. Let $H = \{1, x, x^2, x^3\}$ be a cyclic group with identity element 1. Define

$$\theta : H \times H \to A$$

by

$$\begin{aligned} \theta(a,b) &= \beta, \text{ if } (a,b) = (x^3,x^2), (x^2,x^2) \\ &= -\beta, \text{ if } (a,b) = (x^2,x), (x^3,x), (x^3,x^3), (x^2,x^3) \\ &= 0, \text{ otherwise} \end{aligned}$$

then $L=(H,A,\theta)$ is a nonassociative and noncommutative middle nuclear square loop with nucleus $N(L)=\{(1,a):a\in A\}$.

Proof. The map θ is clearly a factor set. It can be shown as follows

θ	1	x	x^2	x^3
1	0	0	0	0
x	0	0	0	0
x^2	0	$-\beta$	β	$-\beta$
x^3	0	$-\beta$	β	$-\beta$

To show that $L=(H,A,\theta)$ is middle nuclear square loop, we verify equation (2) as follows.

Case i : Since θ is a factor set there is nothing to prove when $g,h,k=1$

Case ii : when $g=x$, (2) becomes

$$\theta\left(h^2,k\right)+\theta\left(x,h^2k\right)=\theta\left(x,h^2\right)+\theta\left(xh^2,k\right) \tag{3}$$

put $h=x$ in (3) we get $\theta\left(x^2,k\right)+\theta\left(x,x^2k\right)=\theta\left(x,x^2\right)+\theta\left(x^3,k\right)$

$k=1\Rightarrow\theta\left(x,x^2\right)=\theta\left(x,x^2\right)$

$k=x\Rightarrow\theta\left(x^2,x\right)+\theta\left(x,x^3\right)=\theta\left(x,x^2\right)+\theta\left(x^3,x\right)\iff-\beta=-\beta$

$k=x^2\Rightarrow\theta\left(x^2,x^2\right)+\theta\left(x,1\right)=\theta\left(x,x^2\right)+\theta\left(x^3,x^2\right)\iff\beta=\beta$

$k=x^3\Rightarrow\theta\left(x^2,x^3\right)+\theta\left(x,x\right)=\theta\left(x,x^2\right)+\theta\left(x^3,x^3\right)\iff-\beta=-\beta$

Put $h=x^2$ in (3) we get $\theta\left(1,k\right)+\theta\left(x,k\right)=\theta\left(x,1\right)+\theta\left(x,k\right)$

$\Rightarrow\theta\left(x,k\right)=\theta\left(x,k\right)$

put $h=x^3$ in (3) we get $\theta\left(x^2,k\right)+\theta\left(x,x^2k\right)=\theta\left(x,x^2\right)+\theta\left(x^3,k\right)$

$k=1\Rightarrow\theta\left(x,x^2\right)=\theta\left(x,x^2\right)$

$k=x\Rightarrow\theta\left(x^2,x\right)+\theta\left(x,x^3\right)=\theta\left(x,x^2\right)+\theta\left(x^3,x\right)\iff\beta=\beta$

$k=x^2\Rightarrow\theta\left(x^2,x^2\right)+\theta\left(x,1\right)=\theta\left(x,x^2\right)+\theta\left(x^3,x^2\right)\iff-\beta=-\beta$

$k=x^3\Rightarrow\theta\left(x^2,x^3\right)+\theta\left(x,x\right)=\theta\left(x,x^2\right)+\theta\left(x^3,x^3\right)\iff\beta=\beta$

which all are true.

Case iii : when $g=x^2$, (2) becomes

$$\theta\left(h^2,k\right)+\theta\left(x^2,h^2k\right)=\theta\left(x^2,h^2\right)+\theta\left(x^2h^2,k\right) \tag{4}$$

put $h=x$ in (4) we get $\theta\left(x^2,k\right)+\theta\left(x^2,x^2k\right)=\theta\left(x^2,x^2\right)+\theta\left(1,k\right)$

$k=1\Rightarrow\theta\left(x^2,x^2\right)=\theta\left(x^2,x^2\right)$

$k=x\Rightarrow\theta\left(x^2,x\right)+\theta\left(x^2,x^3\right)=\theta\left(x^2,x^2\right)\iff\beta=\beta$

$k=x^2\Rightarrow\theta\left(x^2,x^2\right)+\theta\left(x^2,1\right)=\theta\left(x^2,x^2\right)\iff-\beta=-\beta$

$k=x^3\Rightarrow\theta\left(x^2,x^3\right)+\theta\left(x^2,x\right)=\theta\left(x^2,x^2\right)\iff\beta=\beta$

put $h=x^2$ in (4) we get $\theta\left(1,k\right)+\theta\left(x^2,k\right)=\theta\left(x^2,1\right)+\theta\left(x^2,k\right)$

$\Rightarrow\theta\left(x^2,k\right)=\theta\left(x^2,k\right)$

put $h=x^3$ in (4) we get $\theta\left(x^2,k\right)+\theta\left(x^2,x^2k\right)=\theta\left(x^2,x^2\right)+\theta\left(1,k\right)$

$k=1\Rightarrow\theta\left(x^2,x^2\right)=\theta\left(x^2,x^2\right)$

$k = x \Rightarrow \theta\left(x^2, x\right) + \theta\left(x^2, x^3\right) = \theta\left(x^2, x^2\right) \iff \beta = \beta$
$k = x^2 \Rightarrow \theta\left(x^2, x^2\right) + \theta\left(x^2, 1\right) = \theta\left(x^2, x^2\right) \iff -\beta = -\beta$
$k = x^3 \Rightarrow \theta\left(x^2, x^3\right) + \theta\left(x^2, x\right) = \theta\left(x^2, x^2\right) \iff \beta = \beta$
which all are true.

Case iv : when $g = x^3$, (2) becomes

$$\theta\left(h^2, k\right) + \theta\left(x^3, h^2k\right) = \theta\left(x^3, h^2\right) + \theta\left(x^3h^2, k\right) \tag{5}$$

put $h = x$ in (5) we get $\theta\left(x^2, k\right) + \theta\left(x^3, x^2k\right) = \theta\left(x^3, x^2\right) + \theta\left(x, k\right)$
$k = 1 \Rightarrow \theta\left(x^3, x^2\right) = \theta\left(x^3, x^2\right)$
$k = x \Rightarrow \theta\left(x^2, x\right) + \theta\left(x^3, x^3\right) = \theta\left(x^3, x^2\right) + \theta\left(x, x\right) \iff \beta = \beta$
$k = x^2 \Rightarrow \theta\left(x^2, x^2\right) + \theta\left(x^3, 1\right) = \theta\left(x^3, x^2\right) + \theta\left(x, x^2\right) \iff -\beta = -\beta$
$k = x^3 \Rightarrow \theta\left(x^2, x^3\right) + \theta\left(x^3, x\right) = \theta\left(x^3, x^2\right) + \theta\left(x, x^3\right) \iff \beta = \beta$
put $h = x^2$ in (5) we get $\theta\left(1, k\right) + \theta\left(x^3, k\right) = \theta\left(x^3, 1\right) + \theta\left(x^3, k\right)$
$\Rightarrow \theta\left(x^3, k\right) = \theta\left(x^3, k\right)$
put $h = x^3$ in (5) we get $\theta\left(x^2, k\right) + \theta\left(x^3, x^2k\right) = \theta\left(x^3, x^2\right) + \theta\left(x, k\right)$
$k = 1 \Rightarrow \theta\left(x^3, x^2\right) = \theta\left(x^3, x^2\right)$
$k = x \Rightarrow \theta\left(x^2, x\right) + \theta\left(x^3, x^3\right) = \theta\left(x^3, x^2\right) + \theta\left(x, x\right) \iff \beta = \beta$
$k = x^2 \Rightarrow \theta\left(x^2, x^2\right) + \theta\left(x^3, 1\right) = \theta\left(x^3, x^2\right) + \theta\left(x, x^2\right) \iff -\beta = -\beta$
$k = x^3 \Rightarrow \theta\left(x^2, x^3\right) + \theta\left(x^3, x\right) = \theta\left(x^3, x^2\right) + \theta\left(x, x^3\right) \iff \beta = \beta$
which all are true.

Now we show that $L = (H, A, \theta)$ is not associative. For this consider $b \in A$, As $\left(x^3, b\right)\left((x, b)\left(x^3, b\right)\right) = \left(x^3, b\right)(1, 2b) = \left(x^3, 3b\right) \neq \left(x^3, 3b - \beta\right) = (1, 2b - \beta)(x^3, b) = \left(\left(x^3, b\right)(x, b)\right)(x^3, b)$ it follows that $L = (H, A, \theta)$ is nonassociative middle nuclear square loop.

Also L is not commutative because $\left(x^3, b\right)\left(x^2, b\right) = (x, b + \beta) \neq (x, b - \beta) = \left(x^2, b\right)\left(x^3, b\right)$.

Now it remains to show that $N(L) = \{(1, a) : a \in A\}$. For this consider

$$\begin{aligned}
((g, b)(1, a))(h, c) &= (g, b)((1, a)(h, c)) \\
&\iff (g, b + a + \theta(g, 1))(h, c) = (g, b)(h, a + c + \theta(1, h)) \\
&\iff (g, b + a + 0)(h, c) = (g, b)(h, a + c + 0) \\
&\iff (gh, b + a + c + \theta(g, h)) = (gh, a + b + c + \theta(g, h))
\end{aligned}$$

Which is true, so

$$(1, a) \in N_\theta(L)$$

Also

$$\begin{aligned}
((1, a)(g, b))(h, c) &= (1, a)((g, b)(h, c)) \\
&\iff (g, a + b + \theta(1, g))(h, c) = (1, a)(gh, b + c + \theta(g, h)) \\
&\iff (g, a + b + 0)(h, c) = (1, a)(gh, b + c + 0) \\
&\iff (gh, a + b + c + \theta(g, h)) = (gh, a + b + c + \theta(g, h))
\end{aligned}$$

$$\Rightarrow (1,a) \in N_\lambda(L)$$

Finally

$$\begin{aligned}((g,b)(h,c))(1,a) &= (g,b)((h,c)(1,a)) \\ &\Longleftrightarrow (gh, b+c+\theta(g,h))(1,a) = (g,b)(h, a+c+\theta(h,1)) \\ &\Longleftrightarrow (gh, a+b+c+\theta(g,h)+\theta(gh,1)) = (g,b)(h,a+c+0) \\ &\Longleftrightarrow (gh, a+b+c+\theta(g,h)) = (gh, a+b+c+\theta(g,h))\end{aligned}$$

$$\Rightarrow \quad (1,a) \in N_\rho(L)$$

hence

$$(1,a) \in N(L)$$

$$\Rightarrow \{(1,a) : a \in A\} \subset N(L) \tag{6}$$

Conversely: Let $(k,a) \in N(L)$ where $a \in A$ so

$$((g,b)(k,a))(h,c) = (g,b)((k,a)(h,c))$$

$$\Longleftrightarrow (gk, a+b+\theta(g,k))(h,c) = (g,b)(kh, a+c+\theta(k,h))$$

$$\Longleftrightarrow ((gk)h, a+b+c+\theta(g,k)+\theta(gk,h)) =$$

$$(g(kh), a+b+c+\theta(k,h)+\theta(g,kh))$$

And this will be true only if $k=1$, i.e $(k,a) \in \{(1,a) : a \in A\}$

$$\Rightarrow N(L) \subset \ \{(1,a) : a \in A\} \tag{7}$$

From (6) and (7) we get

$$N(L) = \{(1,a) : a \in A\}$$

Which is the required result. □

Example. The smallest group A satisfying the assumptions of Proposition is the 3-element cyclic group $\{0,1,2\}$. Following the construction given in Proposition and taking $\beta = 2$, we get the following nonassociative noncommutative middle nuclear square loop of order 12.

$\cdot$	$(1,0)$	$(1,1)$	$(1,2)$	$(x,0)$	$(x,1)$	$(x,2)$
$(1,0)$	$(1,0)$	$(1,1)$	$(1,2)$	$(x,0)$	$(x,1)$	$(x,2)$
$(1,1)$	$(1,1)$	$(1,2)$	$(1,0)$	$(x,1)$	$(x,2)$	$(x,0)$
$(1,2)$	$(1,2)$	$(1,0)$	$(1,1)$	$(x,2)$	$(x,0)$	$(x,1)$
$(x,0)$	$(x,0)$	$(x,1)$	$(x,2)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$
$(x,1)$	$(x,1)$	$(x,2)$	$(x,0)$	$(x^2,1)$	$(x^2,2)$	$(x^2,0)$
$(x,2)$	$(x,2)$	$(x,0)$	$(x,1)$	$(x^2,2)$	$(x^2,0)$	$(x^2,1)$
$(x^2,0)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$	$(x^3,1)$	$(x^3,2)$	$(x^3,0)$
$(x^2,1)$	$(x^2,1)$	$(x^2,2)$	$(x^2,0)$	$(x^3,2)$	$(x^3,0)$	$(x^3,1)$
$(x^2,2)$	$(x^2,2)$	$(x^2,0)$	$(x^2,1)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$
$(x^3,0)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$	$(1,1)$	$(1,2)$	$(1,0)$
$(x^3,1)$	$(x^3,1)$	$(x^3,2)$	$(x^3,0)$	$(1,2)$	$(1,0)$	$(1,1)$
$(x^3,2)$	$(x^3,2)$	$(x^3,0)$	$(x^3,1)$	$(1,0)$	$(1,1)$	$(1,2)$

continued $\cdots$

$\cdot$	$(1,0)$	$(x^2,1)$	$(x^2,2)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$
$(1,0)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$
$(1,1)$	$(x^2,1)$	$(x^2,2)$	$(x^2,0)$	$(x^3,1)$	$(x^3,2)$	$(x^3,0)$
$(1,2)$	$(x^2,2)$	$(x^2,0)$	$(x^2,1)$	$(x^3,2)$	$(x^3,0)$	$(x^3,1)$
$(x,0)$	$(x^3,0)$	$(x^3,1)$	$(x^3,2)$	$(1,0)$	$(1,1)$	$(1,2)$
$(x,1)$	$(x^3,1)$	$(x^3,2)$	$(x^3,0)$	$(1,1)$	$(1,2)$	$(1,0)$
$(x,2)$	$(x^3,2)$	$(x^3,0)$	$(x^3,1)$	$(1,2)$	$(1,0)$	$(1,1)$
$(x^2,0)$	$(1,2)$	$(1,0)$	$(1,1)$	$(x,1)$	$(x,2)$	$(x,0)$
$(x^2,1)$	$(1,0)$	$(1,1)$	$(1,2)$	$(x,2)$	$(x,0)$	$(x,1)$
$(x^2,2)$	$(1,1)$	$(1,2)$	$(1,0)$	$(x,0)$	$(x,1)$	$(x,2)$
$(x^3,0)$	$(x,2)$	$(x,0)$	$(x,1)$	$(x^2,1)$	$(x^2,2)$	$(x^2,0)$
$(x^3,1)$	$(x,0)$	$(x,1)$	$(x,2)$	$(x^2,2)$	$(x^2,0)$	$(x^2,1)$
$(x^3,2)$	$(x,1)$	$(x,2)$	$(x,0)$	$(x^2,0)$	$(x^2,1)$	$(x^2,2)$

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	2	0	4	5	3	7	8	6	10	11	9
2	2	0	1	5	3	4	8	6	7	11	9	10
3	3	4	5	6	7	8	9	10	11	0	1	2
4	4	5	3	7	8	6	10	11	9	1	2	0
5	5	3	4	8	6	7	11	9	10	2	0	1
6	6	7	8	10	11	9	2	0	1	4	5	3
7	7	8	6	11	9	10	0	1	2	5	3	4
8	8	6	7	9	10	11	1	2	0	3	4	5
9	9	10	11	1	2	0	5	3	4	7	8	6
10	10	11	9	2	0	1	3	4	5	8	6	7
11	11	9	10	0	1	2	4	5	3	6	7	8

We verified the above example with the help of GAP(Group Algorithm Program) package [4].

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